Coalitional Bargaining with Externalities and Endogenous Exit

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January 10, 2003

Abstract

This paper proposes a model of coalitional bargaining based on strategic interaction across coalitions, where players endogenously choose whether to exit. This formulation is general enough to study the formation of coalitions and the distribution of gains from cooperation in a wide variety of economic models with externalities and outside options. We show that when outside options are independent of the actions of other players, the Markov Perfect Equilibria of the coalitional bargaining game converge to efficient outcomes when the players become perfectly patient. On the other hand, we show through an example than in a model with negative externalities where players' outside options depend on the action of other players, the outcome may be inefficient, as players choose to exit before they extract all gains from cooperation.

VERY PRELIMINARY AND INCOMPLETE

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1 Introduction

This paper introduces a coalitional bargaining model combining endogenous exit decisions and externalities across coalitions. This model proposes a realistic description of many instances of collective bargaining, where players are involved in strategic interactions and are free to walk out of the negotiations. For example, President Bush's decision to abandon the Kyoto protocol on the reduction of the emission of greenhouse gases can be represented in our model. Similarly, the decision by Denmark, Sweden and the United Kingdom to opt out of the common currency at the early phases of negotiation over the euro can be understood using our model. Finally, the model can easily be applied to labor bargaining or merger negotiations, which may break off if some players leave the bargaining table. In all these examples, players have the ability to opt out of the negotiations, and their exit decision affects the payoff of all the other players in the game.

The importance and effect of outside options is well-understood in two player bargaining games. Following Rubinstein (1982)'s seminal contribution, Sutton (1986), Shaked and Sutton (1984), Binmore et al. (1986) have studied alternating offer games among two players with outside options. (See also Ponsati and Sakovicz (1998) for a model where both players have outside options and Muthoo (1999, Chapter 5) for a general review.) The "outside options principle" states that the outside option may not be valuable – and not change the outcome of the negotiations– or valuable, in which case agreement is immediately reached and the player with outside option receives a payoff which is approximately equal to her outside option. This paper provides a generalization of the "outside options principle" to multilateral bargaining, and gives an explicit formula characterizing the equilibrium of unanimity bargaining games with outside options.

Existing models of coalitional bargaining typically do not allow players to endogenously choose whether to leave the bargaining table. In the initial models generalizing Rubinstein (1982)'s analysis to coalitions, it is assumed that coalitions leave the game immediately after an agreement has been reached. (See Chatterjee et al. (1993), Okada (1996) and Ray and Vohra (1999)). Under this assumption, coalitional bargaining usually results in inefficient outcomes. Chatterjee et al. (1993) show that efficient outcomes only arise under very stringent conditions on the underlying structure of coalitional gains. Efficient outcomes are even less likely to arise in the economically relevant situation where the formation of a coalitional induces externalities on the other players (Ray and Vohra (1999)). To understand why the assumption of forced exit generates inefficiency, consider the following example. Let there be three players, and the coalitional worths are given by $v(S) = 0$ if $|S| = 1$, $v(S) = 3$ if $|S| = 2$ and $v(S) = 4$ when $|S| = 3$. As players become perfectly patient, the outcome of the bargaining procedure should result in equal sharing of the coalitional surplus among the symmetric players. Hence, an equilibrium where players form the grand coalition results in a payment of $\frac{4}{3}$ for every player. But clearly, players then have an incentive to form only an inefficient coalition of size 2, inducing a payoff of $\frac{3}{2}$ for each of the coalitional members. As the coalition leaves the negotiation table after it has formed, the additional surplus of 1 – moving from a coalition of size 2 to the grand coalition – cannot be divided among the players.

By contrast to these models, Gul (1989), Hart and Mas Colell (1998), Seidmann and Winter (1998) (in the "reversible" model), Montero (2000) and Gomes (2001) propose bargaining procedures where players continuously renegotiate agreements. In these models, where players never leave the bargaining table, subgame perfect equilibrium outcomes are typically efficient. Gul (1989) and Hart and Mas Colell (1998) thus support the Shapley value as the equilibrium payoff of their procedure whereas Montero (2000) and Gomes (2001) specifically consider procedures allowing for externalities across coalitions and show that when the grand coalition is efficient, it is reached as a subgame perfect equilibrium outcome.

In this paper, we consider a procedure where coalitions are neither forced to leave the game immediately after they are formed nor to remain negotiating forever. Instead, we endogenize the players' decision to leave the bargaining table, and thus consider an "intermediate case" between the early models of coalitional bargaining and the recent papers allowing for renegotiation. One of our main interests in this paper is to characterize conditions under which efficient subgame perfect equilibrium outcomes arise in this intermediate model. We show that whenever the degree of externalities is small, equilibrium outcomes are efficient. However, we provide an example to show that when externalities are important, inefficient subgame perfect equilibrium outcomes may arise.

Prior to our study, two models have analyzed coalitional bargaining with endogenous exit. Perry and Reny (1994) propose a very different model, in continuous time, where coalitions choose the moment at which they implement their allocation. The main objective of Perry and Reny (1994) is to provide an extensive form game implementing the core of an underlying game in characteristic form. Seidmann and Winter (1998)'s bargaining game with "irreversible actions" is very close to our model, and was our main source of inspiration. The main difference between our model and Seidmann and

Winter (1998)'s stems from the underlying description of coalitional opportunities. Seidmann and Winter (1998) suppose that gains from cooperation are represented by a coalitional game, whereas we represent coalitional gains through a strategic game played by coalitions. Hence our model accommodates situations where, by opting out of the negotiations, players affect the payoff of other players. Through a series of examples, Seidmann and Winter (1998) show that the game with irreversible actions may result in gradual coalition formation, or even in inefficient partial agreements. The results of our model do not contradict the lesson from those examples. But by analyzing a more complex environment, where coalitional negotiations and strategic actions are intertwined, we get a better understanding of the reasons which lead to inefficient or efficient agreements. This paper embeds coalitional negotiations into a strategic environment, thereby generalizing Seidmann and Winter (1998)'s framework and getting stronger efficiency results.

More precisely, we construct a bargaining game with two distinct phases per period. In the first phase of interaction, as in classical models of coalitional bargaining, a player proposes to form a coalition, and commits to a payment to other coalition members. If all prospective members accept the offer, the coalition is formed, and the proposer gains access to all the resources of the coalition. In the second phase of interaction, all active players simultaneously decide whether to exit the negotiations. Upon leaving the bargaining table, a player chooses an irreversible action which will be played throughout the game. If a player chooses to remain in the game, he selects a temporary action to be played for one period. This specific bargaining procedure is designed to allow simultaneously for the formation of coalitions, strategic interaction across coalitions, and endogenous exit decisions.

The model we consider is general enough to study coalition formation in a wide array of applications, including some that have recently been proposed in the literature. The first application we consider is a unanimity bargaining game with outside options among n players. We show that there exists an efficient equilibrium with immediate agreement, and compute exactly the offers of all the players. It turns out that the player with the largest outside option receives an offer which is exactly his outside option –making him indifferent between exiting or staying in the game, while all other players receive an expected payoff which depends in a nontrivial way on *the entire vector of outside options.* This result enables us to perform comparative statics on the offers received in the game, and to obtain predictions which could be tested in experiments. In other words, our analysis provides an extension of the famous "outside options principle" in two-player bargaining games (Sutton, 1986) to an arbitrary number of players.

Our second application derives from Segal (1999)'s recent formalization of principal-agent relationships with externalities. Segal (1999) considers a situation where a single principal contracts with a set of agents whose trade induce externalities on the other agents. Externalities can either be positive (if the principal's trade with other agents increases the payment of nontraders) or negative (when the principal's trade reduces the payment of nontraders). Segal (1999) shows that when the principal uses bilateral contracts, the level of trade is typically inefficient, as the principal has an incentive to modify trade in order to reduce the outside opportunities of the agents. In our context, the principal's offers are conditional on the acceptance of all the agents with whom she contracts. Not surprisingly, we find that in the case of positive externalities, the principal can implement a fully efficient contract, with every agent receiving an offer equal to the payment he gets in the absence of trade. More interestingly, we show that in the case of negative externalities, the principal is also able to implement an efficient contract, offering to each agent his marginal surplus.

In the last two applications we consider, multiple coalitions can form and agents' payoffs depend on the entire structure of coalitions. We first consider situations with positive spillovers, where the formation of a coalition induces positive externalities on the other players. Typical examples of games with positive spillovers are the provision of pure public goods studied by Ray and Vohra (2001) and mergers on oligopolistic markets analyzed by Bloch (1996) and Ray and Vohra (1999). For both examples, we show that the coalitional bargaining procedure with endogenous exit results in an inefficient outcome – the immediate formation of the grand coalition. This result stands in sharp contrast to the outcome of bargaining procedures with forced exit, which typically yield the formation of inefficient coalition structures. Finally, we consider a specific game with negative spillovers. In this game, oligopolistic firms compete on a market with a fixed entry cost, and benefit from synergies when they form coalitions. For specific parameter values, we show that the coalitional bargaining game induces an inefficient outcome, where firms fail to exploit all the synergies they could create. This example is suggestive of a class of games with negative spillovers, where the outcome of negotiations may be inefficient.

Building on these applications, we are able to provide a sufficient condition guaranteeing efficiency of the equilibrium outcome. Barring difficulties arising from coordination failures when players choose whether to exit, we show that the bargaining procedure always results in efficient outcomes when the payoff of exiting players is independent of the actions of other players.

This condition (that we term "absence of irreversible externalities") is satisfied both in the unanimity bargaining game with outside options, where outside options are given exogenously, and in the principal-agents models with externalities, as agents who refuse to sign the contract are constrained to zero trade. On the other hand, when multiple coalitions can form and compete against another, as in the last two applications, the condition fails, and efficiency of the bargaining procedure is not guaranteed.

The rest of the paper is organized as follows. Section 2 presents the model and Section 3 contains preliminary results on characterization and existence of equilibria. In Section 4, we discuss at length four applications of the model. General sufficient conditions for efficiency are provided in Section 5. Concluding comments are given in the last Section. All proofs are collected in an Appendix.

2 The Model

We consider an infinite horizon bargaining model, with two distinct phases at every period. The *contracting phase* is similar to the coalitional bargaining models proposed by Chatterjee et al. (1993) and Okada (1996). A player is chosen at random to propose a coalition, and a payment to all other coalition members. All prospective members respond in turn to the offer, and the coalition is formed only if all its members unanimously agree to the contract. In the *action phase,* all standing coalitions play a static game, resulting in flow payments for the coalitions. Hence, underlying economic opportunities are captured by the payoffs of a strategic game played by coalitions, instead of a coalitional function, as in Chatterjee et al. (1993) Okada (1996) or Seidmann and Winter (1998), or a partition function, as in Ray and Vohra (1999), Montero (2000) and Gomes (2001). In our view, the interplay between the contracting and the action phases is the main original contribution of the paper, as it enables us to consider simultaneously issues of coalition formation, externalities and endogenous exit decisions.

Formally, let N be a set of players, indexed by $i = 1, 2, .., n$. A coalition S is a nonempty subset of players, and a coalition structure π is a partition of the set N into disjoint coalitions, $\pi = \{S_1, .., S_j, .., S_J\}$. Every coalition S has access to a set of actions, A_S, which can be decomposed into temporary (or reversible) actions and permanent (or irreversible) actions. If a coalition chooses a reversible action, $r_S \in R_S$, it remains at the negotiation table ; if it chooses a permanent action $p_S \in P_S$, it exits the negotiations and is committed to play the permanent action p_S ad infinitum. We suppose that the set of permanent and temporary actions are disjoint, ($R_S \cap P_S = \emptyset$).

A state of the game s will be described by three elements: a collection of coalitions which are still active in the game, $\mathcal{R}(s) = \{S_1, ..., S_J\}$, a collection of coalitions who have already exited, $\mathcal{P}(s) = \{T_1, ..., T_k\}$ and the permanent actions chosen by the coalitions who have already left the game, $p(s) = \{p_{T1}, ..., p_{TK}\}$. At the beginning of the game, all players are singletons, and no player has chosen a permanent action, so $\mathcal{R}(s) = \{\{i\}_{i \in N}\}, \mathcal{P}(s) = \emptyset$.Time is discrete and runs as $t = 0, \Delta, 2\Delta, ...$ Players discount future payoffs according to a common discount factor $0 < \delta < 1$. Each time period is divided into two subperiods: a contracting phase and an action phase.

Contracting phase: At the contracting phase in state s, each of the active coalitions in $\mathcal{R}(s)$ is chosen at random (according to an exogenous rule of order) to propose a contract to other active coalitions. A contract c consists of two parts: a set of coalitions $\mathcal{S} \subset \mathcal{R}(s)$ and a vector of payments to all the coalitions in $\mathcal{S}, x_{\mathcal{S}} = (x_{S1}, ..., x_{Sj})$. Each of the prospective members of $\mathcal{S}$ responds in turn to the offer. If all agree, the coalition $\mathcal{S}$ is formed, each of the coalition members (but the proposer) receives the one time payoff x_{Sj}, and the state moves to s' where $\mathcal{R}(s') = \mathcal{R}(s) \backslash \{S_j\}_{S_j \in \mathcal{S}} \cup \mathcal{S}$, $\mathcal{P}(s) = \mathcal{P}(s')$, $p(s) = p(s')$. If one of the members of $\mathcal{S}$ rejects the offer, the coalition is not formed and the game remains in state s.

Action phase: At the action phase in state s, all active coalitions S_j in $\mathcal{R}(s)$ simultaneously choose an action in A_{Sj}. The action profile is thus given by $a = ((a_{Sj})_{S_j \in \mathcal{R}(s)}, (p_{Tk})_{T_k \in \mathcal{P}(s)})$. We associate to this action profile a vector of flow payoffs (in net present value) for each of the coalitions in $\mathcal{R}(s) \cup \mathcal{P}(s)$, $v_{Sj}(a)$ and $v_{Tk}(a)$. These payoffs correspond to the payoffs obtained by coalitions in the static game where they choose actions. If a subset $\mathcal{T}$ of the active coalitions chooses permanent actions, the state of the game moves to state s' with $\mathcal{R}(s') = \mathcal{R}(s) \backslash \{S_j\}_{S_j \in \mathcal{T}}, \mathcal{P}(s') = \mathcal{P}(s) \cup \{S_j\}_{S_j \in \mathcal{T}}$ and $p(s') = p(s) \otimes (p_{S_j})_{S_j \in \mathcal{T}}$. If no active coalition chooses permanent actions, the game remains in state s. At the end of the action phase, one period of time elapses, and the game starts again in a contracting phase at the next period.

The model we consider departs from traditional game theoretic models in a number ways. First, we suppose that players are *coalitions* and that action spaces A_S and utilities v_S are defined for coalitions rather than individual players. Clearly, when coalitions are restricted to singletons, this formalization is identical to the description of a game in strategic form. Furthermore, our model accommodates as a special case situations where actions of coalitions are just vectors of actions of coalition members

($A_S = \times_{i \in S} A_i$) and utilities of coalitions the sum of utilities of their members ($v_S(a) = \sum_{i \in S} v_i(a)$). Our framework is general enough to encompass situations where coalitions have access to more (or less) opportunities than their members, and where utility inside a coalition is not perfectly transferable. Second, in the dynamic game we consider, the set of player evolves throughout the game, as some coalitions disappear and new coalitions are formed. This makes the formal description of the utility of a player in the game extremely complicated and notation intensive.[1] Instead of writing down an explicit formula for a player's utility, we define the *continuation value* of a coalition S at state s in the contracting phase as $\phi_S^1(s)$ and the continuation value at the action phase as $\phi_S^2(s)$.

We will analyze *stationary perfect equilibria* or *Markov Perfect Equilibria (MPE)* of the coalitional bargaining game described above. A Markov perfect equilibrium is a subgame perfect equilibrium where every coalition adopts a strategy which only depends on the state of the game.

3 Characterization of equilibrium

3.1 An Example

Given the complexity of the notations and of the bargaining procedure, it might be useful to illustrate the model with a very simple example.

Example 3.1

Consider a game with two players, $i = 1, 2$. In the coalition structure [1|2], player 1 only has access to a temporary action, r_1 and player 2 has access to two actions: a temporary action r_2 and a permanent action p_2. The flow payoffs are given by:

	r_2	p_2
r_1	$0, 0$	a, b

In the coalition structure [12], coalition 12 has access to a permanent action, yielding a payoff of v, with $v > a + b$.

According to our definition of the state space, the game has three possible states:

[1] Gul (1989) and Gomes (2001) face the same problem, as they also consider games with renegotiations. Gul (1989) adopts a formulation which keeps track of the sequence in which coalitions are formed, and is already quite complicated. Generalizing Gul (1989)'s procedure to take into account externalities would result in pages of unnecessary notations.

state	$\mathcal{R}$	$\mathcal{P}$
s_1	[1|2]	$\emptyset$
s_2	[1]	[2]
s_3	$\emptyset$	[12]

States s_2 and s_3 are absorbing states: as players have chosen permanent actions, there is no more active coalition in the game. Furthermore, the only state where contracting may occur is state s_1. We suppose that, at state s_1, player 2 proposes with probability q and player 1 with probability $1-q$.

We now proceed to compute the Markov perfect equilibria of the coalitional bargaining game. Clearly, once the grand coalition has been formed the payoff is given by $\phi_N^1(s_3) = \phi_N^2(s_3) = v$. At state s_2, payoffs are given by $\phi_1^1(s_2) = \phi_1^2(s_2) = a$, $\phi_2^1(s_2) = \phi_2^2(s_2) = b$. Consider now state s_1 and compute the expected payoffs of the two players at the action stage:

	r_2	p_2
r_1	$\delta\phi_1^1(s_1), \delta\phi_2^1(s_1)$	a, b

We now consider three possible cases:

Suppose that player 2 chooses to play his temporary action r_2. Then, at the contracting stage, every player proposes to form the grand coalition, player 1 offers x_2 to player 2; player 2 offers x_1 to player 1 and the offers satisfy

$$\begin{aligned}
x_2 &= \delta\phi_2^1, \\
x_1 &= \delta\phi_1^1, \\
\phi_1^1 &= q(v - x_2) + (1-q)x_1 \\
\phi_2^1 &= qx_2 + (1-q)(v - x_1)
\end{aligned}$$

We obtain as a solution $x_2 = \delta qv, x_1 = \delta v(1-q)$. Hence, this strategy forms an equilibrium if and only if $\delta qv \geq b$.

Consider now an equilibrium where player 2 randomizes between actions $p_2$and r_2 and let σ denote the probability of choosing p_2. The offers must then satisfy the system of equations:

$$\begin{aligned}
x_2 &= \delta\phi_2^1 = b \\
x_1 &= \delta(1-\sigma)\phi_1^1 + \sigma a \\
\phi_1^1 &= q(v - x_2) + (1-q)x_1 \\
\phi_2^1 &= qx_2 + (1-q)(v - x_1)
\end{aligned}$$

This systems admits a solution given by

$$x_1 = \frac{\delta qv - \delta qb + \delta b - b}{\delta q}, x_2 = b, \sigma = \frac{b - \delta b - \delta qv + \delta^2 vq}{\delta q(\delta v - b - a)}.$$

The solution satisfies $0 \leq \sigma \leq 1$ if and only if $b \geq \delta vq$ and $\delta \geq \delta_0 = \frac{b}{q(v-a-b)+b}$.

Finally, consider the case where player 2 always chooses the permanent action p_2. The offers then would satisfy

$$\begin{aligned} x_2 &= b, \\ x_1 &= a, \\ \phi_1^1 &= q(v - x_2) + (1-q)x_1, \\ \phi_2^1 &= qx_2 + (1-q)(v - x_1). \end{aligned}$$

This equilibrium exists if and only if $b \geq \delta(qb + (1-q)(v-a))$or $\delta \leq \delta_0 = \frac{b}{q(v-a-b)+b}$.

In conclusion, we have identified all equilibria of the coalitional bargaining game. If $\delta \leq \delta_0$, there exists two possible equilibria: one where player 2 always exits and one where player 2 always remains in the game. For $\delta \geq \delta_0$, either player 2 remains in the game (when $b \leq \delta vq$) or uses a mixed strategy between exiting and staying in the game (when $b \geq \delta vq$).

3.2 General Characterization of Equilibrium

Building on the example, we now provide a complete characterization of equilibrium in the general case. The action stage at state s can be represented by a standard game in strategic form, $\Gamma(s) = (\mathcal{R}(s), \{A_S, u_S(s)\}_{S \in \mathcal{R}(s)})$ where players are the active coalitions at state s, and payoffs $u_S(s)$ are defined as follows. For any action profile a, let $\mathcal{T}(a)$ be the set of coalitions who have chosen permanent actions, and p denote those permanent actions. Define $h(s,a)$ to be the new state reached after the choice of actions a, i.e. $h(s,a) = \{\mathcal{R}(s) \backslash \mathcal{T}(a), \mathcal{P}(s) \cup \mathcal{T}(a), p(s) \otimes p\}$.Then,

$$u_S(s,a) = \delta \phi_S^1(h(s,a)) + (1-\delta) v_S((a_T)_{T \in \mathcal{R}(s)}, (p_T)_{T \in \mathcal{P}(s)}).$$

Hence, utilities are defined as the discounted sum of the flow payoff at the action stage and the continuation value of the game at the state generated by the action choices. A Markov Perfect Equilibrium σ_S must then correspond to a Nash equilibrium of the game $\Gamma(s)$ for any state s .

At the contracting stage at state s, the minimal offer that coalition T accepts is $x_T = \phi_T^2(s)$. Given a contract $c = (\mathcal{S}, x_{\mathcal{S}})$, define the state obtained when the offer is accepted at state s by $g(c,s) = \{\mathcal{R}(s) \backslash \{S_j\}_{S_j \in \mathcal{S}} \cup \mathcal{S}, \mathcal{P}(s), p(s)\}$. In a Markov Perfect equilibrium, if coalition S makes an offer to a collection $\mathcal{S}$ of active coalitions, she will pay the minimal transfer and obtain an expected payoff of

$$\phi_{\mathcal{S}}^2(g(c,s)) - \sum_{T \in \mathcal{S}} \phi_T^2(s).$$

We conclude:

Lemma 3.2 *A strategy profile σ is a Markov Perfect equilibrium if and only if there exists payoffs $\phi_S^1(s), \phi_S^2(s)$ such that (i) at the action stage, σ is a Nash equilibrium of the game $\Gamma(s) = (\mathcal{R}(s), \{A_S, u_S(s)\}_{S \in \mathcal{R}(s)})$ where $u_S(s,a) = \delta \phi_S^1(h(s,a)) + (1-\delta) v_S((a_T)_{T \in \mathcal{R}(s)}, (p_T)_{T \in \mathcal{P}(s)})$, and $\phi_S^2(s)$ is the equilibrium payoff of coalition S at the Nash equilibrium and (ii) at the contracting stage, the contract $c = (\mathcal{S}, x_{\mathcal{S}})$ satisfies:*
$\mathcal{S} \in \arg\max_{\mathcal{S}} (\phi_{\mathcal{S}}^2(g(c,s)) - \sum_{T \in \mathcal{S}} \phi_T^2(s)), x_T = \phi_T^2(s)$.
The expected value at the contracting stage is given by

$$\begin{aligned}
\phi_S^1(s) &= q_S(s) \sum \sigma_S(s)(\mathcal{S})(\phi_{\mathcal{S}}^2(g(c,s)) - \sum_{T \in \mathcal{S}} \phi_T^2(s)) \\
&+ \sum_{T \in \mathcal{R}(s), T \neq S} q_T(s) \sum \sigma_T(s)(\mathcal{S})(\mathbf{1}_{S \in \mathcal{S}} \phi_S^2(s) + \mathbf{1}_{S \notin \mathcal{S}} \phi_S^2(g(c,s))
\end{aligned}$$

where $q_S(s)$ is the probability that coalition S makes an offer at state s and $\mathbf{1}$ the indicator function.

The Lemma gives a complete characterization of equilibrium, and enables us to compute the continuation values of the game at the contracting and action stages. At the action stage, the continuation value is obtained as the equilibrium payoff of a strategic game played by the active coalitions. At the contracting stage, the continuation value is obtained as an expected value, considering three possible situations. With probability $q_S(s)$, coalition S is called to make an offer, and proposes to form any optimal coalition $\mathcal{S}$, obtaining a payoff $(\phi_{\mathcal{S}}^2(g(c,s)) - \sum_{T \in \mathcal{S}} \phi_T^2(s))$. With probability $q_T(s)$, another coalition T is recognized to make an offer, and it either chooses to include coalition S in the offer (in which case S obtains an expected payoff of $\phi_S^2(s)$), or it forms a coalition excluding S (in which case S obtains an expected payoff of $\phi_S^2(g(c,s))$.

We now prove the existence of continuation values $\phi_S^1(s)$ and $\phi_S^2(s)$ satisfying the conditions of the Lemma, to show that the bargaining game admits a Markov Perfect Equilibrium.

Proposition 3.3 *The coalitional bargaining game admits a Markov Perfect Equilibrium.*

4 Applications

In this Section, we develop four applications of the model. In the first application, we construct a Markov Perfect Equilibrium in a multilateral bargaining game with arbitrary outside options. This construction enables us to compute exactly the equilibrium offers as a function of the entire vector of outside options. In the second application, we consider a model of contracting with positive or negative externalities. We show that the coalitional bargaining game always results in efficient trade, and compute exactly the division of the surplus between the principal and the agents. The last two applications deal with more complex situations where multiple coalitions can form and multilateral externalities are present. We first analyze two specific games with positive spillovers (provision of public goods and mergers to monopoly) and show that the bargaining procedure leads to the formation of the grand coalition. We then consider a model with negative externalities (market entry with synergies) and prove that, for some parameter configurations, the coalitional bargaining procedure produces inefficient outcomes.

4.1 Multilateral bargaining with outside options

We consider a unanimity bargaining situation among n players, where the total surplus is equal to v and every player has an outside option v_i.We suppose that the total surplus is larger than the sum of outside options, $v > \sum v_i$. This bargaining situation can easily be recast in the terminology of our general model. Every coalition S has an action space given by $A_S = \{0, 1\}$ where the first action (0) is a permanent action, indicating that they leave the game and the second action (1) a temporary action, indicating that they continue to bargain. The flow payoff received by a coalition S is

0 if all players continue to bargain,$v(S) = \sum_{i \in S} v_i$ if some player opts out and v if the grand coalition is formed and chooses to exit.

This model can be viewed as an extension of two-player alternating-offers bargaining models with outside options (Sutton (1986)). An important difference is that, in our setting, coalitions *simultaneously* choose whether to exercise their outside option. This induces inefficiencies, as we cannot rule out equilibria where all players choose to opt out at the action stage. However, focussing on an equilibrium where players do not simultaneously choose to exit, we obtain the following characterization result.[2]

Proposition 4.1 *Consider a unanimity bargaining game among* n *players with outside options* v_i*, where* $\overline{v} = \max_{i \in N} v_i$ *is the largest outside option, and* $v > \sum_{i=1}^{n} v_i$*. Suppose that every player has an equal probability to make offers. As* δ *converges to* 1*, there exists a Markov Perfect Equilibrium where*

i)If $\overline{v} \geq \frac{1}{n} v$ *the equilibrium payoffs converge to*

$$\phi_i = v_i + \frac{(\overline{v} - v_i)}{\sum_{j=1}^{n} (\overline{v} - v_j)} \left(v - \sum_{j=1}^{n} v_j \right), \text{ for all } i \in N$$

and the player with largest outside option opts out with probability p *such that*

$$\lim_{\delta \to 1} \frac{p}{(1-\delta)} = \frac{n\overline{v} - v}{v - \sum_{j=1}^{n} v_j},$$

ii) If $\overline{v} < \frac{1}{n} v$ *the equilibrium payoffs converge to*

$$\phi_i = \frac{1}{n} v, \text{ for all } i = 1, ..., n,$$

and no player opts out at the investment stage.

It is instructive to compare the equilibrium given in Proposition 4.1 with the equilibria found in two-person bargaining games with outside options. The "outside options principle" applies in both models. When the outside option is not credible, it does not affect the outcome of the bargaining process. When the outside option is credible, the player with the highest outside option is able to extract a higher share of the surplus. But there is

[2] In the next Section, we define precisely a class of equilibria where players do not simultaneously choose to opt out when they don't have an incentive to do so. These equilibria are based on a refinement similar to Selten (1975)'s trembling hand perfection.

a striking difference between the two models. We find that the player with the highest outside option chooses to opt out with a positive probability in equilibrium, whereas in two-player bargaining models, the outside option is never exercised in equilibrium. To understand why the player with the highest outside option chooses to opt out with a probability $0 < p < 1$, notice that if he chose never to opt out ($p = 0$), he would obtain the expected payoff $\frac{v}{n}$ which is dominated by the outside option. If, on the other hand, he chose to opt out all the time, $p = 1$, the outcome would be inefficient and, as $v > \sum_{i=1}^{n} v_i$, the other players would benefit from offering him a payoff greater than his outside option.

When the outside option is credible, the expected payoffs of the players exhibit some interesting features. First, note that the outside options of *all* the players enter the formula describing the expected payoff. This is due to the fact that the game will end up (with some low probability) in an inefficient outcome, where all players obtain their outside option. The offers made at the contracting stage reflect this possibility, and depend on the entire vector of outside options. Furthermore, a close look at the formula provides some interesting predictions, which could be tested in experiments. For example, the model suggests that an increase in the highest outside option increases the sensitivity of a player's payoff with respect to his own outside option, $\frac{\partial^2 \phi_i}{\partial \overline{v} \partial v_i} \geq 0$. Hence, one would expect that, as the highest outside option $\overline{v}$ goes up, the outcomes of bargaining experiments become more and more sensitive to the outside options of all the players.

4.2 Contracting with externalities

Segal (1999) develops a general model of contracting with externalities, which encompasses a wide array of applications. In Segal (1999)'s model, a principal (labeled 0) contracts with n agents, who take actions $a_i \in [0, \overline{a}]$. Externalities among agent's actions are captured by the following utility structure. We suppose that all agenst are identical and that each agent $i = 1, 2, .., n$ that trades with the principal receives a payoff $v_i(a) = a\alpha(A) + \beta(A)$, that is a function of his own trade and the aggregate trade $A = \sum a_i$. The principal receives a payoff $v_0(a) = F(A)$.[3] If agent i does not trade with the principal, he chooses $a_i = 0$, and his payoff $v_i(a) = \beta(A)$ is only a function of the aggregate trade (non-traders do not receive (nor pay) any transfers).

[3] This utility specification is equivalent to the linearity condition (condition L) proposed by Segal (1999, p. 341). Segal (1999) shows that this condition is satisfied in a variety of economic models.

Segal (1999) shows that this general structure encompasses a number of specific models, ranging from models in Industrial Organization (vertical contracting, exclusive dealing, network externalities, mergers to monopoly),to models in Finance (debt restructuring, takeovers) or Public Economics (the provision of public goods and bads). All these models can be divided into two broad categories, according to the sign of the externalities that traders impose on nontraders.

Definition 4.2 *Externalities on nontraders are positive (negative) if their payoff* $v_i(a) = \beta(A)$ *is nondecreasing (nonincreasing) in the aggregate trade* $A = \sum a_i$.

We now recast Segal (1999)'s model in the structure developed in this paper. We now recast Segal (1999)'s model in the structure developed in this paper. If a coalition S does not contain the principal ($0 \notin S$), the action space only contains one temporary action. If a coalition S includes the principal ($0 \in S$), the action space can be decomposed as $A_S \cup \{1\}$ with $A_S = \times_{i \in S \setminus \{0\}}[0, \overline{a}]$. In words, when the principal belongs to the coalition, it can choose either to continue negotiating in the game (the reversible action 1), or to immediately conclude a contract specifying the actions played by all the agents in S. The flow payoffs received by the players are then given as follows. For any coalition structure π, if the coalition containing the principal has not chosen a permanent action, $v_T = 0 \forall T \in \pi$. If the coalition containing the principal has chosen the permanent action p_T,$v_T = [u_0(a_T, 0_{-T}) + \sum_{i \in T} u_i(a_T, 0_{-T})]$ if $0 \in T$ and $v_T = \sum_{i \in T} u_i(0_T, a_{-T})$ if $0 \notin T$. Without any loss of generality, we normalize the non-trade payoffs of principal and agents to zero, $u_0(0) = F(0) = 0$ and $u_i(0) = \beta(0) = 0$.

Whenever a contract is concluded in a coalition of m agents plus the principal, we denote by $A^*(m)$ the trade which maximizes the payoff received by the coalition, $A^*(m) = \arg\max_A F(A) + A\alpha(A) + m\beta(A)$, by $v_c(m)$ the maximum achieved by the coalition, $v_c(m) = \max_A F(A) + A\alpha(A) + m\beta(A)$, and by $v_{-c}(m)$ the payoff received by agents outside the coalition, $v_{-c}(m) = \beta A^*(m)$. We refer to the aggregate payoff as $v(m) = v_c(m) + (n-m) v_{-c}(m)$.

When externalities on nontraders are positive, it is easy to see that the multilateral bargaining procedure reaches an efficient agreement. By offering to form the grand coalition, and offering to each agent his minimal payoff, $v_{-c}(0) = 0$, the principal can extract the entire surplus. As the principal has all the bargaining power and agents' outside options are minimized with no trade, these offers will constitute a subgame perfect equilibrium. (Segal (1999, p. 368) also notes that in games with positive externalities, efficient outcomes can easily be reached by having the principal make an offer conditional on unanimous acceptance.)

When externalities on nontraders are negative, the structure of equilibrium is more complex. First notice that for all $m < n$,

$$\begin{aligned} v_c(m) &= F(A^*(m)) + A^*(m)\alpha(A^*(m)) + m\beta(A^*(m)) \\ &\geq F(A^*(n)) + A^*(n)\alpha(A^*(n)) + m\beta(A^*(n)) \\ &> F(A^*(n)) + A^*(n)\alpha(A^*(n)) + n\beta(A^*(n)) = v(n) \end{aligned}$$

where the first inequality is due to the fact that $A^*(m)$ is the optimal trade of a coalition of size m and the second inequality is due to the fact that externalities are negative. Now consider the sequence of negative numbers, $\frac{v(n)-v_c(m)}{n-m}$ for $m = 1, ..., n-1$ and suppose (in the generic case) that they are all distinct. Let $x^*(m)$ be the unique solution of

$$x^*(m) = \arg\min_{m' \geq m} \left\{ \frac{v(n) - v_c(m')}{n - m'} \right\}$$

and

$$v^*_{-c}(m) = \min_{m' \geq m} \left\{ \frac{v(n) - v_c(m')}{n - m'} \right\}$$

We show below that in the negative externality case contracting takes place in two steps: a first step in which the principal contracts with a subset of agents and a second step where she contracts with the remaining agents.

Proposition 4.3 *As δ converges to 1, there exists a Markov Perfect Equilibrium where initially, the principal contracts with $x^*(1)$ agents and offers them $v^*_{-c}(1)$ and chooses a reversible action at the action stage. In the second step, the principal contracts with the remaining agents, and offers them $v^*_{-c}(1)$.*

Proposition 4.3 relies on a careful description of the Markov Perfect Equilibrium given in the Appendix. At any state where $x^*(m) > m$, the principal contracts with $x^*(m)$ agents, offers them $v^*_{-c}(m)$ and remains in the game. At any state where $x^*(m) = m$, the prinicipal contracts with the $(n-m)$ remaining agents, offers $v^*_{-c}(m)$ and exits the game with a positive probability.

Intuitively, in this equilibrium, the principal proposes to contract first with a subset of agents in order to weaken the bargaining position of the remaining players. In the second stage, she proposes a conditional offer to all the remaining players. In a game with negative externalities, the payoff received by a nontrader is always negative. Hence, the transfer is paid by the agent to the principal. We show that by making a two-step offer, the principal is able to extract from the agents the highest possible transfer, $v^*_{-c}(1)$. Notice finally that the grand coalition is eventually formed in the game, and that the efficiency loss due to delay goes to zero as δ converges to 1.

4.3 Public Good Provision and Mergers to Monopoly

Our third application deals with models of coalition formation with positive externalities. Ray and Vohra (2001) study coalition formation in a pure public good context.[4] Each agent has a utility given by

$$u = Z - c(z),$$

where Z is the total amount of public goods provided and z the quantity produced by the agent, with $c(.)$ strictly increasing and strictly convex. Each coalition chooses cooperatively its level of public good Z_S in order to maximize

$$Z_S - c(\frac{Z_S}{|S|}).$$

The game of public good provision is played noncooperatively across coalitions. Given a Nash equilibrium of this noncooperative game, we define a partition function, $W(S, \pi)$ assigning to each coalition S in the coalition structure π, its equilibrium payoff. This partition function satisfies the following two properties.

(i) for any partition $\pi = \{S_1, S_2, .., S_m\}$ and any pair of coalitions S_i, S_j in π, $W(S_i \cup S_j, \pi \backslash \{S_i, S_j\} \cup \{S_i \cup S_j\}) > W(S_i, \pi) + W(S_j, \pi)$.

and (ii) for any partition $\pi = \{S_1, S_2, .., S_m\}$ and any pair of coalitions S_i, S_j in $\pi, W(S_k, \pi \backslash \{S_i, S_j\} \cup \{S_i \cup S_j\}) > W(S_k, \pi)$ for all $k \neq i, j$.

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[4] An example of coalition formation in public good provision is the emergence of international environmental agreements, where groups of countries choose cooperatively their abatement level for a global pollutant.

In the terminology of our model, the model of public good provision can be rephrased as follows. Every coalition S has access to a single reversible action, 0 (meaning that they do not provide the public good and continue negotiating), and a set of permanent actions, which is the nonnegative quadrant of $\Re^s_+$ (meaning that they provide the public good at level z_S). The flow payoff of a coalition S is given by $v_S(a) = \sum_{i \in S}(Z - c(z_i))$.

We analyze the coalitional bargaining game with three players.[5]

Proposition 4.4 *Consider the model of public good provision with three players with equal probability to make offers. As δ converges to 1, the coalitional bargaining game admits a Markov Perfect equilibrium where every player proposes to form the grand coalition at the initial stage. If externalities are large enough,[6] players opt out at the action stage with positive probability. Otherwise, no player opts out at the action stage and payoffs converge to $\frac{W(N)}{3}$.*

Proposition 4.4 shows that, contrary to Ray and Vohra (2001)'s analysis, efficient outcomes are always obtained at the equilibrium of the game of public good provision. To understand the difference between the two results, note that in Ray and Vohra (2001), players choose sequentially whether to opt out of the game – they opt out immediately after the contracting stage–, whereas we suppose that choices of irreversible actions are taken simultaneously. Hence, the first mover advantage which leads early players to opt out of the game and free ride on the public goods provided by subsequent players vanishes in our model, and efficiency is restored. It is also instructive to compare the equilibrium of the public goods model with the equilibrium of the unanimity bargaining game.[7] In both cases, the efficient outcome is obtained, and when externalities are not too strong, both games give rise to an equal distribution of the payoff among the three players. If externalities (as measured by the difference $W(1,1|23) - \frac{W(N)}{3}$) are strong, players opt out of the game with positive probability at the action stage, whereas in unanimity games, the probability that the large player drops out converges to zero as δ converges to 1.

[5] We have been unable to generalize the result beyond three players. With an arbitrary number of players, it becomes difficult to show that the best strategy of a player is to form the grand coalition immediately.

[6] Specifically, if $W(1,1|23) \geq \frac{W(N)}{3}$

[7] Gomes (2000a) analyzes three-player games with externalities in a related model, where renegotiation occurs, but where players do not choose endogenously whether to leave the game. His work shows the importance of the externalities parameter in the determination of equilibrium payoffs.

As a second example of coalition formation with positive externalities, we consider the merger game studied by Bloch (1996) and Ray and Vohra (1999). There are n identical firms in a linear oligopoly characterized by a demand

$$P = 1 - Q$$

and a zero constant marginal cost. Firms may merge, and for any given coalition structure π, we can easily compute the profit of every firm as

$$W(\pi) = \frac{1}{(|\pi| + 1)^2},$$

where $|\pi|$ denotes the number of elements in the partition π, i.e. the total number of firms present on the market after the mergers.

We can transpose the merger game in our model as follows. We suppose that firms only start producing when they leave the merger game, and that they commit to a contingent production plan, which depends on the number of competitors they face on the market. Hence every firm has access to a single reversible action, 0, (meaning that they continue negotiating and do not produce), and a set of contingent permanent actions, $q(t)$, where $q(t) \in [0, 1]$ is the production level when the firm faces t competitors on the market. Payoffs are then given as follows. If at state s, k firms have left the merger game, each of them obtains a flow payoff.$v_i(s) = q_i(k)(1 - \sum_j q_j(k))$. Firms which remain active in negotiations receive a payoff of 0.

We now prove:

Proposition 4.5 *In the merger game, as δ converges to 1, there exists a Markov Perfect Equilibrium where an initial offer to form a monopoly is accepted by all the players. If $n \leq 5$, no player opts out at the action stage, and offers converge to $x = \frac{1}{4n}$. If $n > 5$, all players opt out with positive probability at the action stage.*

The proof of the Proposition, contained in the Appendix, provides a careful construction of the Markov Perfect equilibrium supporting this outcome. It is important to highlight the difference between our analysis and the coalitional bargaining models of Bloch (1996) and Ray and Vohra (1999). In the earlier models, it was assumed that players moved sequentially, and opted out immediately after a coalition was formed. In our model, a player cannot choose to leave the game and free ride on the coalitions formed by subsequent players. Instead, we support the efficient outcome, because players know that if they reject the offer, the game moves to an action stage where all players simultaneously choose to opt out with a positive probability.

4.4 Market Entry with Synergies

We now consider a model where the formation of a coalition induces negative externalities on the other players. Suppose that three symmetric firms contemplate entering a market with fixed entry costs. By making a prior agreement, firms can benefit from synergies which will reduce their entry cost. We assume that individual firms face a cost F, that a coalition of two firms faces an entry cost $G < F$ and that the coalition of three firms face a zero entry cost. If a single firm enters the market, she obtains a gross monopoly profit π ; if two or three firms enter the market simultaneously, price competition results in zero profit for all the firms which have entered the market.

The equilibrium we construct has the following structure. At the initial contracting stage, any firm proposes to form a coalition of two firms, and enters the market with positive probability at the action stage. Hence, the firms do not fully exploit all the synergies they could create and the bargaining procedure results in an inefficient outcome. Notice that these strategies will form an equilibrium if and only if two conditions are satisfied:

- at the initial stage, a firm prefers to form a coalition of two firms rather than the grand coalition,

- at the action stage, the coalition of two firms chooses to leave the negotiation table and enter the market with a positive probability.

For these two conditions to hold, we need to impose specific conditions on the parameters of the model. We suppose:

Condition 4.6 *The parameters of the model satisfy:* $\pi > F > \frac{2}{3}\pi$, $\frac{1}{6}\pi > G > 0$, $\pi > F + G$.*and* $-3\pi^2 + 28G\pi - 20FG - 30G^2 < 0$.

Condition 4.6 is satisfied for low values of the fixed cost G and high values of the fixed cost F.

Proposition 4.7 *Consider the model of entry with synergies and suppose that condition 4.6 is satisfied There exists a Markov Perfect equilibrium of the coalitional bargaining game where, as* δ *converges to* 1*, any firm proposes to form a coalition of two firms at the initial contracting stage and enters the market with positive probability at the action stage.*

Proposition 4.7 shows that inefficient outcomes may arise in a model of coalitional formation with negative externalities. This result stands in contrast to the analysis in Gomes (2001), where players never leave the negotiation table and always reach efficient outcomes. When exit is endogenous, players may optimally choose to leave the game before the total surplus is realized. It is important to note that this result can only be obtained for specific parameter values, guaranteeing (i) that the difference between the total surplus and the surplus of a two-player coalition is low enough (G close to zero), and (ii) that two-player coalitions impose negative externalities on the isolated firm (F much larger than G). In conclusion, this last application shows that endogenous exit is not sufficient to guarantee efficient outcomes in coalitional bargaining games with externalities.

5 Absence of Irreversible Externalities and Efficiency

In this Section, we provide sufficient conditions for the coalitional bargaining procedure to yield efficient outcomes. Building on the examples of the previous Section, we define games *without irreversible externalities* as situations where players' exit payoff are independent of the actions chosen by other players. Formally:

Condition 5.1 *The underlying game v is a game* without irreversible externalities *if and only if for all coalitions S choosing $p_S \in P_S, v(a_{-S}, p_S) = v(a'_{-S}, p_S)$, $\forall a_{-S}, a'_{-S} \in A_{-S}$.*

Two applications of the previous Section satisfy this condition: the multilateral bargaining game with outside options and the contracting game with externalities.

We show that games without irreversible externalities always admit Markov Perfect Equilibria that are approximately Pareto efficient if the interval of time between periods is insignificant (or δ is close to one). We also show that when externalities on players' exit payoff are negligible, the coalitional bargaining procedure results in approximately efficient outcomes.

However, note that as players choose *simultaneously* whether to exit, they may face coordination failures at the action stage, with inefficiencies arising in equilibrium just because if they assume that other players will opt out then opting out may be their best response.

In many circumstances though this equilibrium may not be the most natural. For example, consider a standard two player bargaining with outside option (like the one in Section 4.1). While the most interesting equilibrium seems to be the one we characterized before in which opt-out occurs with zero (or negligible) probability, there is also another equilibrium in which both players opt-out for sure (and, at the contracting stage, proposers offer to extract all the surplus).

We use a refinement concept in the spirit of Selten (1975)'s trembling-hand perfection to eliminate these uninteresting equilibria..

Definition 5.2 *For any $\varepsilon > 0$, an $\varepsilon-$constrained coalitional bargaining game is a game, where at every state s where no coalition has opted out, any coalition $S \subset N$ plays a reversible action with probability greater than or equal to ε, $\sum_{a_S \in R_S} \sigma_S^2(s)(a_S) \geq \varepsilon$. An ε- constrained Markov Perfect equilibrium is a Markov Perfect Equilibrium of the $\varepsilon-$ constrained game.*

A simple adaptation of Proposition 3.3 shows that ε- constrained Markov Perfect equilibria exist.

Lemma 5.3 *For any $\varepsilon > 0$, an $\varepsilon-$constrained Markov Perfect Equilibrium exists.*

Interestingly, our next result show that despite the constrained optimality nature of the equilibrium above, for games without irreversible externalities, such solution is also a Markov Perfect Equilibria of the unconstrained game. Moreover, the constrained equilibria is approximately efficient (the aggregate equilibrium value at both the contracting and action stages converges to the efficient payoff, as the interval between offers become negligible).

Proposition 5.4 *Let $\varepsilon > 0$ and consider a game without irreversible externalities where the grand coalition is strongly efficient ($\max_{a_N} v_N(a_N) > \max_a \sum_{S\in\pi} v_S(a)$, $\forall \pi \neq \{N\}$). There exists $\delta(\varepsilon) > 0$ such that for all $\delta \geq \delta(\varepsilon)$, all $\varepsilon-$ constrained Markov Perfect Equilibria are also Markov Perfect Equilibria of the original game. Moreover, these Markov Perfect Equilibria are such that as δ converges to 1, the equilibrium converges to the efficient outcome and the probability that any coalition opts out early on converges to zero.*

The reason for which the $\varepsilon-$constrained equilibrium is also an unconstrained equilibrium is because the restriction is never binding. Suppose by contradiction that the restriction were binding, so that exiting is a strict better response than staying at the negotiation table for some coalition at some subgame where no coalition has opted out. At the action stage of this subgame, an inefficient outcome would then be realized (with probability at least $1-\varepsilon$) and the players' aggregate payoff would be bounded away from the efficient payoff.

But note that if the coalition deviated and stayed on, she would be the proposer next period with some probability, and any proposer is able to extract all the efficiency loss for herself. This is accomplished by making an offer to all other coalitions to move to the efficient allocation with an offer equal to the inefficient outcome they would get if they rejected it. This leads to a contradiction because the value from staying on would surpass the value of exiting.

Notice that this reasoning works because the game has no irreversible externalities: once a coalition exits, its outside option is independent of the action of other coalitions. If the exit payoff of a coalition depended on the action of other coalitions, there would be no guarantee that this coalition could increase its payoff by remaining in the game (see the market synergies example). However, the next result shows that, as long as irreversible externalities are small enough, coalitions still prefer to choose reversible actions, and the outcome of the bargaining procedure remains approximately efficient.

Proposition 5.5 *Let v be a game such that the grand coalition is strongly efficient, and*

$$\max_{S,p_S,a_{-S},a'_{-S}} |v(a_{-S},p_S)-v(a'_{-S},p_S)| = \eta(v).$$

There exists $\eta_\varepsilon > 0$ and $E_\varepsilon > 1$ such that if $\eta(v) < \eta_\varepsilon$ all ε-constrained Markov Perfect Equilibria are Markov Perfect Equilibria of the original game. Furthermore, equilibrium payoffs are bounded above by $\max_{a_N} v_N(a_N) - E_\varepsilon \eta M$ where $M = \max\{p_i(s)^{-1} : i \in \mathcal{R}(s)$ with $P_i \neq \emptyset$ and all states $s\}$.

6 Conclusion

This paper proposes a model of coalitional bargaining based on strategic interaction across coalitions, where players endogenously choose whether to exit. This formulation is general enough to study the formation of coalitions and the distribution of gains from cooperation in a wide variety of economic models with externalities and outside options. We show that when outside options are independent of the actions of other players, the Markov Perfect

Equilibria of the coalitional bargaining game converge to efficient outcomes when the players become perfectly patient. On the other hand, we show through an example than in a model with negative externalities where players' outside options depend on the action of other players, the outcome may be inefficient, as players choose to exit before they extract all gains from cooperation.

These results highlight the difference between our model and previous models of coalitional bargaining. In a setting with externalities, Ray and Vohra (1999) show that when players cannot renegotiate, the outcome of coalition formation is typically inefficient, as players have an incentive to leave the game before extracting all the surplus. On the contrary, Gomes (2001) establishes that when renegotiation occurs and players cannot choose to exit, the outcome is always efficient. Our study shows that in a model with endogenous exit, the outcome can either be efficient or inefficient, depending on the type of externalities imposed on other players.

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8 Appendix

Proof of Proposition 3.3: We define a correspondence $F : \Phi \times \Phi \times \Sigma^1 \times \Sigma^2 \to\to \Phi \times \Phi \times \Sigma^1 \times \Sigma^2$ whose fixed points are the MPE. Let Φ be the set of coalition values $\phi = (\phi_S(s))$, where $\phi \in \Phi$ if and only if each $\phi_S(s) \in R$ is bounded below by

$$\min\{v_S(a) : \text{for all states } s' \text{ s.t. } S \in \mathcal{C}(s') \text{ and } a \in (A_T)_{T\in\mathcal{C}(s')}\},$$

and the sum $\sum_{S\in\mathcal{C}(s)} \phi_S(s)$ is bounded above by

$$\max\left\{\sum_{T\in\mathcal{C}(s')} v_T(a) : \text{for all states } s' \text{ and } a \in (A_T)_{T\in\mathcal{C}(s')}\right\}.$$

Let Σ^1 be the set of proposers' strategies σ^1 at the contracting stage, so $\sigma^1_T(s)$ is a probability distribution over the set $\{\mathcal{S} \subset \mathcal{R}(s) : \ T \in \mathcal{S}\}$ (we omit the transfers from the strategy profile, because we already know that proposers always propose a transfer equal to the value of the coalition at the action stage). Let Σ^2 be the set of strategies σ^2 at the action stage, so $\sigma^2_T(s)$ is a probability distibution over A_T.

The correspondence F is such that $(\varphi^1, \varphi^2, \mu^1, \mu^2) \in F(\phi^1, \phi^2, \sigma^1, \sigma^2)$ if and only if, for all states s and coalitions $S \in \mathcal{C}(s)$, the following holds:

$$\begin{aligned}
\varphi^1_S(s) &= q_S(s)\left(\sum_{\mathcal{S}} \sigma^1_S(s)(\mathcal{S})(\phi^2_{\mathcal{S}}(g(\mathcal{S},s)) - \sum_{T\in\mathcal{S}} \phi^2_T(s)\right) \\
&\quad + \sum_{T\in\mathcal{R}(s)} q_T(s) \sum_{\mathcal{S}} \sigma^1_T(s)(\mathcal{S})(\mathbf{1}_{S\in\mathcal{S}}\phi^2_S(s) + \mathbf{1}_{S\notin\mathcal{S}}\phi^2_S(g(\mathcal{S},s)), \\
\varphi^2_S(s) &= u_T(s,\sigma^2)(\phi^1,\phi^2,\sigma^1,\sigma^2),
\end{aligned}$$

and (*supp* denotes the support of a probability distribution)

$$\begin{aligned}
supp\left(\mu^1_T(s)\right) &\subset \operatorname*{arg\,max}_{\mathcal{S}\subset\mathcal{R}(s) \text{ s.t. } T\in\mathcal{S}} \left\{(\phi^2_{\mathcal{S}}(g(\mathcal{S},s)) - \sum_{T\in\mathcal{S}} \phi^2_T(s))\right\}, \\
supp\left(\mu^2_T(s)\right) &\subset \operatorname*{arg\,max}_{a_T\in A_T} \left\{u_T(s,a_T,\sigma^2_{-T})(\phi^1,\phi^2,\sigma^1,\sigma^2)\right\},
\end{aligned}$$

where $z = (\phi^1, \phi^2, \sigma^1, \sigma^2)$ and for any $\mu^2 \in \Sigma^2$,

$$u_T(s,\mu)(z) = \sum_{a=(a_R)_{R\in R(s)}} \prod_{R\in R(s)} \mu^2_R(s)(a)\, u_S(s,a)(z),$$

and $u_S(s,a)(z) = \left(\delta\phi_S^1(h(s,a)) + (1-\delta)\, v_S(a \otimes p(s))\right)$.

According to Lemma 2, the fixed points of F are the MPE. To show that F has a fixed point we use the Kakutani fixed point theorem. It can be easily verified using standard arguments that $Z = \Phi \times \Phi \times \Sigma^1 \times \Sigma^2$ is a compact and convex finite-dimension set, that $F(Z) \subset Z$, $F(z)$ is a convex (and non-empty) set for all $z \in Z$, and F has a closed graph (or is u.h.c.). Thus the Kakutani fixed point theorem applies. Q.E.D.

Proof of Proposition 4.1: We construct the candidate equilibrium. At any state s where some players have opted out, the equilibrium strategy is for all active players to opt out. At any state s where no player has opted out, with m active coalitions, let S_1 be the coalition with the highest outside option, (and suppose that there is a single coalition with the highest outside option). At the contracting stage, every player proposes to form the grand coalition, and to offer x_j to other coalitions, resulting in expected equilibrium payoffs ϕ_j. At the action stage, player S_1 opts out of the game with probability p and the other players continue to negotiate. The variables $p(s), x_i(s)$ and $\phi_i(s)$ are given by the following equations.

If $v_1 \geq \frac{1}{m}v$ then

$$\begin{aligned}
p(s) &= (1-\delta)\frac{mv_1 - \delta v}{\delta\left(\delta v - \delta v_N - v_1(1-\delta)\right)}, \qquad (1)\\
\phi_j(s) &= \frac{v_1 + \frac{p}{(1-\delta)}\delta^2 v_j}{\delta\left(1+\delta\frac{p}{(1-\delta)}\right)} \text{ and } x_j(s) = \phi_j(s) - \frac{(1-\delta)\,v_1}{\delta} \text{ for } j = 2,...,m,\\
\phi_1(s) &= \frac{v_1}{\delta} \text{ and } x_1(s) = v_1,
\end{aligned}$$

If $v_1 < \frac{1}{m}v$ then

$$\begin{aligned}
p(s) &= 0,\\
\phi_i(s) &= \frac{1}{m}v \text{ and } x_i(s) = \frac{1}{m}\delta v \text{ for } i = 1,...,m. \qquad (2)
\end{aligned}$$

We now show that this strategy profile forms a Markov perfect equilibrium. Consider any state where all players are active. If $v_1 \geq \frac{1}{m}v$, at the action stage, player 1 is indifferent between opting out and continuing, as $v_1 = \delta\phi_1$. For players $j = 2,...,m$,

$$\delta\phi_j - v_j = \frac{v_1 - v_j - \frac{p}{(1-\delta)}\delta(1-\delta)v_j}{\left(1+\delta\frac{p}{(1-\delta)}\right)}.$$

For δ close enough to 1, $\delta\phi_j - v_j > 0$, so no player wants to opt out. If now $v_1 < \frac{1}{m}v$, for all players $j = 1, 2, ..., m$, $\delta\phi_j - v_j = \frac{\delta v}{m} - v_j > 0$ for δ large enough. So no player wants to opt out either.

Now consider the contracting stage. First suppose that $v_1 \geq \frac{1}{m}v$. By Proposition 1, if the grand coalition is formed, the offers must satisfy

$$\begin{aligned} x_j &= (1-p)\,\delta\phi_j + p\delta v_j \text{ for } j = 2, ..., n, \\ \phi_i &= \frac{1}{m}(v - x_N) + x_i, \\ x_1 &= v_1 = \delta\phi_1, \end{aligned} \tag{3}$$

Combining the last two equations,

$$\phi_1 - x_1 = \frac{(1-\delta)\,v_1}{\delta} = \frac{1}{m}(v - x_N),$$

so

$$x_i = \phi_i - \frac{(1-\delta)\,v_1}{\delta} \text{ for } i = 1, ..., m,$$

Replacing the value of x_i in the first equation yields

$$\phi_j - \frac{(1-\delta)\,v_1}{\delta} = (1-p)\,\delta\phi_j + p\delta v_j,$$

whose unique solution is the ϕ_j given in the proposed strategy profile. Adding all equations for $j = 2, ..., n$ results in

$$x_N - x_1 = (1-p)\,(\delta\phi_N - \delta\phi_1) + p\delta\,(v_N - v_1),$$

and since $\phi_N = v$, $x_N = \phi_N - m\frac{(1-\delta)v_1}{\delta}$, $\delta\phi_1 = v_1$, and $x_1 = v_1$ we can solve for p,

$$p = (1-\delta)\,\frac{mv_1 - \delta v}{\delta\,(\delta v - \delta v_N - v_1\,(1-\delta))},$$

as we claimed. Notice that $p \geq 0$ if and only if $\delta \geq \delta_0$ where $\delta_0 = \frac{v_1}{v - v_N + v_1} < 1$ because $v - v_N > 0$, and for δ close enough to 1, $p \leq 1$.

Now suppose that $v_1 < \frac{1}{m}v$. By Proposition 1, if the grand coalition is formed,

$$\begin{aligned} x_i &= \delta\phi_i \text{ for all } i \\ \phi_i &= \frac{1}{m}(v - x_N) + x_i \end{aligned} \tag{4}$$

and it is straightforward to verify that the unique solution of the system of equations is given by the formula in the description of the strategy profile.

It remains to verify that no player wants to deviate by forming a subcoalition at the contracting stage. Consider a deviation by which some of the active players form a subcoalition $S \subset N$, and let ϕ'_S be the continuation value of coalition S, $v_S = \sum_{j \in S} v_j$ the outside option of S, and $\phi_S = \sum_{j \in S} \phi_j$, the sum of equilibrium payoffs obtained if the grand coalition is formed. If $v_S \geq v_1$ then the payoff of coalition S converges for δ large enough to $\phi'_S = v_S < \phi_S$ (since $\phi_i \geq v_i$ with strict inequality for at least one $i \in S$) and thus all players $i \in S$ are better off not deviating. Similarly, if $v_S < v_1$, all players $j \notin S \cup \{1\}$ benefit from the deviation, since their new payoff ϕ'_j is obtained by replacing m by $m - |S| + 1$ in the formula for equilibrium payoffs, and $\phi'_j > \phi_j$. Now, as $\sum_{j \in N} \phi'_j = \sum_{j \in N} \phi_j = v$ and $\phi'_1 = \phi_1, \phi'_S < \phi_S$and players in S are better off not deviating.

Our analysis only deals with the case where there is a unique player with the highest outside value at any state. The result can be generalized to situations with multiple players with highest values as follows. Suppose that there are m players, $j = 1, ..., m$, such that $v_j = \max_{i \in N} v_i$. Perturb the payoffs, by adding a random vector ε to all the payoffs, and construct the equilibrium for the perturbed game, where all values are different. As ε goes to zero, because equilibrium payoffs and strategies are upper hemi continuous in the parameters of the game, one can obtain limit equilibrium payoffs and strategies for the original game. Q.E.D.

Proof of Proposition 4.3: We give an explicit construction of the Markov Perfect Equilibrium. Let m be the number of agents who have already contracted with the principal.

At a subgame m where $x^*(m) = m$, the principal offers $\phi^2_{-c}(m)$ to all the remaining agents, and the agents accept accept any offer greater than or equal to $\phi^2_{-c}(m)$. At the action stage, the principal exist with a positive probability q. The values q and $\phi^2_{-c}(m)$ are computed as solutions to the equations:

$$\begin{aligned}
\phi^1_c(m) &= v\,(n) - (n-m)\,\phi^2_{-c}(m) \\
\phi^1_{-c}(m) &= \phi^2_{-c}(m) \\
\phi^2_c(m) &= \delta \phi^1_c(m) = v_c\,(m) \\
\phi^2_{-c}(m) &= (1-q)\,\delta \phi^1_{-c}(m) + q v_{-c}(m)
\end{aligned}$$

At a subgame m where $x^*(m) > m$, the principal offers to contract with $x^*(m) - m$ of the $n - m$ remaining agents (all agents are chosen with equal probability). She offers $\phi^2_{-c}(m)$ to all the agents, and agents accept any offer greater than or equal to $\phi^2_{-c}(m)$. At the action stage, the principal never exits. The value $\phi^2_{-c}(m)$ is computed as a solution to the equations:

$$
\begin{aligned}
\phi^1_c(m) &= \phi_c(x^*(m)) - (x^*(m) - m)\,\phi^2_{-c}(m) \\
\phi^1_{-c}(m) &= \frac{(x^*(m) - m)}{n - m}\phi^2_{-c}(m) + \frac{(n - x^*(m))}{n - m}\phi^2_{-c}(x^*(m)) \\
\phi^2_c(m) &= \delta\phi^1_c(m) \\
\phi^2_{-c}(m) &= \delta\phi^1_{-c}(m)
\end{aligned}
$$

Note that, as δ converges to 1, the equilibrium offers $\phi^2_{-c}(m)$ converge to $\frac{v(n)-v_c(x^*(m))}{n-x^*(m)} = v^*_{-c}(m)$ and $\frac{q}{(1-\delta)}$ converges to the positive value $\frac{v_c(m)-v(n)}{v(n)-v_c(m)-(n-m)v_{-c}(m)}$.

To show that this strategy profile forms a subgame perfect equilibrium, we first consider subgames satisfying $x^*(m) = m$. By construction, the principal's exit decision at the action stage and the agents' responses at the contracting stage are optimal. It remains to check that the principal's offer at the contracting stage is optimal. Suppose by contradiction that the principal makes an acceptable offer to $m' < n$ agents. She would then receive a payoff $\phi^2_c(m') - (m' - m)\,\phi^2_{-c}(m)$ instead of $v(n) - (n-m)\phi^2_{-c}(m)$. Two cases must be distinguished. If $x^*(m') = m'$, then $\phi^2_c(m') = v_c(m')$. But because $x^*(m) = m$,

$$
\phi^2_{-c}(m) < \frac{v(n) - v_c(m')}{n - m'},
$$

and hence

$$
v_c(m') < v(n) - (n - m')\phi^2_{-c}(m),
$$

establishing that the deviation is unprofitable. If now $x^*(m') > m'$. In the continuation game, the prinipal proposes to form a coalition of size $x^*(m')$ and then moves to the grand coalition. Overall, she thus offers $v^*_{-c}(m')$ to the remaining $(n - m')$ agents and $\phi^2_c(m') = v(n) - (n - m')v^*_{-c}(m')$. But because $x^*(m) = m, v^*_{-c}(m') > v^*_{-c}(m)$ and hence,

$$
v\,(n) - \left(n - m'\right) v^*_{-c}(m') - \left(m' - m\right) v^*_{-c}(m) < v\,(n) - (n - m)\, v^*_{-c}(m),
$$

estbalishing that the deviation is unprofitable.

Consider now a subgame satisfying $x^*(m) > m$. We first show that, at the action stage, staying in the game is the optimal action of the principal. By exiting, the principal obtains a payoff of $v_c(m)$ and by staying a payoff of $\phi_c(m) = v(n) - (n-m)\, v^*_{-c}(m)$. As $x^*(m) \neq m$,

$$v^*_{-c}(m) < \frac{v(n) - v_c(m)}{n-m},$$

so that the optimal strategy is to choose a temporary action. At the contracting stage, the agents' response is optimal by construction, and by an argument similar to the argument in the case $x^*(m) = m$, the principal has ni incentive to offer to form a coalition of size $m' \neq x^*(m)$. Q.E.D.

Proof of the properties of the partition function in the public good example: To prove property (i), it suffices to show that

$$(|S_i| + |S_j|) \max_z (|S_i| + |S_j|)z - c(z) > |S_i| \max_z |S_i|z - c(z) + |S_j| \max_z |S_j|z - c(z).$$

Let $z_i^* = \arg\max\, |S_i|z - c(z), z_j^* = \arg\max\, |S_j|z - c(z)$. Define

$$z^* = \frac{|S_i|}{|S_i| + |S_j|} z_i^* + \frac{|S_j|}{|S_i| + |S_j|} z_j^*.$$

By construction, $(|S_i| + |S_j|)z^* = |S_i|z_i^* + |S_j|z_j^*$ and by strict convexity of the cost function,

$$(|S_i| + |S_j|)c(z^*) < |S_i|c(z_i^*) + |S_j|c(z_j^*),$$

establishing the result.

Now suppose $|S_i| \geq |S_j|$ and let Z_j^* be the optimal total contribution of coalition S_j. By convexity of the cost function,$|S_i|c(\frac{Z_j^*}{|S_i|}) \leq |S_j|c(\frac{Z_j^*}{|S_j|})$. Hence,

$$|S_i| \max_z |S_i|z - c(z) +$$

To prove property (iii), consider the first order condition for the optimal contribution:

$$c'(z) = |S_i|.$$

As c is strictly convex, an increase in the size of the group increases the per capita contribution, and hence also increases the total group contribution. We conclude that the merger of two groups increases their global contribution, to the benefit of all the other players. Q.E.D.

Proof of Proposition 4.4: To prove the proposition, we construct the Markov Perfect Equilibrium. At states where players form the grand coalition, [123], it is clear that the optimal action is to choose the optimal level of public goods provision. At the state where all players are active and the coalition [12|3] has been formed, the game is similar to a two-player unanimity game with outside options. According to Proposition 4.1, the Markov perfect equilibrium of the game either yields utility $\frac{w(N)}{2}$ to both players or $W(12, 12|3)$ to player 12 and $W(N) - W(12, 12|3)$ to player 3.or $W(N) - W(3, 12|3)$ to player 12 and $W(3, 12|3)$ to player 3. If one of the players – say player 3–has opted out, by property (i), the remaining two players should form a coalition, and divide equally the payoff $W(12, 12|3)$. Consider now the initial stage, [1|2|3] where no player has opted out. In the postulated equilibrium, every player gets a symmetric value $\phi_i = \frac{w(N)}{3}$. Define $z^* = \arg\max_z(z - c(z))$. At the action stage, a player obtains

$$\begin{aligned}
\phi^2 &= W(1,1|2|3)\sigma^2 + 2\sigma(1-\sigma)((z^* - c(z^*))(1-\delta) + W(1,1|2|3)\delta) \\
&\quad +(1-\sigma)^2((z^* - c(z^*))(1-\delta) + W(1,1|23)\delta) \text{ if she opts out} \\
\phi'^2 &= \sigma^2(2z^*(1-\delta) + W(1,1|2|3)\delta) + 2\sigma(1-\sigma)(z^*(1-\delta) + \frac{W(12,12|3)}{2}\delta) \\
&\quad +(1-\sigma)^2\frac{W(N)}{3}\delta \text{ if she stays.}
\end{aligned}$$

By posing $\phi^2 = \phi'^2$, and letting δ converge to 1, we obtain a solution

$$\sigma = \frac{W(1,1|23) - \frac{W(N)}{3}}{W(1,1|23) - \frac{W(N)}{3} + 2(\frac{W(12,12|3)}{2} - W(1,1|2|3))}.$$

As long as $W(1,1|23) - \frac{W(N)}{3} \geq 0$, there exists an equilibrium with a positive value σ. If $W(1,1|23) - \frac{W(N)}{3} < 0$, there is an equilibrium where none of the player opts out and each obtains an expected payoff $\frac{W(N)}{3}$. The offer x is obtained by solving

$$x = \phi^2 = \phi'^2.$$

It remains to show that the proposer has no incentive to announce a different coalition. If she makes an unacceptable offer, then the game moves to the action stage, and she gets $\phi^2 < W(N) - 2\phi^2$. If she offers to form a coalition 12, her expected payoff becomes $\delta\frac{W(N)}{2}$ (or $\delta(W(N) - W(3, 12|3) < \delta\frac{W(N)}{2})$ if $W(12, 12|3) < \frac{W(N)}{2}$ and $\delta W(12, 12|3)$.if $W(12, 12|3) \geq \frac{W(N)}{2}$. We now show that $\max\{W(12, 12|3), \frac{W(N)}{2}\} - \phi^2 < W(N) - 2\phi^2$. First, by Property (ii), $W(1, 1|23) > W(1, 1|2|3)$. Hence, $\phi^2 < W(1, 1|23)$, and by Property (i), $W(12, 12|3) + W(1, 1|23) < W(N)$. We conclude that $W(12, 12|3) + \phi^2 < W(N)$. Now, as $\phi^2 < \frac{W(N)}{3}, \frac{W(N)}{2} + \phi^2 < W(N)$, completing the proof. Q.E.D.

Proof of Proposition 4.5: We construct the Markov Perfect Equilibrium. Let a state s be characterized by two parameters: the number of players who have already left the game, k, and the number of active players, m.

At any state where $m \leq (1 + k)^2$, the unique equilibrium strategy prescribes that players make unacceptable offers and leave immediately. The proof is by induction on the number of active players. Suppose that $m = 2 \leq (1+k)^2$. As $\frac{1}{(k+2)^2} < \frac{2}{(k+3)^2}$, both players optimally choose to leave the game. Now consider a state with $m \leq (1 + k)^2$ active players, and suppose that for all states with $m' < m$, players leave the game. At the action stage, if any player leaves the game, the state moves to a state with $m' < m$ active players, and all players leave the game and obtain a payoff $\frac{1}{(k+m+1)^2}$. Hence, the continuation value at state s satisfies: $\phi^1(s) \geq \frac{1}{(k+m+1)^2}$. Suppose by contradiction that $\phi^1(s) \geq \frac{1}{(k+m+1)^2}$. Then at the contracting stage, every player must make an acceptable offer of $\phi^2(s)$ to the other players. This implies that there exists a value r for which:

$$\frac{1}{(k + m - r + 2)^2} - r\phi^2(s) > 0.$$

But notice that

$$\frac{1}{(k + m - r + 2)^2} - r\phi^2(s) \leq \frac{1}{(k + m - r + 2)^2} - \frac{r}{(k + m + 1)^2}.$$

It is easy to check that the function

$$f(r) = \frac{1}{(k + m - r + 2)^2} - \frac{r}{(k + m + 1)^2}$$

is a convex function which attains its maximum either at $r = 1$ or $r = m$. As $m \leq (1+k)^2$, the maximum is obtained at $r = 1$, and hence

$$\frac{1}{(k+m-r+2)^2} - \frac{r}{(k+m+1)^2} < 0,$$

contradicting the assumption that the player made an acceptable offer. We conclude that all players leave the game and

$$\phi^1(s) = \phi^2(s) = \frac{1}{(k+m+1)^2}.$$

Next consider states with $m > (1+k)^2$ and $m - 1 \leq (2+k)^2$. These are states where, whenever one player leaves the game, all other players leave the game. Hence, the game is similar to a symmetric unanimity bargaining game, where the payoff of the grand coalition is given by $\frac{1}{(k+2)^2}$ and the outside option of every player is $\frac{1}{(k+m+1)^2} < \frac{1}{m(k+2)^2}$. By Proposition 4.1, in equilibrium, every player obtains an expected payoff of $\frac{1}{m(k+2)^2}$.

Finally, consider states with $m - 1 > (2+k)^2$. (Notice that when $n > 5$, the initial state will satisfy this inequality, as $m = n, k = 0$.) At the action stage, all players are symmetric and we construct a symmetric mixed strategy equilibrium of the game, characterized by a common probability σ of leaving the game. At the contracting stage, we suppose that every player makes an acceptable offer to all other active players. We now check that these strategies form a Markov Perfect Equilibrium.

Let t^* be the first integer satisfying:

$$m - t \leq (k+t+1)^2.$$

In words, t is the minimal number of players, who, by leaving the game, induce all other players to leave the game. By construction, $t^* \geq 2$. If a player stays in the game at state s, her expected payoff is given by:

$$\begin{aligned} A &= \delta \sum_{t=0}^{t^*-1} \left(\frac{t}{m-1}\right) \sigma^t (1-\sigma)^{m-1-t} \frac{1}{(m-t)(k+t+2)^2} + \\ &\quad \delta \sum_{t=t^*}^{m-1} \left(\frac{t}{m-1}\right) \sigma^t (1-\sigma)^{m-1-t} \frac{1}{(k+m+1)^2}, \end{aligned}$$

because all players leave the game if $t \geq t^*$ and if $t < t^*$, by construction of the equilibrium, every player makes an acceptable offer to all other active players. If a player leaves the game at state s, her expected payoff is:

$$\begin{aligned}B &= \sum_{t=0}^{t^*-2} \binom{t}{m-1} \sigma^t (1-\sigma)^{m-1-t} [(1-\delta)\frac{1}{(t+k+2)^2} + \delta \frac{1}{(t+k+3)^2}] + \\ &\quad \sum_{t=t^*-1}^{m-1} \binom{t}{m-1} \sigma^t (1-\sigma)^{m-1-t} [(1-\delta)\frac{1}{(t+k+2)^2} + \delta \frac{1}{(k+m+1)^2}],\end{aligned}$$

as the player makes a flow profit for the current period which depends on the number of players who produce $k+t+1$, and next period, either obtains a profit anticipating that all remaining players merge (if $t+1 > t^*$), or that remaining players leave the game (if $t+1 \leq t^*$). Now, letting δ converge to 1 we compute

$$\begin{aligned}B - A &= \sum_{t=0}^{t^*-2} \binom{t}{m-1} \sigma^t (1-\sigma)^{m-1-t} [\frac{1}{(t+k+3)^2} - \frac{1}{(m-t)(k+t+2)^2}] + \\ &\quad \binom{t^*-1}{m-1} \sigma^{t^*-1} (1-\sigma)^{m-t^*} [\frac{1}{(k+m+1)^2} - \frac{1}{(m-t^*+1)(k+t^*+1)^2}].\end{aligned}$$

As σ converges to 1, $B - A$ converges to $\frac{1}{(k+m+1)^2} - \frac{1}{(m-t^*+1)(k+t^*+1)^2} < 0$. As σ converges to 0, $B-A$ converges to $\frac{1}{(k+3)^2} - \frac{1}{m(k+2)^2} > 0$ as $m > (k+2)^2$. Hence, as $B - A$ is a continuous function of σ, there exists a value σ^* for which $B - A = 0$.

It remains to check that, at the contracting stage, every player has an incentive to make an acceptable offers to all remaining active players. We first establish the following result. Let

$$g(t) = \frac{1}{(m-t)(k+t+2)^2}.$$

We compute:

$$g'(t) = \frac{k+3t-2m}{(m-t)^2(k+t+2)^3}.$$

For $t < t^*$, $m - t > (k+t+1)^2$, implying that

$$k + 3t - 2m < k + t - 2(k+t+1)^2 < 0.$$

Hence $g'(t) < 0$ and for all $t < t^*$, $\frac{1}{(m-t)(k+t+2)^2} < \frac{1}{m(k+2)^2}$. Hence, $\phi^2(s) = A < \frac{1}{m(k+2)^2}$.

This result immediately establishes that no player has an incentive to make an unacceptable offer, since by offering to form the grand coalition, she obtains:

$$\frac{1}{(k+2)^2} - (m-1)\phi^2(s) > \phi^2(s).$$

Now, suppose that the player proposes to form a subcoalition of $r < m$ firms. Two cases must be distinguished. If $m - r < (k+2)^2$, then following the merger, all players leave the game. If $m - r \geq (k+2)^2$, following the merger, the game moves to a state where players continue to negotiate.

If $m - r \geq (k+2)^2$, the game moves to a new state s' where $k' = k$ and $m' = m - r + 1$. Hence $\phi^2(s') < \frac{1}{(m-r+1)(k+2)^2}$. We now show :

$$\frac{1}{(k+2)^2} - (m-1)\phi^2(s) > \phi^2(s') - (r-1)\phi^2(s).$$

To this end, note that

$$\frac{1}{(k+2)^2} - \phi^2(s') > \frac{m-r}{m-r+1}\frac{1}{(k+2)^2} \geq \frac{m-r}{m}\frac{1}{(k+2)^2} > (m-r)\phi^2(s).$$

Next, suppose that $m - r < (k+2)^2$. Then, the game moves to a state s' with $\phi^2(s') = \frac{1}{(k+m-r+2)^2}$. Now,

$$\frac{1}{(k+2)^2} - \phi^2(s') = \frac{(m-r)}{(k+2)^2}\frac{(m-r+2k+4)}{(k+m-r+2)^2}.$$

Note that

$$\begin{aligned} m(m-r+2k+4) - (k+m-r+2)^2 &= m(k+2) - (k+m-r+2)(k-r+2) \\ &> (k+2)^3 - (k-r+2)(k+2)(k+3), \end{aligned}$$

using the fact that $m > (k+2)^2$ and $(m-r) < (k+2)^2$.

Now,

$$(k+2)^3 - (k-r+2)(k+2)(k+3) = (k+2)[(k+2)^2 - (k+2-r)(k+3)] > 0$$

for all $r \geq 1$. Hence, we obtain

$$\frac{1}{(k+2)^2} - \phi^2(s') = \frac{(m-r)}{(k+2)^2}\frac{(m-r+2k+4)}{(k+m-r+2)^2} > \frac{(m-r)}{m(k+2)^2} > (m-r)\phi^2(s),$$

concluding the proof of the Proposition. Q.E.D.

Proof of Proposition 4.7: Consider the action stage, after two players have formed a coalition. Let player 1 denote the coalition of two firms, and player 2 the single firm. The payoffs of the players are given by the following matrix.

	stay out	enter the market
stay out	$\delta\phi_1, \delta\phi_2$	$0, \pi - F$
enter the market	$\pi - G, 0$	$-G, -F$

where ϕ_1 and ϕ_2 denote the continuation values, at the contracting stage with a coalition of two firms (player 1) and a single firm (player 2). We will construct an equilibrium where player 1 enters the market with a probability $0 < p_1 < 1$ and player 2 enters the market with a positive probability $0 < p_2 < 1$. In this completely mixed equilibrium, both players are indifferent between staying out and entering the market:

$$(1 - p_2)\delta\phi_1 = (1 - p_2)\pi - G, \tag{5}$$
$$(1 - p_1)\delta\phi_2 = (1 - p_1)\pi - F. \tag{6}$$

Furthermore, at the contracting stage where player 1 faces player 2, the grand coalition will necessarily be formed. To see this, note that if one of the players (say player 1) made an unacceptable offer, the following two incompatible conditions would need to be satisfied: $\phi_1^1 = \phi_1^2$ (as player 1 makes an unacceptable offer) and $\phi_1^2 = (1 - p_2)\delta\phi_1^1$ (as both players play a completely mixed equilibrium at the action stage). Now, let x_i denote the minimal offer that player i is willing to accept

$$\begin{aligned} x_1 &= (1 - p_2)\delta\phi_1, \\ x_2 &= (1 - p_1)\delta\phi_2. \end{aligned}$$

Finally, let

$$\begin{aligned} \phi_1 &= \frac{1}{2}(\pi - x_2) + \frac{1}{2}x_1, \\ \phi_2 &= \frac{1}{2}(\pi - x_1) + \frac{1}{2}x_2. \end{aligned}$$

Solving the last four equations for $\delta = 1$, we obtain:

$$x_1 = \frac{\pi p_1(1-p_2)}{p_1+p_2}, x_2 = \frac{\pi p_2(1-p_1)}{p_1+p_2}$$

Replacing in equations (5) and (6), we obtain two quadratic equations defining p_1 and p_2 :

$$\begin{aligned}
\pi p_2(1-p_2) &= G(p_1+p_2), \\
\pi p_1(1-p_1) &= F(p_1+p_2).
\end{aligned}$$

We will show that this system of equations admits a solution for p_1 and p_2 in $(0,1)$. After some manipulations, we can rewrite the system as

$$\begin{aligned}
F(p_2) &\equiv -\pi^2 p_2^3 + 2p_2^2\pi(\pi - G) + p_2(G\pi + FG - G^2 - \pi^2) + G(\pi - F - G) = 0. \\
p_1 &= p_2(\frac{\pi(1-p_2)}{G} - 1).
\end{aligned}$$

It can easily be checked that as $\pi > F+G, F(0) > 0$. Since furthermore $F(1) < 0$, the equation $F(p_2) = 0$ admits at least one solution in $(0,1)$. Furthermore, note that

$$p_1(1-p_1) = \frac{F}{G} p_2(1-p_2).$$

Hence, for any equilibrium value $p_2^* \in (0,1)$, the corresponding p_1^* must satisfy $p_1^* \in (0,1)$. Note also that, as

$$-3\pi^2 + 28G\pi - 20FG - 30G^2 < 0,$$

we have $F(\frac{1}{5}) < 0$, so there exists a solution $p_2^* < \frac{1}{5}$.

Now consider the action stage if no coalition has been formed. We will show that it is a weakly dominant strategy for every firm to stay out of the market. By staying out, a firm obtains either $\delta\phi_i^1$ if no other firm enters the market, or 0 if another firm enters the market. By entering the market, the firm either gets $\pi - F$ if no other firm enters the market, or 0 otherwise. Hence, we need to show

$$\delta\phi_i^1 \geq \pi - F. \tag{7}$$

To compute ϕ_i^1, we consider the initial contracting stage when all firms are independent. In the equilibrium we construct, we suppose that any firm proposes to form a coalition with one of the two other firms (choosing any of the two firms with probability $\frac{1}{2}$) and offering x to that firm. Let v_1 denote the continuation value for a firm who has merged with another one and v_2 the continuation value for the independent firm. From our previous computation, when δ converges to 1,

$$v_1 = \frac{\pi p_1^*(1-p_2^*)}{p_1^*+p_2^*}, v_2 = \frac{\pi p_2^*(1-p_1^*)}{p_1^*+p_2^*}$$

Now,

$$\begin{aligned}
\phi_i^1 &= \frac{1}{3}(v_1 - x) + \frac{1}{3}x + \frac{1}{3}v_2 = \frac{v_1+v_2}{3} \\
&= \frac{\pi}{3} - 2\frac{p_1^* p_2^*}{p_1^*+p_2^*}.
\end{aligned}$$

Letting δ converge to 1, inequality (7) rewrites as:

$$3F(p_1^*+p_2^*) > 2\pi(p_1^*+p_2^*+p_1^*p_2^*).$$

At equilibrium,

$$F(p_1^*+p_2^*) = \pi p_1^*(1-p_1^*).$$

Hence inequality (7) is equivalent to:

$$p_1^* - 3p_1^{*2} - 2p_2^* - 2p_1^*p_2^* > 0. \tag{8}$$

Rewriting $p_2^* = p_1^*(\frac{\pi}{F}(1-p_1^*)-1)$, we obtain

$$\frac{3-p_1^*}{2(1-p_1^{*2})} > \frac{\pi}{F}.$$

This last condition is always satisfied, as $\frac{3-p_1^*}{2(1-p_1^{*2})}$ is an increasing function over $[0,1)$ and $\frac{\pi}{F} < \frac{3}{2}$.

Finally, we check that it is an optimal strategy for a firm to form a coalition with one of the two other firms at the initial contracting stage. Given the preceding step, at the action stage, the continuation value of each firm is given by

$$x = \delta\frac{v_1+v_2}{3}.$$

If a firm makes an unacceptable offer it obtains the continuation value x. If it proposes to form a coalition of two firms, it gets $v_2 - x$. Now, to show $v_2 \geq 2x$, it suffices to establish

$$v_2 \geq 2v_1.$$

After some algebra, letting δ converge to 1,we may rewrite this inequality as:

$$p_1^* - 2p_2^* + p_1^* p_2^* > 0.$$

But this last inequality is weaker than inequality (8) and hence is always satisfied.

If now a firm makes an offer to form the grand coalition, it obtains a value $\pi - 2x$. Hence, to show that the firm prefers to form a coalition of 2, we need to establish:

$$v_2 - x \geq \pi - 2x,$$

or

$$v_2 + x \geq \pi.$$

After some algebra, and letting δ converge to 1, this inequality becomes

$$p_1^* - 5p_2^* - 5p_1^* p_2^* > 0.$$

As $p_1^* = p_2^*(\frac{\pi}{G}(1 - p_2^*) - 1)$, and using the fact that $p_2^* < \frac{1}{5}$,we obtain

$$\frac{\pi}{G} > \frac{6 - 5p_2^*}{(1 - p_2^*)(1 - 5p_2^*)}.$$

As $\frac{6-5p_2^*}{(1-p_2^*)(1-5p_2^*)}$ is a decreasing function over the interval $[0, \frac{1}{5})$ and $\pi > 6G$, this inequality is always satisfied. Q.E.D.

Proof of Lemma 5.3: Define the correspondence $F_\varepsilon : \Phi \times \Phi \times \Sigma^1 \times \Sigma_\varepsilon^2 \rightarrow\rightarrow \Phi \times \Phi \times \Sigma^1 \times \Sigma_\varepsilon^2$ as in proposition 3.3 with two minor modifications: (i) the set $\Sigma_\varepsilon^2 \subset \Sigma^2$ is such that $\sigma^2 \in \Sigma_\varepsilon^2$ if and only if $\sum_{a \in A(s)} \sigma_S^2(s)(a) \geq \varepsilon$ for all states s where no coalitions have opted out yet and $S \in \mathcal{C}(s)$; (ii) $(\varphi^1, \varphi^2, \mu^1, \mu^2) \in F_\varepsilon(\phi^1, \phi^2, \sigma^1, \sigma^2)$ if and only if φ^1, φ^2, μ^1, and μ^2 are as in the proof of proposition 3.3, except that at states s where no coalitions have opted out yet and for all $S \in \mathcal{C}(s)$, the following holds:

$$u_S(s, \mu_S^2, \sigma_{-S}^2)(z) = \max \left\{ u_S(s, \widetilde{\mu}_S^2, \sigma_{-S}^2)(z) : \text{ s.t. } \left(\widetilde{\mu}_S^2, \sigma_{-S}^2\right) \in \Sigma_\varepsilon^2 \right\},$$

where $z = (\phi^1, \phi^2, \sigma^1, \sigma^2)$ and the function u are as defined before.

The correspondence F_ε have fixed points because all the conditions for the Kakutani fixed point theorem applies, and the fixed points are an $\varepsilon-$constrained MPE. Q.E.D.

Proof of Proposition 5.4: Suppose by contradiction that an $\varepsilon-$constrained MPE (σ^1, σ^2) with payoffs $\left(\phi^1, \phi^2\right)$ is not an MPE of the unconstrained problem. Then there exists a state s where no coalitions have opted out yet and a coalition $S \in \mathcal{R}(s)$ where the constraint is binding,

$$v_S := \max_{x_S \in P_S} v_S(x_S, \sigma^2_{-S}) > u_S(s, a_S, \sigma^2_{-S}) \text{ for all } a_S \in R_S, \tag{9}$$

so that $\sum_{a_S \in R_S} \sigma^2_S(s)(a_S) = \varepsilon$. Let K the minimum aggregate level of efficiency lost when at least one coalition opts out

$$K := \min\left\{\left|v_N(a) - v_N(a^*_N)\right| : \forall a \in A(s), \text{ such that } C(s) > 1\right\}. \tag{10}$$

where $v_N(a^*_N) = \max_{a_N \in A_N} v_N(a_N)$. By assumption $K > 0$.

Since coalition S opts out with probability $(1-\varepsilon)$, and once S opts out the aggregate payoff $v_N\left(a^k\right) \le v_N(a^*_N) - K$ for all future action profiles a^k, then $\phi^2_N(s) \le v_N(a^*_N) - (1-\varepsilon)K$.

We now estimate the lowest payoff that coalition S can obtain in the $\varepsilon-$constrained game. He can guarantee for itself at least v_S by opting out, but in the $\varepsilon-$constrained game he can only opt with probability at most equal to $1-\varepsilon$. The strategy of, at the investment stage, opting out of the game at every period with probability $1-\varepsilon$, and choosing x_S such that $\underline{v}_S = \min_{x_S \in R_S} \min_{x_{-S} \in X_{-S}} v_S(x_S, x_{-S})$ with prob ε, and, at the contracting stage, not making any offers and rejecting any offers made, yields coalition S at least

$$\underline{\phi}_S = \frac{\underline{v}_S(1-\delta) + \delta(1-\varepsilon)v_S}{1-\delta\varepsilon},$$

which converges to v_S when δ converges to 1 (the formula comes from the evaluation of $E\left[\sum_{t=0}^{\infty} \delta^t (1-\delta) v_S\left(a^t\right)\right]$). Thus $\phi^k_S(s) \ge \underline{\phi}_S$, for $k = 1, 2$.

In addition,

$$\begin{aligned}
\phi^1_S(s) &\ge (1 - q_S(s))\,\underline{\phi}_S + q_S(s)\left(v_N(x^*_N) - \sum_{j \in N(s)\setminus S} \phi^2_S(s)\right) = \\
&= (1 - q_S(s))\,\underline{\phi}_S + q_S(s)\phi^2_S(s) + q_S(s)\left(v_N(x^*_N) - \sum_{j \in N(s)} \phi^2_S(s)\right),
\end{aligned}$$

which implies

$$\phi_S^1(s) \geq \underline{\phi}_S + q_S(s)(1-\varepsilon)K.$$

But coalition S's payoff $u_S(s, a_S, \sigma_{-S}^2)$, for any $a_S \in R_S$, is at least equal to

$$\lambda\left(\delta\phi_S^1(s) + (1-\delta)\underline{v}_S\right) + (1-\lambda)\left(\delta\underline{\phi}_S + (1-\delta)\underline{v}_S\right),$$

where $\lambda \geq \varepsilon^{C(s)-1}$ is the probability that all remaining $C(s)\backslash S$ coalitions choose reversible actions. Combining our findings so far, we then get

$$\liminf_{\delta\to 1} u_S(s, a_S, \sigma_{-S}^2) \geq \lambda\left(v_S + p_S(s)(1-\varepsilon)K\right) + (1-\lambda)v_S) = v_S + \lambda \cdot p_S(s)(1-\varepsilon)K > v_S$$

which is in contradiction with ineq. (9).

The proof of the efficiency part is by induction on the number of coalitions in $C(s)$. The proposition obviously holds for states s with $C(s) = 1$ (since the contract including all players will lead to the choice of an efficient action). Let the induction hypothesis be that if the proposition holds for all states s with $C(s) < m$ then it holds for all states with $C(s) = m$ coalitions.

Suppose by contradiction the proposition does not hold for some state s with $C(s) = m$ coalitions (but holds for all s' with $C(s') < m$). So there is a subsequence $\delta_k \to 1$ with equilibrium satisfying $\phi_{N,k}^2(s) \leq v_N(a_N^*) - K$, where $K > 0$. A similar argument as above shows that the payoff for not opting out is, for k large enough, greater than $v_S + \lambda \cdot q_S(s)K$ which is greater than the payoff if the coalition opts out. Thus in equilibrium no coalition should opt out, and $\phi_{S,k}^1(s) = \phi_{S,k}^2(s)$.

Consider coalitions' strategies at the contracting stage of state s. The gain of forming a coalition is strictly positive for all players, because if the grand coalition is formed, that yields an excess of $v_N(a_N^*) - \phi_{N,k}^2(s) \geq K > 0$. But then no proposers are not willing to pass up its chance to form a coalition and, with probability 1, there will be a coarsening of the coalition structure. By the induction hypothesis $\phi_{N,k}^2(s') \to v_N(a_N^*)$ for all s' with $C(s') < m$, and since the aggregate payoff $\phi_{N,k}^1(s)$ is equal to linear combination of the values $\phi_{N,k}^2(s')$ this implies that $\phi_{N,k}^1(s) = \phi_{N,k}^2(s) \to v_N(a_N^*)$ (contradiction). Q.E.D.

Proof of Proposition 5.5: We first show that if $\eta < \eta_\varepsilon$ (where $\eta_\varepsilon > 0$ is yet to be determined) then all $\varepsilon-$constrained Markov perfect equilibrium are MPE. This proof has the same structure of the proof of proposition 5.4 and thus we just include the steps that are different.

Inequality 9 changes to

$$v_S + \eta/2 \geq \max_{x_S \in P_S} v_S(x_S, x_{-S}) > u_S(s, a_S, \sigma^2_{-S}) \text{ for all } a_S \in R_S,$$

and let $K > 0$ be like in definition 10.

A similar argument shows that $\liminf_{\delta \to 1} \underline{\phi}_S^{(\delta)} \geq v_S - \eta/2$. In addition,

$$\liminf_{\delta \to 1} \phi_S^1(s) \geq v_S - \eta/2 + q_S(s)(1-\varepsilon) K.$$

But coalition S's payoff $u_S(s, a_S, \sigma^2_{-S})$, for any $a_S \in R_S$, is at least equal to

$$\lambda \left(v_S - \eta/2 + q_S(s)(1-\varepsilon) K\right) + (1-\lambda)\left(v_S - \eta/2\right),$$

Combining our findings so far, we then get

$$\liminf_{\delta \to 1} u_S(s, a_S, \sigma^2_{-S}) \geq v_S - \eta/2 + \lambda \cdot q_S(s)(1-\varepsilon) K > v_S + \eta/2$$

leads to a contradiction if

$$\eta < \eta_\varepsilon := \frac{\varepsilon^{n-1}(1-\varepsilon) K}{M}.$$

Now we show that all $\varepsilon-$constrained Markov perfect equilibrium are MPE and the equilibrium payoffs are bounded away by $\liminf_{\delta \to 1} \phi_N^{k,(\delta)}(s) \geq v_N(a_N^*) - M \cdot E_\varepsilon \cdot \eta$ (where E_ε is yet to be determined). One inequality is

$$\phi_{S,k}^1(s) \geq v_S + \eta\left(E + \frac{1}{2}\right) + q_S(s) K$$

so the payoff for not opting out is, for k large enough, greater than

$$\lambda\left(v_S + \eta\left(E + \frac{1}{2}\right) + K\right) + (1-\lambda)\left(v_S - \eta/2\right) \geq v_S + \frac{1}{2}\eta$$

if

$$E := \frac{\left(1 - \varepsilon^{n-1}\right)}{\varepsilon^{n-1}}$$

The rest of the argument is the same. Q.E.D.