Concave Value of Information in Multiple Decision Problems

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General motivation

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- How are investments into information acquisition taken? Trade off between costs and benefits.

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- How does this value evolve with the resources invested?

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- How does this value evolve with the resources invested?
- Increasing, or decreasing returns to scale?

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- Recall a result on nonconcavity of the value of information.

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- Recall a model with concave value of information.

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- Recall a result on nonconcavity of the value of information.
- Recall a model with concave value of information.
- Study the value of information in a “canonical” model of information acquisition.

Information acquisition before decision

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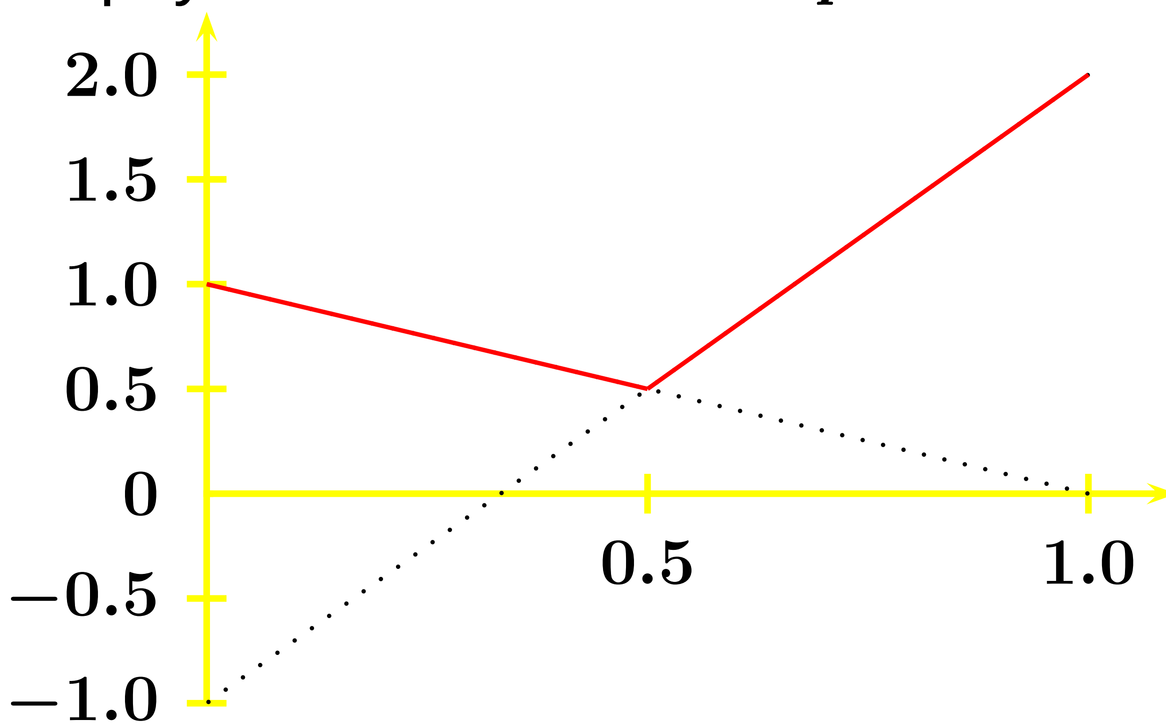
- Payoff $g(x, k)$ also depends on some random state of nature k, of known prior distribution.
- A statistical experiment μ gives a probability over signals S for each value of k.
- The decision maker can choose among statistical experiments prior to decision of x, at a cost that depends on the experiment chosen.

Example of decision problem

	k_1	k_2
x_1	2	-1
x_2	0	1

Prior is $P(k_1) = \frac{1}{3}$.

Expected payoffs as a function of $p =$ *belief in* k_1.



Example of statistical experiments

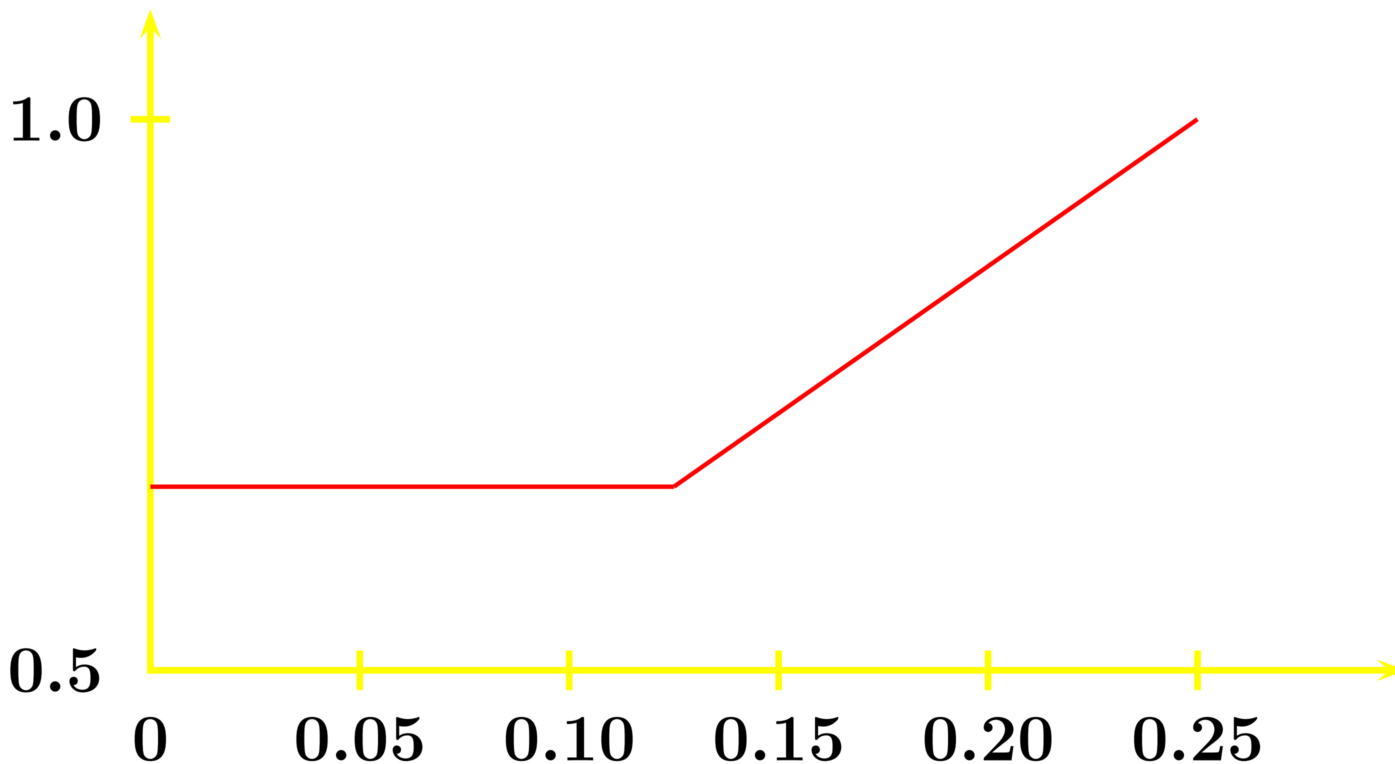
	k_1	k_2
s_1	$\frac{1}{2} + 2\alpha$	$\frac{1}{2} - \alpha$
s_2	$\frac{1}{2} - 2\alpha$	$\frac{1}{2} + \alpha$

$$0 \leq \alpha \leq \frac{1}{4}$$

$$P(s_1) = \tfrac{1}{2},\ P(k_1|s_1) = \tfrac{1}{3} + \tfrac{4}{3}\alpha$$
$$P(s_2) = \tfrac{1}{2},\ P(k_1|s_2) = \tfrac{1}{3} - \tfrac{4}{3}\alpha$$

"Splitting" around original beliefs.

Expected payoff as a function of α



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How does the payoff the D.M. can obtain evolve with α?

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Value of information is not concave

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- Zero marginal value of “small” information not very intuitive.
- Paradox: If you never agree to pay “for a little more” information, how can you ever agree to pay at all?
- Little choice in the model about “what” information to obtain.

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Obtained value of information is concave

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Radner and Stiglitz make an assumption, apparently innocuous but not met in Arrow's model.

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  - Can the concavity result be extended to other utility functions?

Model: nature and decisions

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Payoffs are discounted, λ discount factor.

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D.M. can ask basic questions at any stage to a research group, or to an expert, about states of nature

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We examine perfect answers first, "noisy" answers later.

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$$\begin{cases} q_{t+1} &= \sigma_{t+1,q}((k_1, a_1), \ldots, (k_t, a_t)) \\ x_{t+1} &= \sigma_{t+1,x}((k_1, a_1), \ldots, (k_t, a_t)) \\ a_{t+1} &= q_{t+1}(k_\infty) \end{cases}$$

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The information cost is

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In general

$$\underline{\boldsymbol{v}} \leq \max_{\boldsymbol{\sigma}} \boldsymbol{V}_{\boldsymbol{\lambda}}(\boldsymbol{\sigma}) \leq \overline{\boldsymbol{v}}$$

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3. We then derive the optimal information acquisition rate.

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Σ_r is the set of strategies of acquisition rate r.

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What can we say about $v_\lambda(r)$?

Or about $\lim_{\lambda \to 1} v_\lambda(r)$?

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X, Y pair of discrete random variables

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It is the *mutual information* $\boldsymbol{I}(\boldsymbol{X}, \boldsymbol{Y})$ between $\boldsymbol{X}$ and $\boldsymbol{Y}$.

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where the maximum is taken over pairs of r.v.'s (x, k) s.t. $k \sim \nu$ and $I(x, k) \leq r$.

The limit value is thus the value of a one-shot decision problem in which price of information is given by mutual information.

Example

In the decision problem

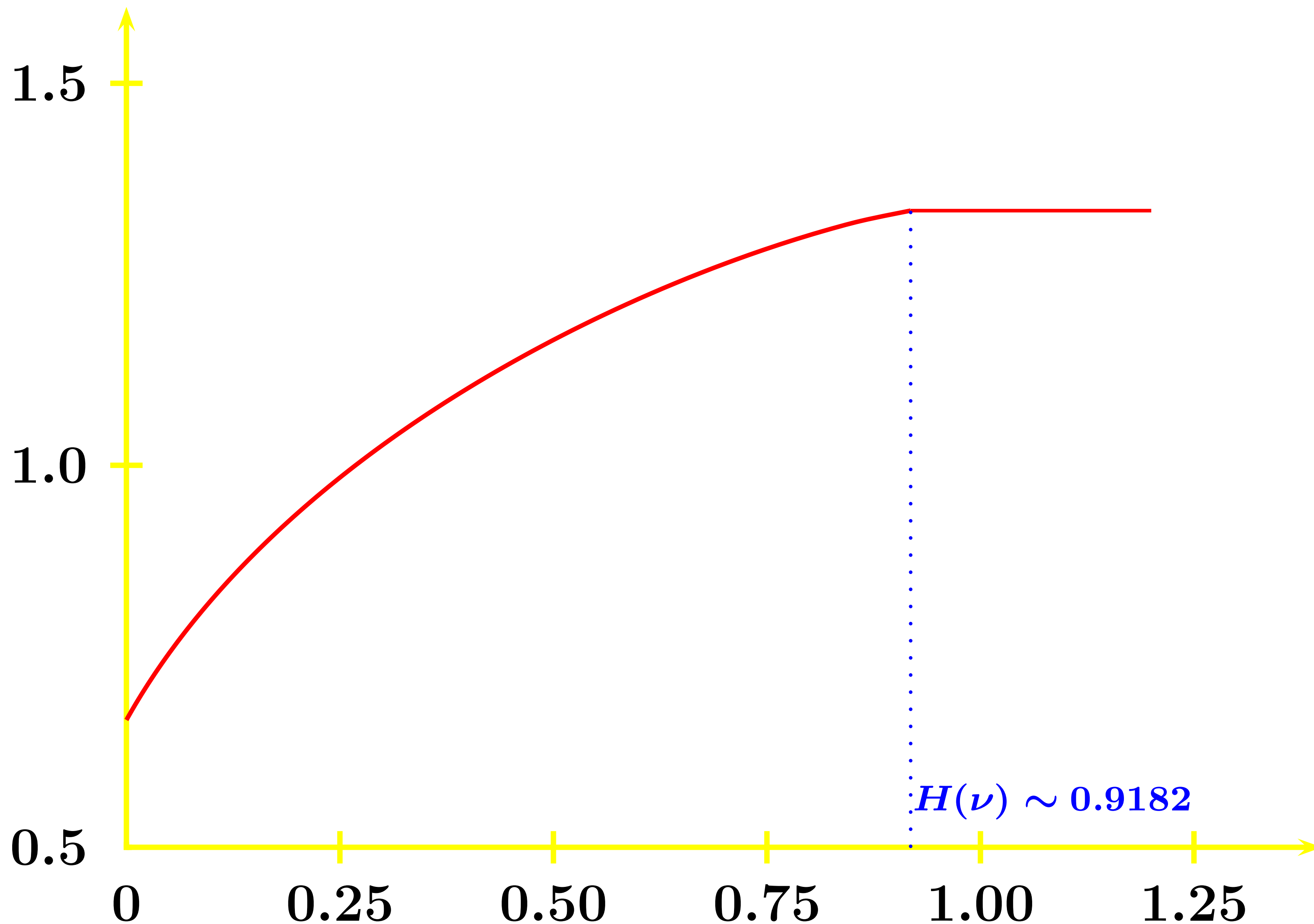
	k_1	k_2
x_1	2	-1
x_2	0	1

experiments that maximize expected payoff and minimize information costs correspond to “splittings”

$$\begin{array}{ll} P(s_1) = \lambda & P(k_1|s_1) = x \\ P(s_2) = 1-\lambda & P(k_1|s_2) = 1-x \end{array}$$

$$\lambda x + (1-\lambda)(1-x) = \frac{1}{3}$$

The derived graph of $v(r)$



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4. v is increasing on $[0, r_0]$.
5. The left derivative of v at r_0 is 0.

Noisy answers: model

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The set of basic answers is $\boldsymbol{A_0}$, a basic question is a map

$$\boldsymbol{q_0\colon K^{\mathbb{N}} \longrightarrow A_0}$$

Noise is given by a probability transition $\boldsymbol{N}$ from $\boldsymbol{A_0}$ to $\boldsymbol{B_0}$.

If $\boldsymbol{A_0}$ is the (true) answer, D.M. observes $\boldsymbol{b_0 \in B_0}$ with probability $\boldsymbol{N(a_0)(b_0)}$. Noise is independent between questions.

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Sampling, where the same basic question is asked an arbitrary number of times, is a particular case.

Noisy answers: value of information

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Let

$$C = \max_{a_0} I(a_0, b_0)$$

C may be larger or smaller than 1.

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Main Result

$$v(r) = \lim_{\lambda \to 1} v_\lambda(r) = \max \mathrm{E} g(x, k)$$

where $k \sim \nu$ and $I(x, k) \leq Cr$.

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Results extend to more general processes, e.g. Markov, with or withour observation of past states of nature.

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3. Demand never reaches r_0, which means that full information never obtains.