A Dynamic Oligopoly Model with Assymetric Information.
(incomplete, work in progress)

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1 Introduction

In considering the stability of price fixing arrangement the emphasis in the literature is on the incentives that each firm has to deviate from the price fixing arrangement. A collusive agreement would be sustainable if the gain from deviation can be offset by long term punishment. Different authors pointed out that there are other threats for the stability of the collusive behavior. Scherer (1980, p.199) writes "different sellers are likely to have at least slightly divergent notions about the most advantageous price. Especially with homogenous products, these conflicting views must be reconciles if joint profits are to be held near the potential maximum". A similar view has been expressed by Tirole (1989, p.242) "It is often felt that heterogeneity in both costs and products may make coordination on a given price difficult". The Antitrust literature points out on a similar concern and Posner (1976, p.64) writes "Among the obstacles to fixing a mutually satisfactory price are the conflicting interests of sellers having different costs".

The empirical literature also identifies firms heterogeneity as a potential source of cartel instability. In studying the collusive behavior among Bromine producers between 1885-1914 Levenstein (1997) pointed out that only half of the price wars that occurred in this industry can be interpreted as a punishment phase that followed deviations from the prescribed collusive behavior. The other half of the price wars, and in particular the more severe ones, occurred when one of the firms demanded a more favorite allocation of the collusive market shares. Interestingly, in these cases the price cuts were announced ahead of time, there was no surprise deviation from the prescribed prices and no attempt to gain some short run profits. McMillan (1991) described the price fixing and market sharing in Japan's construction industry. In such an symmetric industry market shares were allocated in meetings and side payments were part of the practice and as McMillan pointed out: "These mechanisms do not always work to everyone's satisfaction: periodically, a dango is exposed when a member, dissatisfied with the share of the spoils he is offered, leaks details of the dango to the press". That is, a firm that demanded a larger share of the collusive pie initiated a penalty or a punishment for all the members of the cartel.

Much of the theoretical literature on collusive behavior emphasis the need of the cartel to balance the short run gains deviating (or from undercutting one's competitors) and the expected long run losses from entering the punishment phase (see Friedman (1971), Rubinstein (1979) and Abreu (1986)). Capturiong this aspect of collusive behavior one can indeed adopt a setup with a symmetric industry. For explaining the occurance of price wars it is not sufficient to consider the possibility of deviation from the collusive behavior but as Green and Porter (1984) pointed out we need to introduce imperfect monitoring and stochastic demand.[1]

Collusion among asymmetric firms has been discussed in several articles; see Scmalensee (1987), Harrington (1989), Fudenberg, Levine and Maskin (1994) and Compte, Jenny and Rey (1999). In all these papers the asymmetry between the firms was exogenously given and did not change throughout time. In recent two papers Athey and Bagwell (2001) and Athey, Bagwell and Sanchirico (2001) consider also a collusive market with incomplete information. Part of the optimal collusive behavior is that at every period the most efficient firm does all the production. The information assymetry makes this coordination problematic and as a result collusive behavior gives rise to price

[1] In such a case firms do not observe the actions taken by their competitors and at the same time cannot deduce from their own profits or market price the actions taken by their competittors. Consequently, firms can deviate from the collusive agreement without leaving a "direct evidence" for such a deviation. Price wars occur whenever there are sufficient "indirect evidence" for a deviation (i.e. very low prices or profits). However, it is important to point out that on the equilibrium path there are no deviations yet firms use price wars (i.e., punishments) to enforce such a behavior. Abreu, Pearce and Stacchetti (1986) extended Green and Porter (1984) and considered the optimal cartel behavior.

rigidity and on the other hand may involve in some circumstances occasional price wars.[2]

In a recent paper Fershtman and Pakes (2000) considered an asymmetric collusive market in a dynamic setup in which firms invest and thus the firms' relative position is endogenously changed over time and entry and exit are determined endogenously. When the industry has an asymmetric structure such that some firms are relatively disadvantageous (have low level of capital) there is a high probability that they will stay in the market only for couple period and exit afterwards. Consequently, the ability to punish such firms is limited and without the severe punishment it is not possible to sustain the collusive behavior. Note however that the explanation of price wars in this model is very much in the tradition of the whole IO literature. which .and therefore like in the rest of the literature if you cannot punish you cannot sustain collusion. Thus the paper demonstrates that asymmeteries and in particular endogenous asymmeteries play an important role in the stability of collusive agreement and on what an industry may achieve by strategy coordination.

Going back to the example of the Bromine cartel the traditional IO theory can explain only 50% of the price wars. The other price wars in this industry had been triggered by a disagreement between the firms on how to allocate the market. There are two aspects of this issue. First it is possible to have a price war at the initial stage of the game when there is an assymetry between the firms but for this we need to have incomplete information regarding the type of each firm. This type of effect resembles the analysis of bargaining with incomplete information. But the fact that such price wars repeat themselves in several period implied that a sitaution of bargaining with incomple information may occur in several period. Thus a theory that emphasizes periodic disagreement on the collusive arrangement would require a setting with endogenous asymmetric firms and changing environment.[3]

We consider in this paper a dynamic oligopolistic market with entry and exit in which firms invest in improving their relative position but the realization of this investment is private information. Prices are coordinated in (costly) meetings during which all relevant information is revealed. In peri-

[2]The industry asymmetry in these papers however is within a period. At each period ex-ante all the firms are identical. thus the setup is a repeated game with asymmetric information rather than dynamic game setup which is employed in our paper.

[3]Also if the environment is unchanged then once they reach an agreement then there is no reason to change it.

ods following a meeting firms maintain their agreement and charge the prices that have been agreed upon they invest but observe only the realization of their own investment. Firms, however, maintain the right to demand a meeting in which information about the firms' relative position is revealed and the collusive prices are recalculated. While the timing of such a demand will be endogenously determined it will occur when a firm believe that its relative position sufficiently improve to warrant a costly recalculation of collusive pricing.[4] We do not allow for deviations from the collusive arrangement and the costly meeting (or price wars) are the outcome of a dissatisfaction of one or more firms with the existing collusive arrangement.[5] We (numerically) analyze such a market and examine the timing and the frequency of meetings, the effect of the collusive arrangement on entry, exit, investment and productive efficiency. We compare the equilibrium behavior in our model to the behavior in a similar model but with complete information regarding the firms relative position and discuss the effect of incomplete information on the market equilibrium. The numerical method that we used in this paper is the stochastic algorithm method (see Pakes 2001) which we adjust to take account of incomplete information.

2 The Model.

We adapt the framework presented in Ericson and Pakes (1995) and modify it to allow for asymmetric information and collusion. In each period there are n_t incumbent firms which differ in their physical characteristics, in say $\omega_{i,t}$. $\omega_{i,t}$ is, for example, a characteristic of the firm cost function, and it evolves over time with the outcomes of an investment process. Positive outcomes

[4]Our interpretation of costly meeting is a period with a costly price war. We assume however that during such price wars the firms relative position becomes public information. this capture our intuition that during price wars there are mechanisms that provides information regarding the firms' position. However for simplicity such mechanisms are beyond the scope of this paper.

[5]We thus do not explain the occurance of price wars as we impose a breakdown of the collusive agreement whenever one of the firms demand a change. a more complete description would be to allow firms to demand a change of the collusive arrangement and then let the competitors either agree to such a change or to call for a meeting. Clearly a strategy in which firms always conceed to such demands will not be a part of equilibrium behavior as in such a case firms will always demand changes in their favor regardless of their position.

lead to states in which the firm can make more profits.

Every period firms set their output for that period, say $q_{i,t}$, and observe their profits for that period, then firms decide simultaneously and independently on exit, entry, and investment. I.e. incumbents decide whether to exit and potential entrants decide whether to enter. Entrants who do enter pay a sunk cost of entry and enter at a particular state in the following period (it takes one period to set up their plant and equipment). The firms that stay in the market sets their investment, $x_{i,t}$. The outcomes of these investment activities becomes apparent at the beginning of the next period.

These quantities, together with the firms' state variables determine the profits of each active firm, say $\pi_i(\omega, q) = \pi(\omega_{i,t}, q_{i,t}, q_{-i,t})$. Note that the firm's profit does not depend on $\omega_{-i,t}$,this is due to our interpretation of the state variable as the cost position of the firm.[6]

Investments are directed at improving the firm's "physical" state, their ω value which evolves over time with the outcome of the firm's investment process and an industry specific exogenous common shock. We assume that the firm's investment output is only observable by the firm in question. That is, in a typical period each firm knows its own ω_i but does not know the ω of the other firms. The oligopoly game is therefore a dynamic game with asymmetric information.

We allow, however, that the firms' state variables would be periodically observed by rival firms. Such information would be revealed in periods in which there are "meetings". During such a meeting information about ω is fully revealed and the incumbent firms decide on a vector of output.

Meeting are either called by one of the firms that wish to change the collusive arrangement i.e., market shares or collusive prices, or when there is a change in the market structure that requires an adjustment of the collusive arrangement. Meeting are costly

We distinguish between two types of situations that lead to a meeting.

[6]Thus in a Cournot quantity competition the cost of one firm does not affect the profits of its competitors. The issue here is that profits, conditional on quantities, does not reveal anything about the current value of the competitor's states. This would happen in differentiated product models also, if there were many state variables per firm, and current profits did not depend on the state variable for which there is asymmetric info. However then we would need at least two state variables per firm; one generating known differences among firms and one generating differences that are asymmetrically known. A good e.g. might be the oil cartel, where the cost of extraction are asymmetric but the locations and the quality of the oil are known.

The first is due to either a call for a meeting by one (or more) of the incumbents or new entry. The cost of such a meeting is FK. The other instance is when one of the firms exit the market, in which case the remaining firms need to realign their collusive quantities. The cost of these meetings is given by K (which could be zero).

A "meeting" in our model is not just a mechanism that reveals information about the firms' state variables, it is also determines how the market will be shared for some (endogenously determined) time. That is, like in the OPEC meeting, the firms agree on a quota system for all the incumbent firms.

2.1 The Profit Function.

Firms compete in a Cournot type competition. We assume a linear market demand function; i.e. $P = a - bQ$ where P is market price and Q is the total quantity produced by all firms. We assume a constant marginal cost which varies with the firm's productivity (ω), i.e. $c(\omega_i, q_i) = c(\omega_i)q_i$, where

$$c(\omega_i) \equiv \frac{\gamma exp[-\omega_i/3]}{1 + exp[-\omega_i/3]}.$$

For notational simplicity we write c_i as a short form for $c(\omega_i)$.

The resultant per period profit function is

$$\pi(\omega_i, q) = (a - b\sum_j q_j)q_i - c_i q_i. \tag{1}$$

2.1.1 Collusive Profits

We allow firms to collude in setting quantities. The quantities are agreed to at specific, endogenously determined, times, to be called "meeting" times. We want the quantity vector agreed to in these meetings to reflect the firms relative position in the market.

When all firms are identical it is natural to focus on collusive arrangements in which the gains from collusion are equally distributed among firms. There is less agreement on collusive rules when firms differ from one another and side payments are not possible; except perhaps for the conditions that the collusive agreement should increase all firms' profits and leave "better" firms better off. Subject to these constraints there is no conceptual problem with taking any solution for the vector of collusive profits conditional on ω and using it here. We focus on one of the suggestions of Schmalensee(1987), and assume that relative market shares are determined by the Nash solution, but total output maximizes the sum of profits conditional on these share [7].

I.e. if we let $q^N(\omega)$ be the Cournot equilibrium output vector, and $s^N(\omega)$ be the Cournot equilibrium market shares, then total output ($Q^c = \sum q_i$) is determined as

$$Q^c(\omega) = argmax_Q \sum_i \pi(\omega_i, s^N(\omega)Q)$$

where,

$$\pi(\omega_i, s^N(\omega), Q) = (a - bQ)s_i^N(\omega)Q - c_i s_i^N(\omega)Q. \tag{2}$$

and[8]

$$s_i^N(\omega) = \frac{(N+1)(a - c_i)}{\sum_j (a - c_j)} - 1. \tag{3}$$

Note that all shares are positive if $a/N > \gamma$, and that in this case

$$Q^c(\omega) = \frac{a - \sum_i s_i c_i}{2b}. \tag{4}$$

We impose the condition that collusive profits are higher than the Nash equilibrium profits, that is; $\pi(\omega_i, s^N(\omega), Q^c(\omega)) > \pi(\omega_i, q^N(\omega))$. This condition will be satisfied for all the range of parameters that we use in our analysis.

We assume that when a meeting occurs all the state variables become common knowledge (though precisely how this happens is a "black box" in our framework). Then the collusive output vector is calculated and implemented. However as the game proceeds firms observe only the update of their own state variable and not the updated state variable of the other firms in the industry or their investment levels. Consequently firms continue to follow the collusive output vector that was agreed upon in the last meeting until it is either challenged by one of the incumbent firms or there is entry or exit.

[7] An alterntive might be to assume a bargaining solution as in Fershtman and Pakes, 2000. The only disadvantage of this is that it is computationally more burdensome.

[8] When $s_i^N(\omega) < 0$ we assume that the firm does not produce and then we recalculate the market shares.

2.1.2 Meetings.

There are two types of meetings depending on the circumstances that trigger them. If a continuing incumbent calls a meeting, or their is entry, then market shares are reallocated and the collusive level is determined. In such a case all firms bear the cost of FK.[9] However, whenever there is only a single firm in the industry this firm does not bear the above costs as in such a case there is no need for coordination and the firm may adjust its output without paying extra costs for such adjustments.

Meetings also occur when a firm exits, since exit triggers the need to reallocate market shares. These meetings result in an immediate realignment of market shares to the collusive level that results from the current vector of $\omega's$. We allow the meeting to be costly, and allocate a cost of K to each continuing incumbent allowing for K<FK. However, as before if after the exit there is a single firm that remain in the industry, this firm does not bear this cost as there is no need to realignment with other firms.

2.2 Physical States, Investment, and Entry and Exit.

As in EP(1995) we assume that $\omega_{i,t}$ is an integer, so $\omega_{i,t} \in \Omega \subset Z$, so $\omega_t = (\omega_{1,t}, \ldots, \omega_{n_t,t}) \in \Omega^{n_t} \subset (Z)^{n_t}$. $\omega_{i,t}$ evolves over time with the outcomes of the firm's investment process, say $\eta_{i,t}$, and an industry specific exogenous process that affects the profit opportunities facing the industry in a given period, say ν_t. ν_t captures the effects of factor prices and improvements in competition from products outside the industry; factors which generate positive correlation among the profits of our competing firms. Thus

$$\omega_{i,t+1} = \omega_{i,t} + \eta_{i,t+1} - \nu_{t+1} \tag{5}$$

Both η and ν will be nonnegative random variables, and the distribution of $\eta_{i,t+1}$ will be better, in the stochastic dominance sense, the larger is investment, our $x_{i,t}$. That is the distribution of η is determined by the family

$$P = \{P_\eta(.|x), x \in R_+\},$$

[9] An alternative formulation would be to assume that the costs is a proportion of the profits but in such a case results may be different as high cost firms will pay lower penalties.

which is assumed stochastically increasing in x. The distribution of ν is given exogenously.

If an incumbent decides to exit it gets a sell-off value of ϕ_i dollars, exits in the next period and never reappears again. We assume a stochastic scrap value, $\widetilde{\phi}_i$ which is uniformly distributed over $[\phi_l, \phi_h]$. We further assume that the realization of $\widetilde{\phi}_i$ is observed by firm i prior to deciding on investment/exit/meeting but that this realization is not observed by other players. We let $\chi_{i,t} \in \{0, 1\}$ indicate whether a firm exits ($\chi_{i,t} = 0$) or continues ($\chi_{i,t} = 1$).

Potential entrants decide whether to enter or not. To enter they must pay a sunk cost of x^e. We assume stochastic entry cost; $\widetilde{x}^e$ which is uniformly distributed over $[x_l^e, x_h^e]$. The realization of $\widetilde{x}^e$ is observed by the potential entrant prior to the entry decision, but it is not observable by other players. An entrant appears in the following period as an incumbent at an $\omega_{i,t+1} = \omega^e - \nu_{t+1}$ where ω^e is given. For simplicity we assume there is at most one entrant in every period, and indicate whether entry occurs by the indicator function $\chi^e = \{0, 1\}$, $\chi^e = 1$ indicating entry.

2.3 Details of the Game.

2.3.1 Timing of Decisions.

Information becomes available in the following sequence.

- At the beginning of the period there is a realization of the outcome of the investment processes of the last period and a realization of the entry cost $\widetilde{x}^e$and the scrap value for each firm $(\widetilde{\phi}_1, ..., \widetilde{\phi}_n)$.

- If either; (i) a meeting was called in the previous period, or if (ii) an entry decision was taken during the previous period, or if (iii) one or more firms decide to exit during the previous period, then there is a meeting. The meeting allocates quantities according to the rules above. If there is no meetings quantities are the same as in the last period.

  Note; At this point in the period all incumbents and potential entrants have the information required to make the decisions needed for this period. When we refer to the information set of period t we will be referring to the information available at this point in the period.

- Simultaneously, profits are allocated and decisions are made on: entry, exit, calling a meeting, and investment.
- Simultaneously; (i) firms who choose to exit announce that, (ii) entrants announce, pay the cost of entry, and are ready to operate at the beginning of the next period, and (iii)firms who wish to call a meeting do so.
- The period ends.

2.3.2 Information Sets.

- We let J_t be the information that is common to all incumbents and potential entrants at the time quantity decisions were made (i.e. after a meeting if there was one). Then

$$J_t = \{\omega_{t-\tau(t)}, \hat{\tau}(t), \nu(\hat{\tau}(t))\} \in \mathcal{J},$$

  where,

  - $\tau(t)$ is the number of periods that have passed since the meeting (if there was a meeting this period $\tau(t) = 0$). To ease the computational burden we assume imperfect recall; i.e. firms can recall information from at most $\overline{\tau}$ periods. Thus we set $\hat{\tau}(t) \equiv min\{\tau(t), \overline{\tau}\}$. So at period t firms only recall the precise date of the last meeting if that meeting was less than $\overline{\tau}$ periods before, otherwise they just realize it was more than $\overline{\tau}$ years ago. Note however that $\omega_{t-\tau(t)}$ is in the information set in every period since it was always used in period $t-1$ (and if there was no meeting there is no new information on these variables).
  - $\nu(\hat{\tau}(t)) \equiv (\nu_{t-\hat{\tau}(t)+1}, \ldots, \nu_t)$, with the understanding that if a meeting was called in the current period $\nu(\hat{\tau})$ is empty. As above firms recall at most the common shocks over the last $\overline{\tau}$ periods.
  - $J = \Omega^N \times \{1, \ldots, \overline{\tau}\} \times \{0,1\}^{\overline{\tau}}$, where $\omega \in \Omega$ and N is the maximum number of firms ever active. We invoke regularity conditions similar to those in Ericson and Pakes (1995), to insure that both N and $\#\Omega$ are finite.

- $J_{i,t}$ will be firm i's information set in t (i.e., the public information and the firm's own state).

$$J_{i,t} = \{\omega_{i,t}, J_t\} \in \Omega \times \mathcal{J}.$$

- The information set of a potential entrant, say $J_{e(t)} = (J_t, x^e)$.

2.3.3 Strategies.

- Quantity decisions: $q_{i,t} = q^c(\omega(t - \tau_t))$. That is the firms always produce according to the last agreement. Thus firms do not change their output if there is no new meeting that specifies a new quota vector.

- The following decisions are made simultaneously (that is each decision is only based on $J_{i,t}$ and the firms' private information regarding their scrap value ϕ_i or entry costs x^e).

  - Exit. $\chi_{i,t} : (J_{i,t}, \phi_i) \to \{0, 1\}$.
  - Entry. The potential entrant knows the publicly available information, so its decision is $\chi^e_t : (J_t, x^e) \to \{0, 1\}$.
  - Calling a meeting: $m_{i,t} : (J_{i,t}, \phi_i) \to \{0, 1\}$.
  - Investment. $x_{i,t} : (J_{i,t}, \phi_i) :\to R^+$. Note a firm's investments are not observed by its competitors.

Note that the strategies $m_{i,t}$ and $x_{i,t}$ are also functions of the realization of the value of the scrap value, ϕ_i, since a firm that choose to exit cannot call a meeting (or invest).

2.3.4 Transitions of State variables.

Recall that after these decisions it is common knowledge whether or not there will be a meeting in the next period. There will be a meeting if

1. $\chi^e_t = 1$, or
2. $\sum m_{i,t} > 0$, or
3. $\sum(1 - \chi_{i,t}) \neq 0$,

In the first two cases the meeting is associated with an exogenous cost of FK. In the third case the meeting is accompanied by the exogenous cost, K . Note that both cases these costs will be zero if there is a single firm in the industry.

The transitions for the individual states are given by

- If $\chi_{i,t} = 0$, then the firm disappears forever.
- If $\chi_{i,t} = 1$, then $\omega_{i,t+1} = \omega_{i,t} + \eta_{i,t+1} - \nu_{t+1}$, where $\eta_{i,t+1} = 1$ with probability $p(x_{i,t})$ and zero elsewhere.
- If $\chi^e_{t-1} = 1$, then there is a new entrant at $\omega_{i,t} = \omega^e - \nu_t$.

2.4 Equilibrium: Applied Markov Perfect Equilibrium.

Let $\mathcal{S} = \Omega^N \times \mathcal{J}$ and let s be its generic element. Note that an $s \in \mathcal{S}$ defines a tuple $(J, J_1, \ldots, J_{n(J)})$, where J provides the information available to the potential entrant and each $J_i = J \times \omega_i$ provides the information available to the different incumbents. We will say that J_i (or J) is a component of s if there is a J_{-i} such that $(J, J_i, J_{-i}) = s$.

Definition. An equilibrium is a tuple:

1. $W = \{(W(\eta|J_i, m),\ \forall J_i \in \Omega \times \mathcal{J},\ \eta \in (0,1),\ m \in (0,1)\}$,
2. $\mathcal{R} \subseteq \mathcal{S}$,
3. Policies,$(x(J_i, \phi_i), \chi(J_i, \phi_i), m(J_i, \phi_i))$ or $\chi^e(J, x^e)$, for each J and J_i which is a component of an $r \in \mathcal{R}$
4. and an initial condition, say $r_0 \in \mathcal{R}$,

such that the following conditions hold:

1. Let $\mathcal{P}_{\mathcal{R}} \equiv \{p^e(s'|r, policies),\ r \in \mathcal{R}\}$. $\mathcal{P}_{\mathcal{R}}$ is the Markov process constructed from the polices generated by the $r \in \mathcal{R}$. Then $\mathcal{R}$ is a recurrent class of $\mathcal{P}_{\mathcal{R}}$[10],

[10]That is if $r \in \mathcal{R}$ communicates with s' then $s' \in \mathcal{R}$, and every $(r, r') \in \mathcal{R}^2$ communicates with each other.

2. If J_i is a component of $r \in \mathcal{R}$ then the policies are optimal given $W(\cdot|J_i, m)$[11] ,

3. If J_i is a component of $r \in \mathcal{R}$ then

   (a) if $\chi(J_i, \phi) = 1$ and $x(J_i) > 0$, then $W(\cdot|J_i, m(J_i))$ satisfies

   $$W(\eta|J_i, m(J_i)) = E_{p^{e,i}(J_i'|J_i,\eta)} \left[\pi(J_i') - Z(J_i, J_i') + \beta V(J_i') \mid \eta\right]$$

   for $\eta = [0, 1]$, where $Z(\cdot)$ is FK, K or 0 depending on whether there is a meeting, exit, or no reallotment of quantities during a period, $p^{e,i}(J_i'|J_i, \eta_i)$ is the empirical distribution generated by the optimal policies and a give η_i, and

   $$V(J_i') =$$

   $$E_{\tilde{\phi}} max \left\{ \tilde{\phi}, max_m sup_x \left[W(1|J_i', m)p(x) + W(0|J_i', m)(1 - p(x)) - x\right] \right\},$$

   while if $x(J_i) = 0$ this condition must be satisfied only for $\eta = 0$ (see the remarks below).

   (b) if $\chi^e(J, x^e) = 1$, then

   $$W(0|J_e, m = 1) =$$

   $$E_{p^{e,\omega(e)}(J'_{\omega(e)}|J,0,m=1)} \left[\pi(J'_{\omega(e)} - FK + \beta V(J'_{\omega(e)}) \mid \eta\right]$$

   with $p^{e,\omega(e)}(\cdot|\cdot)$ and $V(J'_{\omega(e)})$ defined as above. ♠

Remarks.

- Note that $p^{e,i}(\cdot|\cdot, \eta)$ is an objective distribution, i.e. it provides the probabilities of J_i' given any (J_i, η) that is actually observed by the agent. These probabilities are not defined when an element in the conditioning set is not observed when using equilibrium policies. For example $p^{e,i}(\cdot|\cdot)$ are not defined for m different from $m(J_i)$, and consequently there is no condition $(3a)$ for $W(\eta|J_i, m \neq m(J_i))$.

[11] Note that $W(0|J_e, m = 1)$, where $J_e \equiv J \times \omega_e$ determines the value of entry, so that an optimal entry policy is $\chi_e = 1 \Leftrightarrow W(0|J_e, m = 1) > x^e$.

- However the $W(\eta|J_i, m)$ that are not constrained by condition (3a) are constrained by condition (2), or the optimality condition (by weak inequality constraints). For e.g. if $x(J_i) = 0$ then the only condition we have on $W(1|J_i, m(J_i))$ is the condition that derives from the optimality condition for x which is

$$[W(1|J_i, m(J_i)) - W(0|J_i, m(J_i))] \leq [\frac{\partial p(x)}{\partial x}|_{x=0}]^{-1}.$$

- One can also define an equilibrium for our game without relying on the empirical distribution of outcomes. Such a definition will specify $W(\cdot|\cdot)'s$, policies, and perceptions $p(\cdot|\cdot)'s$ for all possible information sets and actions such that; policies are optimal given $W(\cdot|\cdot)$, the $W(\cdot|\cdot)$ are consistent with policies and perceptions, and the perceptions are consistent with policies whenever possible[12].

- Though the alternative definition of equilibrium given above may be more complete, were we to numerically (or empirically) analyze behavior from any $r \in \mathcal{R}$, conditions (1) to (3) in our definition would be the only conditions that would be testable. Moreover had we required the fuller set of conditions the burden of computing an equilibria would be orders of magnitude larger. The increased computational burden would result from both requiring computations on a larger state space, and from requiring that the perceptions be consistent with strategies whenever possible.

3 Computing the Equilibrium

We begin with the Bellman equation with the information available just before all decisions are made and profits allocated. This will define equilibrium values and policies.

If we let $\pi(J_{i,t}) \equiv \pi(\omega_{i,t}, q(\omega(t - \tau_t))$, and omit the (i, t) index for notational convenience, then the Bellman equation

$$V(J) = \pi(J) + E_{\widetilde{\phi}_i} max \left\{ \beta\phi_i; max_{m\in\{0,1\}} sup_{x\geq 0} \left[-x + \sum_\eta W(\eta|J, m) p(\eta|x) \right] \right\} \tag{6}$$

[12]Note that if we interpret our empirical distributions as perceptions then whenever they are defined they satisfy consistency with respect to policies.

where

$$W(\eta^*|J,m) \equiv \beta E_{(J')}\left\{V(J')|\eta' = \eta^*, J, m, \chi = 1\right\}. \qquad (7)$$

Writing the Bellman equation in this way makes it easy to see that were we to compute the fixed point defining equilibrium behavior iteratively, i.e. if we were to compute $V(\cdot)$ at each iteration of a successive approximation routine, we would have to evaluate

$$E_{(J')}\left\{V(J')|\eta' = \eta^*, J, m, \chi = 1\right\}$$

explicitly at each point at each iteration. This would require explicit calculation of "posterior" probabilities of the form

$$Pr(\omega_{-i,t} = \omega^*_{-i}|J_{i,t}, m_{i,t}, \chi_{i,t} = 1)$$

at each point. These probabilities would have to be calculated recursively and kept in memory. Moreover the cardinality of the set of ω_{-i} with positive probabilities (and hence the number of probabilities that would have to be kept in memory and integrated over) would increase exponentially in τ.

What we are about to do is present a computational algorithm which never requires us to calculate this expectation, and hence never requires us to calculate and retain posterior probabilities. Instead we will compute the $W(\eta|J,m)$ in (7) iteratively using techniques analogous to those used in the stochastic approximation literature [13]. That is we will start with an initial values of $W(\eta|J,m)$; for$\{(\eta, J, m) \in \{0,1\} \times (J \times \Omega) \times \{0,1\}$, and update them through an iterative process that never requires the calculation of $V(\cdot)$. I.e. in each iteration we will only need our estimates of $W(\cdot)$ and the policies obtained by substituting those estimates into equation (9). The next section describes this iterative process.

Note that if we knew our iterative process had converged the equilibrium values of $W(\cdot)$ we could calculate equilibrium behavior and the resulting values directly from equation(). However we are left with the problem of determining whether a candidate set of $W(\cdot)$ outputted by our algorithm do in fact satisfy the equilibrium conditions. We could verify whether any particular set of $W(\cdot)$ satisfy these conditions by calculating the expectation in

[13] This literature dates back to Robbins and Monroe, ; for a more modern treatment see Beertsekas and Tsikilis,???? ; and for applications to dynamic games see Pakes and McGuire,2001.

equation (7) for those $W(\cdot)$ and ensuring the boundary conditions analogous to those introduced in Pakes and McGuire2001 are satisfied. This, however, would require us to calculate the posteriors implicit in those $W(\cdot)$, albeit only at a test (instead of at every) iteration. Still as demonstrated by Pakes and McGuire(2001) on a much "smaller" problem, just the test iterations computation increases the computational burden of the algorithm enormously. So below we will introduce a test procedure which never requires us to calculate the fixed point explicitly – not even in the test iteration.

Before going on to introduce the iterative stochastic algorithm it will be useful to rewrite equation (9) as

$$V(J_{i,t}) = E_{\tilde{\phi}_i} max\left\{\beta\phi_i + \pi(\omega_{i,t}, q(\omega(t-\tau_t)); max_{m_{i,t}\in\{0,1\}} V^c(J_{i,t}, m_{i,t})\right\}, \tag{8}$$

where

$$V^c(J_{i,t}, m_{i,t}=1) = \pi(\omega_{i,t}, q(\omega(t-\tau_t))) \tag{9}$$
$$+max_{x\in R^+}\{E[-x - \beta FK(J_{i,t+1}) + \beta V(J_{i,t+1})] | J_{i,t}, x, m_{i,t}=1\},$$

and

$$V^c(J_{i,t}, m_{i,t}=0) = \pi(\omega_{i,t}, q(\omega(t-\tau_t)) \tag{10}$$
$$+max_{x\in R^+}\left\{E\left[-x - \beta Z_1 FK(J_{i,t+1}) - \beta Z_2 K(J_{i,t+1}) + \beta V(J_{i,t+1}) | J_{i,t}, x, m_{i,t}=0\right]\right\}$$

where

$FK(J_{i,t+1}) = FK$ (similarly $K(J_{i,t+1}) = K$) if the #firms$(J_{i,t+1}) > 1$ and $FK(J_{i,t+1}) = 0$ (and $K(J_{i,t+1}) = 0$) otherwise.

and Z_1 and Z_2 capture hostile and friendly renegotiation respectively such that

$$Z_1 \equiv \{\sum_{j\neq i} m_{j,t} > 0 \ or \ \chi_e(J_t) = 1\}$$

$$Z_2 = [1 - Z_1]\{\sum_{j\neq i}(1 - \chi_{j,t}) > 0\}$$

and $\{\cdot\}$ is notation for the indicator function which takes the value one if the condition "$\cdot$" is satisfied, and zero elsewhere.

Note that if

$$W(\eta | J_{i,t}, m_{i,t}=1) \equiv \beta E[(V(J_{i,t+1}) - FK(J_{i,t+1})) | \eta_{i,t+1} = \eta, J_{i,t}, m_{i,t}=1]$$

for $\eta \in \{0,1\}$, then

$$V^c(J_{i,t}, m_{i,t}=1) = \pi(\omega_{i,t}, q(\omega(t-\tau_t))) \quad (11)$$
$$+max_{x\in R^+}[-x + W(1|J_{i,t}, m_{i,t}=1)p(x) + W(0|J_{i,t}, m_{i,t}=1)(1-p(x))].$$

Similarly if

$$W(\eta|J_{i,t}, m_{i,t}=0) \equiv$$
$$E\left[(-\beta Z_2 K(J_{i,t+1}) - \beta Z_1 FK(J_{i,t+1}) + \beta V(J_{i,t+1}))|\eta_{i,t+1}=\eta, J_{i,t}, m_{i,t}=0\right]$$

for $\eta \in \{0,1\}$, then

$$V^c(J_{i,t}, m_{i,t}=0) = \pi(\omega_{i,t}, q(\omega(t-\tau_t))) \quad (12)$$
$$+max_{x\in R^+}[-x + W(1|J_{i,t}, m_{i,t}=0)p(x) + W(0|J_{i,t}, m_{i,t}=0)(1-p(x))].$$

Next we move on to the Bellman equation for a potential entrant.

$$V_e(J_t) = \beta E[V(\omega^e - \nu_{t+1}, J_{t+1}) - FK(J_{i,t+1})|\chi_t^e = 1, J_t].$$

4 A Stochastic Algorithm.

An iteration, which will be indexed by k, is defined by a location, say L_k, and by a memory, which will be designated M_k.

4.1 Storage and Policies at Iteration k

The location is defined as a tuple

$$L^k = \{J^k, \omega_1^k, \ldots, \omega_{n(J(k))}^k\}$$

where ω_j^k is the current ω of the j^{th} largest firm in the $\omega_{t-\tau(t)}$ specified in J^k, and as before,

$$J^k = \{\omega_{t-\tau(t)}^k, \hat{\tau}_t^k, \nu^k(\hat{\tau}(t))\}$$

There is the possibility of storage in memory at each possible L. Distinct objects are stored at J. Further for each J there can be items stored at each of the triples couples, $\{J, j, \omega\}$ for $j = 1, \ldots, n(J)$ and $\omega \in \{1, \ldots, \Omega\}$ (of course at any iteration some of these will have no information stored). I.e. for each J^k, we begin with the largest firm in the ω tuple defining J^k, find out its current ω, and list objects under the triple $(J^k, 1, \omega)$.

- For each J^k we will have $M(J^k)$ stored at J^k where $M(J^k)$ is
  - The number of times we have visited J^k, or $h^k(J^k)$, and if $h^k(J^k) > 0$ we have
  - $V_e^k(J)$ the k^{th} iteration's estimate of the value of entry at J
  - $q(J^k, j)$ for $j = 1, \ldots, n(J^k)$.
  - entry policy $\chi_e^k(J^k)$. Note that $\chi_e^k(J^k)$ is the **probability of entry**, which corresponds to realizations of $\widetilde{x}_e$.

  Note that if $h^k(J^k) = 0$, nothing is in memory for that point.

- Then for each $\{J^k, j, \omega\}$, where it is understood that ω here refers to the ω of the j^{th} firm in the order determined by J^k at the k^{th} iteration, we have $M(J^k, j, \omega)$, or the information stored in memory at (J^k, j, ω) as
  - $h^k(J^k, j, \omega)$ the number of times we have hit (J^k, j, ω), and if $h^k(J^k, j, \omega) > 0$
  - $W^k(\eta | J^k, j, \omega, m)$ for $m \in \{0, 1\}$ and $\eta \in \{0, 1\}$.
  - $\pi(J^k, j, \omega)$.
  - $V^k(J^k, j, \omega)$.
  - $x^k(J^k, j, w)$, $\chi^k(J^k, j, w)$, $m^k(J^k, j, w)$. Note that $\chi^k(J^k, j, w)$ is the probability of staying, which corresponds to realizations of $\widetilde{\phi}_j$.

4.2 Updating and Initialization.

The update required is as follows;

- Update $L^k = \{J^k, \omega_1^k, \ldots, \omega_{n(J(k))}^k\} \to L^{k+1} = \{J^{k+1}, \omega_1^{k+1}, \ldots, \omega_{n(J(k+1))}^{k+1}\}$
- Update $M(J^k)$. Here we update only $V_e^k(J^k)$ and $h^k(J^k)$, see below.
- Update $M(J^k, j, \omega)$, for $j = (1, \ldots, n(J^k))$. Here we update

  $$W^k(\eta | J^k, j, \omega, m) \to W^{k+1}(\eta | J^k, j, \omega, m)$$

  for $\eta \in \{0, 1\}$ and $m \in \{0, 1\}$, and $h^k(J^k, j, \omega)$, see below.

In doing these updates we will use the operator $V(\cdot|W)$ defined as

$$V(J,j,\omega|W) \equiv E_{\tilde{\phi}_i} max \{\beta\phi + \pi(J,j,\omega), V^c(J,j,\omega|W)\},$$

where

$$V^c(J,j,\omega|W) \equiv$$
$$\left\{\pi(J,j,\omega) + max_{m\in\{0,1\}} max_x[-x + W(1|J,j,\omega,m)p(x) + W(0|J,j,\omega,m)(1-p(x))]\right\}.$$

While we do the update we initialize if required. That is

- If $h^k(J^{k+1}) = 0$ compute $q(\omega^{k+1}) = s^N(\omega)Q^c(\omega)$ (from equations 2 and 4) and put it in memory for J^{k+1}. Also if we have to initialize we set

  $$V_e^k(J^k) = W^k(0|J^k + I(\omega_e), j(\omega_e, J^k + I(\omega_e))), \omega_e, m = 1),$$

  where here and below

  $$\omega^{k+1} + I(z)$$

  adds an $\omega = z$ to the ω^{k+1} vector, and then reorders it in the natural order, and where $j(z,J)$ provides the order of the $\omega = z$ element in the ω vector defined by J.

- If $h^k(J^{k+1}, j, \omega) = 0$, calculate $\pi(J,j,\omega) = \pi(\omega, s^N(\omega)Q^c(\omega))$ (from equation 3) and put in memory. Also initialize

  $$W^1(\eta|J,j,\omega,m=0) = \pi(J,j,\omega+\eta)/([1-\beta], \text{ for } \eta = \{0,1\},$$

  and

  $$W^1(\eta|J,j,\omega,m=1) = \beta\pi(J-I(j,J)+I(\omega+\eta), j(J-I(j,J)+I(\omega+\eta)), \omega+\eta)/([1-\beta],$$

  for $\eta = \{0,1\}$.

4.2.1 Update 1: Policies.

Get the realization of $\widetilde{x}^e$ and $(\widetilde{\phi}_1, ..., \widetilde{\phi}_n)$.

- Determine whether $\chi_e^k(J^k) = 1 \Leftrightarrow V_e^k(J^k) \geq x_e$.
- Choose $x(J^k, j, \omega, m)$ as

$$argmax_x[-x + \sum_\eta W^k(\eta|J^k, j, \omega, m)p(\eta|x)],\ for\ m \in \{0, 1\},$$

- Calculate

$$V^c(J^k, j, \omega, m) = \pi(J^k, j, \omega) - x^k(J^k, j, \omega, m) + \sum_\eta W^k(\eta|J^k, j, \omega, m)p(\eta|x(J^k, j, \omega, m))$$

  for $m \in \{0, 1\}$.

- Calculate $m(J^k, j, \omega) = argmax_{m\in\{0,1\}}V^c(J^k, j, \omega, m)$ and $V^c(J^k, j, \omega) = max_{m\in\{0,1\}}V^c(J^k, j, \omega, m)$
- Calculate $\chi(J^k, j, \omega) = 0 \Leftrightarrow V^c(J^k, j, \omega) \leq \beta\phi_i + \pi(J^k, j, \omega)$

Now we have all the policies. For each (J^k, j, ω) we have calculated $x(J^k, j, \omega, m)$ and then $V^c(J^k, j, \omega, m)$ for $m \in \{0, 1\}$; then $m(J^k, j, \omega)$, and finally $\chi(J^k, j, \omega)$.

4.2.2 Update 2: Finding The New Location.

To obtain the new location we need the draws from distributions determined by the policies just calculated. So we make the draws first, keep them in the working file, and use them below.

- Draw ν^{k+1}. Here $\nu^{k+1} = 1$ with probability δ and $\nu^{k+1} = 0$ with probability $1 - \delta$.
- For each (J^k, j, ω) such that $\chi(J^k, j, \omega) = 1$, use $x(J^k, j, \omega, m(J^k, j, \omega))$ to draw η_j^{k+1} and calculate $\omega^{k+1}(J, j) = \omega^k(J, j) + \eta_j^{k+1} - \nu^{k+1}$ (note $\chi^k(J_j^k) = 0 \Rightarrow \omega_j^{k+1} = 0$). Note $\eta_j^{k+1} = 1$ with probability $\alpha x(m)_j/(1 + \alpha x(m)_j)$ and $\eta_j^{k+1} = 0$ with probability $1/(1 + \alpha x(m)_j)$.

Now we have all the needed random draws. This enables us to update L^k.

- If $\sum_i [1-\chi(J_i^k)] = 0$ and $\sum_j m(J_j^k) = 0$ and $\chi_e(J^k) = 0$, then

$$J^{k+1} = \{\omega_{k-\tau(k)}, \hat{\tau}(k+1) = min[\hat{\tau}(k)+1, \overline{\tau}], \nu(\hat{\tau}(k+1))\}$$

- Otherwise form ω^{k+1} by taking $\omega^{k+1}(J^k, j)$ for each j at which $\chi(J^k, j) = 1$ and $\omega^e - \nu^{k+1}$ if $\chi^e(J^k) = 1$, and ordering the result (in the natural order)

$$J^{k+1} = \{\omega^{k+1}, 0, 0\},$$

- $L^{k+1} = (J^{k+1}, \omega^{k+1})$.

4.2.3 Update 3: $M(J^k)$.

First set

$$h^{k+1}(J^k) = h^k(J^k) + 1.$$

Next set

$$V_e^{k+1}(J^k) - V_e^k(J^k) = [h^k(J^k)+10]^{-1}[\beta(V(J^e(\omega^{k+1}) - FK(J_{i,t+1})), j(\omega_e - \nu^{k+1}, J^e(\omega^{k+1})), \omega_e - \nu^{k+1}$$

((((we need to check the indexes here of $FK(J_{i,t+1})$.....)))))
where

$$J^e(\omega^{k+1}) =$$

- J^{k+1} if $\chi_e^k = 1$ while
- $(\omega^{k+1} + I(\omega_e - \nu^{k+1}), 0, 0)$ otherwise.

4.2.4 Update 4: $M(J^k, j, \omega)$.

First we update

$$h^{k+1}(J_j^k) = h^k(J_j^k) + 1.$$

We now have to update $W^k(\eta|J^k, j, \omega, m)$ for $\eta \in \{0,1\}$ and $m \in \{0,1\}$. The update can differ with

$$\chi_j^k, m_j^k, \eta_j^{k+1}, \sum_{i \neq j} m_i^k, \sum_{i \neq j} \chi_i^k, \text{ and } \chi_e^k.$$

All updates will be of the form

$$W^{k+1}(\eta|J^k, j, \omega, m) - W^k(\eta|J^k, j, \omega, m) = [h^k(J_j^k)+2]^{-1}[V^\dagger(*) - W^k(\eta|J^k, j, \omega, m)],$$

and what we have to do is provide the form of $V^\dagger(*)$. Note that there are four updates for each of 2^6 possibilities. They reduce however to three generic cases.

Case 1: We evaluate a situation in which there is a price war, i.e. either $m = 1$;or$\sum_{i \neq j} m_i^k \geq 1$; or $\chi_e^k = 1$

There are two possible J^* in this case, one for $\chi_j^k = 1$ and one for $\chi_j^k = 0$ and the $V^\dagger(*)$ will depend on J^*. If $\chi_j^k = 1$, then

$$J^*(\cdot) = ((\omega^{k+1} - I(\omega_j^k + \eta_j^{k+1} - \nu^{k+1}) + I(\omega_j^k + \eta - \nu^{k+1}), 0, 0).$$

If $\chi_j^k = 0$, then

$$J^*(\cdot) = ((\omega^{k+1} + I(\omega_j^k + \eta - \nu^{k+1})), 0, 0).$$

In either case

$$V^\dagger(*) = \beta V(J^*(\cdot), j(\omega_j^k + \eta - \nu^{k+1}, J^*), \omega_j^k + \eta - \nu^{k+1}|W^k) - \beta FK(J^{k+1}).$$

Case 2: In the position we are evaluating there is no price wars, but we update the ω vector due to exit; i.e. $m = 0, \chi_e^k = 0, \sum_{i \neq j} m_i^k = 0$, but $\sum_{i \neq j}[1 - \chi_j^k] \neq 0$.

Then using the two definitions of J^* above (one for $\chi_j^k = 1$ and one for $\chi_j^k = 0$) we have

$$V^\dagger(*) = -\beta K(J^{k+1}) + \beta V[J^*, j(\omega_j^k + \eta - \nu^{k+1}, J^*), \omega_j^k + \eta - \nu^{k+1}|W^k]$$

Case 3: In the position we are evaluating there is no meeting at all, i.e. $m = 0, \chi_e^k = 0, \sum_{i \neq j} m_i^k = 0$ **and** $\sum_{i \neq j}[1 - \chi_j^k] = 0$**.**

Note that these are the cases when the new states are not fully revealed in the state we are evaluating.. Here it does not matter what χ_j^k is. Either way.

$$V(*) = \beta V[(\omega_{k-\tau(k)}, min(\hat{\tau}(k) + 1, \overline{\tau}), \nu(\hat{\tau}(k) + 1), j, \omega_j^k + \eta - \nu_j^{k+1} | W^k].$$

After Updating the $W^k(\cdot)$ then update $V^k(\cdot)$ where

$$V^{k+1}(\cdot) = V(\cdot | W^{k+1})$$

using the $V(\cdot)$ operator defined above.

4.3 The Structure of Memory.

The memory is structured in layers. The first layer specifies an ω vector, say ω^*. It corresponds to

$$J = (\omega^*, \hat{\tau} = 0, 0),$$

and is ordered with the trinary tree arrangement of Pakes-McGuire (forthcoming).

Next we move "horizontally" to two more J, and they are

$$J = (\omega^*, \hat{\tau} = 1, \nu(+1) = 1), \text{ and } J = (\omega^*, \hat{\tau} = 1, \nu(+1) = 0).$$

>From each of these points we move horizontally again, updating $\hat{\tau}$ by one in the move, and adding a $\nu(\cdot)$. Thus moving along the horizontal chain we have 2^τ common information sets when there is τ periods since the last meeting. Of course this stops at $\hat{\tau} = \overline{\tau}$. So the most we can have in a horizontal dimension is 2 to the power $\overline{\tau}$. Note that at each of these points we have stored $M(J)$.

Now at each J we have a layer below it listing the ordered positions $(J, 1)$ to $(J, n(J))$. Under each $(J, j(\cdot, J))$ we have listed the possible positions $(J, j(\omega, J), \omega)$ for the up to $\#\Omega$ positions that could be listed here (listed by some order in ω). By each $(J, j(\omega, J), \omega)$ we list $M(J, j(\omega, J), \omega)$.

5 Testing For an Equilibrium.

To be completed..

6 Results.

To be completed.

References

Abreu, D., D. Pearce, and E. Stacchetti (1986), "Optimal Cartel equilibria with Imperfect Monitoring" *Journal of Economic Theory* 39(June), 251-269.

Abreu, D. (1986), "Extremal Equilibria of Oligopolistic Supergames", *Journal of Economic Theory*, Vol. 39 pp.191-225.

Athey S and Bagwell, K (2001) "Optimal Collusion with Private Information"

Athey S and Bagwell, K and C. Sanchirico (2001) "Collusion and Price Rigidity"

Bagwell, K. and R.W. Staiger (1997), "Collusion Over the Business Cycle", *The Rand Journal of Economics*, Vol 28, pp 82-106.

Caplin, A. and B. Nalebuff (1991), "Aggregation and Imperfect Competition: the Existence of Equilibrium," *Econometrica* V. 59 pp. 25-59.

Compte, O., F. Jenny and P.Rey (1999); "Capacity Constraints, Mergers and Collusion", *mimeo*, University of Tolouse.

Davidson, C. and R. Deneckere (1990), "Excess Capacity and Collusion" *International Economic Review*, Vol. 31, pp.521-542.

Ellison G. (1994), "Theories of Cartel Stability and the Joint Executive Committee", *Rand Journal of Economics,* Vol.25. pp. 37-57.

Ericson R. and A. Pakes, (1995) "Markov-Perfect Industry Dynamics: A Framework for Empirical Work" *Review of Economic studies*, 62, 53-82.

Fershtman C. and N. Gandal (1994), "Disadvantageous Semicollusion" *International Journal of Industrial Organization*, vol.12, pp. 141-154.

Fershtman C. and A. Pakes

Friedman, J. W. (1971), "A Noncooperative Equilibrium for Supergames" *Review of Economic Studies*, 38, pp. 1-12.

Fudenberg, D., D. Levine, and E. Maskin (1994), "The Folk Theorem with Imperfect Public Information" *Econometrica*, 62(5) pp. 997-1039.

Hatiwanger, J. and J.E. Harrington (1991), "The Impact of Cyclical Demand Movements on Collusive Behavior", *The Rand Journal of Economics*, Vol. 22, pp. 89-106.

Harrington, J. (1989), "Collusion among Asymmetric Firms: The Case of Different Discount Factors" *International Journal of Industrial Organization*, 7, 289-307.

Kandori, M. (1991), "Correlated Demand Shocks and Price Wars During Booms." *Review of Economic Studies*, Vol. 58, pp 171-180.

Green, E.J. and Porter, R. (1984), "Noncooperative Collusion Under Imperfect Price Competition" *Econometrica*, Vol. 52, pp.87-100.

Levenstein M. (1997), "Price Wars and the Stability of Collusion: A Study of the Pre-World War I Bromine Industry", *Journal of Industrial Economics*, pp. 117-147.

Maskin, E. and J. Tirole (1988), "A Theory of Dynamic Oligopoly: I and II", *Econometrica*, 56, pp.549-600.

McMillan J. (1991) "Dango: Japan's price fixing conspiracies" *Economics and Politics* Vol 3 pp.201-218.

Nash, J.F. (1950), "The Bargaining Problem," *Econometrica* 18, pp 155-162.

Pakes, A. (1994); "Dynamic Structural Models: Problems and Prospects", Chapter 5 in C. Sims (ed.) *Advances in Econometrics Sixth World Congress*, Vol. II, pp 171-260, Cambridge University Press.

Pakes, A. and P. McGuire (1994), "Computing Markov Perfect Nash Equilibrium: Numerical Implication of a Dynamic Differentiated Product Model", *The Rand Journal of Economics*, 25, pp. 555-589.

Pakes, A. and G. Gowrisankaran, and P. McGuire (1995), "Implementing the Pakes-McGuire Algorithm for Computing Markov Perfect Equilibria", *mimeo*, Yale University.

Porter, R.H. (1983a), "Optimal Cartel Trigger Price Strategies" *Journal of Economic Theory*, Vol 29, pp. 313-338.

Porter, R.H. (1983b), "A Study of Cartel Stability: The Joint Executive Committee, 1880-1886" *Bell Journal of Economics*, Vol. 14, pp. 301-314.

Rotemberg, J.J. and Saloner, G. (1986), " A supergame-theoretic model of price wars during booms" *American Economic Review*, Vol.76, pp. 390-407.

Rubinstein, A. (1979), "Equilibrium in Supergames with the Overtaking Criterion", *Journal of Economic Theory,* 21, pp. 1-9.

Scherer, F. (1980), "*Industry Market Structue and Economic Performance*", Chicago:Rand McNally.

Schmalensee, R. (1987), "Competitive Advantage and Collusive Optima" *International Journal of Industrial Organization* 5, pp. 351-367.

Staiger, R.W. and F.A. Wolak (1992), "Collusive Pricing with Capacity Constraint in the Presence of Demand Uncertainty." *The Rand Journal of Economics*, Vol. 23, pp. 203-220.

Starr, A.W. and Y.C. Ho (1969), "Nonzero-Sum Differential Games," *Journal of Optimization Theory and Application* Vol 3, pp.1984-208.

Stigler, G.J. (1964), "A Theory of Oligopoly" *Journal of Political Economy*, Vol. 72, pp.44-1.

Stocking, G. and Watkins, M.W. (1946) "*Cartels in Action. Case Studies in International Business Diplomacy*" New York; The Twentieth Century Fund.

Symeonidis, G. (1999), "Are Cartel Laws Bad for Business? Evidence from the UK", *mimeo*, the University of Essex.