Endogeneous Communication Networks*

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1 Introduction

Networks shape economic and social activities. Sociologists, anthropologists and economists have long stressed the importance of social networks in a wide variety of contexts including job referrals, exchanges of favors, diffusion of ideas. [1] Communication networks become ubiquitous and transform the ways in which human beings conduct transactions and social activities. Economic networks play an increasing role in businesses and public policies.

An emerging literature in economics analyzes the strategic formation of networks, assuming that rational economic agents form bilateral links with other agents in order to maximize their utility. Following Jackson and Wolinsky (1996), this literature has stressed a fundamental tension between *efficient* networks, which maximize the sum of agents'utilities, and *stable* networks which are immune to deviations by pairs of players, who can either sever links or form a new link.

The literature on strategic network formation models the formation of a link as a 0-1 decision: either two agents are linked or they are separate. This modeling assumption fails to capture the richness of economic and social networks. In reality, not all links have the same strength: some are weak and others are strong. Friends can either be close or distant, communication networks can either be reliable or not, alliances among firms can be superficial or far-reaching. In order to account for this wide variety of links, one has to model explicitly the strength of bilateral ties. In formal terms, economic and social networks need to be modeled as weighted (or value) graphs rather than graphs represented by a simple binary relation. [2]

Our objective in this paper is precisely to propose a model of strategic network formation where agents choose the intensity of their links. We suppose that each agent is endowed with a fixed amount of resources that she can allocate among various relationships. For example, she could either choose to invest in a small number of strong links (e.g. build a small number of strong relationships), or to invest in a large number of weak links (build a large number of weak relationships). The resulting network structure is a weighted graph, whose efficiency and stability can be analyzed using the standard tools introduced by Jackson and Wolinsky (1996).

The description of social relations as weighted links introduces new mod-

[1]A small sample of this literature includes Granovetter (1974), Boorman (1975), Montgomery (1991), Calvo-Armengol and Jackson (1994).

[2]Goyal(2004) also emphasizes the importance of modelling the *quality* of links .

eling issues and we are faced at the outset with different modeling choices. How do personal investments translate into strengths of relationships? How do agents benefit from the relations in the network, and how do those benefits depend on the quality of links? Concerning the first question, we focus largely on the case where the total strength of any given link is the sum of *convex* transformations of individual investments in the link. A special case is when individual investments are *perfect substitutes*. In order to illustrate the different implications of alternative modelling assumptions, we also consider the polar case where agents' investments are *perfect complements*.

Concerning the last two questions, we focus in this paper on *communications networks*, and assume that an agent derives positive benefits from all the agents with whom she is connected, either directly or indirectly. The value of these benefits however depends on the "reliability" of the path connecting the two agents. This in turn depends on the strengths of links. In one version of the model (the product reliability version), we suppose that the reliability of a path is simply the product of the strengths of all links in the path. This corresponds to the case where the strength of each link also represents the probability that information can be transmitted through the link. Under this assumption, the value of an indirect link is simply the reliability of the most reliable path. In a second version of the model, (the min-reliability version), we suppose that the value of an indirect link depends on the bottleneck of the communications network, and is equal to the strength of the weakest link on the optimal path.

Our first task is to characterize efficient networks. Efficient networks must be *trees* when the strength of a link is the sum of convex transformations of individual investments. In the perfect substitutes case, we show that the efficient network is always a tree with symmetric investments along all the links. In the product-reliability model, the only optimal architecture is a star. Hence, every agent establishes a communication link with only one agent (the hub of the star), thereby maximizing the value of all indirect connections in the graph. When investments are perfect complements, the picture becomes notably more complex. In the product-reliability model, we show that trees cannot be efficient, and that the circle is efficient for small numbers of agents. However, an example shows that the efficient architecture is much more complex than the circle for larger numbers of agents. In the min-reliability case, the efficient architecture is either the circle or the line, which maximize the strengths of links in the network.

These results highlight the difference implications of our modeling assumptions on the efficient architectures. Except when investments are perfect substitutes, there is a trade-off between the need to increase direct and indirect benefits. For instance, if the strength of a link is the sum of strictly convex tranformations of individual investments, then direct benefits are maximised if each agent allocates all his investment on one link. This may reduce indirect benefits in the product reliability model sicne that may require equalisation of strengths across all links. In the min-reliability model, the distance between agents is irrelevant, and the only important feature of the network is the presence of bottlenecks. The optimal network architecture thus requires that links have the highest possible strength, which means that the network is sparse (typically a tree), and that the quality of all links are equal (which would typically obtain in a line or a circle, where all nodes have a very low degree). The line emerges as an efficient network both in the perfect substitutes and perfect complements case, and we conjecture that this might be an optimal architecture for more general formulations.

In the product-reliability model, the value of indirect benefits depends not only on the strength of the links, but also on the distance between nodes. The optimal architecture then results from a complex trade-off, as the addition of new links might decrease distance between agents, but at the cost of decreasing the strength of existing links. In the two knife-edge cases we consider (perfect substitutes and perfect complements), the trade-off is resolved in different ways. In the perfect substitutes case, the star arises as an efficient network, because it minimizes distances of indirect links, and the investment of peripheral agents can compensate for the low investment of the hub of the star. By contrast, in the perfect complements case, all agents are required to make large investments, and our analysis suggests that efficient networks will be regular, and display a lot of symmetry.[3]

We then turn attention to stability of the networks. We call a network Nash stable if no individual can improve his utility by through a deviation. A network is strongly pairwise stable if pairs of agents wants to establish a new link. Our results are mixed - as in Jackson and Wolinsky (1996)'s study of 0-1 networks, there is no clear connection between efficiency and stability. If the strength of a link is given by the sum of strictly convex transformations of individual investments, then the efficient architecture in ither version of reliability need not be Nash stable. In the product relaibility

[3]We note however that we have not been able to obtain a general characterization of efficient networks when investments are perfect complements.

model, if investments are perfect substitutes, then the efficient architecture is Nash stable. Whether it is strongly pairwise stable or not depends on the level of inevestment available to each agent.

In the min-reliability model, we easily show that efficient network architectures in the perfect substitutes and perfect complements cases are both Nash stable and strongly pairwise stable. In fact, any reallocation of investments from the symmetric tree or the line and circle will end up in forming links with asymmetric values, thereby decreasing the smallest value in the network.

Our analysis can be interpreted as a generalization of the communications networks studied by Jackson and Wolinsky (1996) , Bala and Goyal (2000a) and Bala and Goyal (2000b). [4]As in these models, agents derive positive benefits from all agents they are linked with, and the value of these benefits depends on the structure of the communications network. The major difference between these models and ours is that they assume an exogenous decay in the benefits, whereas we endogenously determine the benefits of indirect connections. In Jackson and Wolinsky (1996) and Bala and Goyal (2000a), indirect benefits depend only on the distance in the graph, with a fixed decay parameter δ ; Bala and Goyal (2000b) introduce a fixed reliability parameter and model the value of an indirect link as a function of this exogenous parameter. By contrast, in this paper, the reliability of links is endogenous, and the value of indirect links is a choice variable of the agents in the network.

Prior to our study, a small literature has analyzed networks with endogenous link strength in specific settings. Goyal and Moraga Gonzales (2001) consider the formation of strategic alliances among oligopolistic firms, with a two-stage process. Firms initially choose which links to establish, and then endogenously choose the effort they put in cost reduction along every link. Durieu, Haller and Solal (2004) construct a model of nonspecific networking, where players choose the intensity of the links, but cannot discriminate among other players and either establish links to all the players or to none. Brueckner (2003) addresses the same question as us – the formation of friendship links where agents can endogenously choose the intensity of the link– but assumes that indirect benefits only arise from nodes at distance two. If agents are connected by indirect paths of length greater than two, no benefit

[4]In fact, the case of perfect complements is closely related to Jackson and Wolinsky (1996) as the formation of link requires efforts from both parties, whereas the case of perfect substitutes is closer to Bala and Goyal (2000), as agents can unilaterally form links. The general model is in between these two extremes.

is obtained. His main focus is not on the characterization of efficient and stable networks, but on the computation of effort levels for fixed network structures.

The rest of the paper is organized as follows. We introduce the model and notations in Section 2. In Section 3, we study optimal network architectures. Section 4 is devoted to the analysis of pairwise stable networks.

2 Model and Notations

Let $N = \{1, 2, ..., n\}$ be a set of individuals. Individuals derive benefits from links to other individuals. These benefits may be the pleasure from friendship, or the utility from (non-rival) information possessed by other individuals, and so on. In order to fix ideas, we will henceforth interpret benefits as coming from information possessed by other individuals. Each individual has a total resource (time, money) of $X > 0$, and has to decide on how to allocate X in establishing links with others.

Let x_i^j denote the amount of resource invested by player i in the relationship with j. Then, the *strength* of the relationship between i and j is a function of x_i^j and x_j^i. Let s_{ij} denote this strength as a function of the amounts x_i^j and x_j^i. Let this functional dependence be denoted as

$$s_{ij} = f(x_i^j, x_j^i)$$

with $s_{ij} \in [0, 1]$.

Alternative assumptions can be made about the functional form of f. Throughout much of the paper, we will focus on the following case.

Assumption 1: For each $i, j \in N$, $f(x_i^j, x_j^i) = \phi(x_i^j) + \phi(x_j^i)$, where ϕ is strictly increasing, convex, with $\phi(0) = 0$.

A special case is when $\phi(x) = x$ for all x. In this case, the contributions of any i and j to the link ij are *perfect substitutes.*

An alternative assumption is the following.

Assumption 2: For each $i, j \in N$, $f(x_i^j, x_j^i) = \min(x_i^j, x_j^i)$.

Assumption 2 says that the contributions of i and j are *perfect complements.*

We say that individuals i and j are linked if and only if $s_{ij} > 0$. Each pattern of allocations of X, that is the vector $\mathbf{x} \equiv (x_i^j)_{\{i,j \in N, i \neq j\}}$ results in a weighted graph, which we denote by $g(\mathbf{x})$. [5]

Individuals i and j each get a *direct* benefit of s_{ij} from the link ij. However, as in JW, individuals also derive benefits from indirect links. We describe the form of these indirect benefits.

Given any g, a path between individuals i and j is a sequence $i^0 = i, i^1, ..., i^m, ..., i^M = j$ such that $i^{m-1}i^m \in g$ for all m. Two individuals are *connected* if there exists a path between them. Connectedness defines an equivalence relation, and we can partition the set of individuals according to this relation. Blocks of that partition are called components, and we let $\mathcal{N}(g)$ denote the set of components of the graph g.

Suppose i and j are connected, but not directly linked in the graph, that is $ij \notin g$. Then, the (indirect) benefit that i derives from j depends on the *reliability* with which i can access j's information. Given this interpretation and our definition of *direct* benefits, notice that the reliability of a direct link ij is simply the strength of the link ij.

For any pair of individuals i and j and graph g let $P(i,j)$ denote the set of paths linking i to j. Define

$$\begin{aligned} p^*(i,j) &= \arg \max_{p(i,j) \in P(i,j)} s_{ii^1} ... s_{i^{m-1}i^m} ... s_{i^{M-1}j}, \\ R^p(i,j) &= \max_{p(i,j) \in P(i,j)} s_{ii^1} ... s_{i^{m-1}i^m} ... s_{i^{M-1}j}. \end{aligned}$$

So, for any two individuals i and j, $p^*(i,j)$ denotes the path which has the highest reliability, as measured by the products of strengths of the links on that path. The reliability of that path is denoted $R^p(i,j)$.

As an alternative, we will also define reliability of any path in terms of the strength of the weakest link in that chain. Hence, we would have

$$R^m(i,j) = \max_{p(i,j) \in P(i,j)} \min_{s_{i^{m-1}i^m} \in p(i,j)} s_{i^{m-1}i^m}$$

Henceforth, we will refer to R^p and R^m as *Product-Reliability* and *Min-Reliability* respectively.

Individual i's payoff from a graph g is then given by

$$U_i(g) = \sum_{j \in N \setminus i} R^p(i,j) \quad \text{or} \quad U_i(g) = \sum_{j \in N \setminus i} R^m(i,j)$$

The value of a weighted graph g is given by $V(g) = \sum_i U_i(g)$.

[5] To simplify notation, we will sometimes ignore the dependence of g on the specific pattern of allocations.

Definition 1 *A graph g is efficient if $V(g) \geq V(g')$ for all g'.*

Our model is related to the original communications model of JW. However, there are significant differences. First, JW assumed that a link between i and j either exists or does not. That is, either $s_{ij} = 1$ or $s_{ij} = 0$. In contrast, we allow s_{ij} to take on any value between 0 and 1. Second, JW assume that i and j each pay an exogenously given cost c if they form the link ij. As we have described earlier, there are no exogenous costs of forming a specific link in our model. Instead, individuals have a "budget constraint" - they can "costlessly" invest in as many links as they want so long as the total investment does not exceed X. Lastly, JW's reliability function is also different from ours - they assume that the indirect benefit that i derives from j is δ^{d-1}, where d is the geodesic distance between i and j, while $\delta \in (0, 1)$ is a parameter.

As we have remarked before, JW initiated the analysis of the potential conflict between efficiency and "stability" of endogenously formed networks. This is also the main focus of our paper. We now describe the concepts of stability that will be used in this paper.

Given any pattern of investments $\mathbf{x}$, and individual i, $(\mathbf{x_{-i}}, x'_i)$ denotes the vector where i deviates from x_i to x'_i. Similarly, $(\mathbf{x_{-i,j}}, x'_{i,j})$ denotes the vector where i and j have jointly deviated from (x_i, x_j) to (x'_i, x'_j).

Definition 2 *A graph $g(\mathbf{x})$ is Nash stable if there is no individual i and x'_i such that $U_i(g(\mathbf{x_{-i}}, x'_i)) > U_i(g(\mathbf{x}))$.*

So, a graph g induced by a vector $\mathbf{x}$ is Nash stable if no individual can change her pattern of investment in the different links and obtain a higher utility.

Definition 3 *A graph $g(\mathbf{x})$ is Strongly Pairwise Stable if there is no pair of individuals (i, j) and joint deviation (x'_i, x'_j) such that*

$$U_i(g(\mathbf{x_{-i,j}}, x'_{i,j})) + U_j(g(\mathbf{x_{-i,j}}, x'_{i,j})) > U_i(g(\mathbf{x})) + U_j(g(\mathbf{x}))$$

So, a graph is strongly pairwise stable if no pair of individuals can be jointly better off by changing their pattern of investment. Notice that we define a pair to be better off if the *sum* of their utilities is higher after the deviation. This leads to a stronger definition of stability than a corresponding definition with the requirement that both individuals be strictly better off after the deviation. The current definition implicitly assumes that individuals can make side payments to one another. The availability of side payments is consistent with our definition of efficiency - a graph is defined to be efficient if it maximises the *sum* of utilities of all individuals.

JW define a weaker notion of stability - *pairwise stability.* Basically, JW restrict deviations by assuming that only one link at a time can be changed.[6]

Some specific network architectures will be important in subsequent sections. We define these below.

Definition 4 *A graph g is a star if there is some $i \in N$ such that $g = \{ik | k \in N, k \neq i\}$.*

The distinguished individual i figuring in the definition will be referred to as the "hub".

The degree of a node i in graph g is $\#\{j | ij \in g\}$. That is, the degree of a node equals the number of its neighbours.

Definition 5 *A k-regular graph is a graph where every node has degree k. A circle is the unique 2-regular graph.*

Definition 6 *A graph g is a line if there is a labelling of individuals $i_1, \ldots, i_N$ such that $g = \{i_k i_{k+1} | k = 1, \ldots, N-1\}$.*

3 Efficiency

In this section, we discuss the nature of efficient graphs under Assumptions 1-2, and under alternative specifications of the reliability function.

It will become clear in what follows that the efficient architecture will typically involve a trade-off between the graphs which maximise direct and

[6]Our current definition corresponds to the definition of pairwise stability used by Dutta and Mutuswami(1997). JW used the weaker definition because they wanted to prove that it is posisble that no efficient graph is pairwise stable. So, their weaker notion translated to a stronger impossibility result. In contrast, we want to show that efficient graphs *can* be stable, and so we use a stronger notion of stability.

indirect benefits. For instance, suppose f is strictly convex. [7] Then, each i maximises direct benefit by concentrating all her resource on just one link. But, as we point out below, this pattern of resource allocation may reduce aggregate indirect benefits. Of course, if $\phi(x) = x$ for all x, then direct benefits are maximised *whenever* all resources are utilised - the actual pattern of resource use does not matter. In this case, there are no trade-offs between direct and indirect benefits - the efficent graph is one which maximises indirect benefits. This enables us to get particularly sharp characterisations of efficient graphs in this case.

3.1 Efficiency under Product Reliability

Throughout this subsection, we assume that R^p is the reliability function. This is not mentioned explicitly in what follows.

Although we have defined a path as a sequence of nodes, it will be convenient to abuse notation, and say that the link kl belongs to a path $p(i, j)$ in g if k, l are successive nodes in the sequence constituting the path $p(i, j)$.

Notice that a star connecting everyone has two important properties. First, it is a tree, and so minimises the number of direct connections amongst all connected graphs. This increases the average strength of direct links. Also, the fewer the number of direct links, the greater is the scope for indirect benefits. Second, it also minimises the distance between all nodes which do not have a direct connection. Given the product form of the reliability function, the latter also increases the scope of higher indirect benefits. Thus, for all these reasons, a star should be the most desirable architecture. The following theorem verifies this intuition.

Theorem 1 *Suppose Assumption 1 holds. Then,*

(i) The unique efficient graph is a star with every peripheral node investing fully in the arc with the "hub".

(ii) Moreover, if $\phi(x) = x$ for all x, then each arc in the star has strength $X + \frac{X}{n-1}$.

Proof. : We prove the first part of the theorem in two steps. Consider any g which is feasible, connected[8] and not a star.

[7] The trade-off is particularly stark if $f(x_i^j, x_j^i)$ is a *strictly concave* function. In this case, direct benefits are maximised when the numbr of links are maximised- that is when the complete graph forms. But, there are no indirect benefits at the complete graph!

[8] It is easy to show that graphs which are not connected cannot be efficient.

Step 1: We construct a star $\hat{S}$ with higher aggregate utility than g.

Step 2: If $\hat{S}$ is infeasible, then we construct a feasible star which has higher aggregate utility than $\hat{S}$.

Proof of Step 1: Let g have $K \geq n-1$ links. For all $ij \in g$, let us denote $s_{ij} \equiv z_i$ where $i < j$. Without loss of generality, assume that

$$z_1 \geq z_2 \geq \ldots \geq z_K$$

We construct the star $\hat{S}$ with hub n as follows. For each $i \neq n$, define

$$\bar{x}_i^n = \min(\phi^{-1}(z_i), X), \bar{x}_n^i = \phi^{-1}(z_i) - \bar{x}_i^n$$

If g is a tree, then $K = n-1$. Let $\hat{x}_i^j = \bar{x}_i^j$ for all i, j, and $\hat{S} \equiv g(\hat{\mathbf{x}})$.

If g is not a tree, then $K > n-1$, and the investments involved in $\{z_n, \ldots, z_K\}$ have not been distributed.

Since $z_i \geq z_{i+1}$, $\bar{x}_i^n \geq \bar{x}_{i+1}^n$. If $\bar{x}_i^k = X$ for all $k = 1, \ldots, n-1$, then from convexity of ϕ, $\sum\limits_{i=1}^{n-1} \bar{x}_n^i < X$. In this case, let $\bar{x}_i^n = \hat{x}_i^n$ for all $i = 1, \ldots, n-1$, and $\hat{x}_n^1 = X - \sum\limits_{k=2}^{n-1} \bar{x}_n^k > \bar{x}_n^1$, and $\hat{x}_n^k = \bar{x}_n^k$ for $k = 2, \ldots, n-1$. Let $\hat{S} \equiv g(\hat{\mathbf{x}})$.[9]

Otherwise, let $\tilde{k} \geq 1$ be the smallest integer such that $\bar{x}_{\tilde{k}}^n < X$. Define $\hat{x}_i^n = \bar{x}_i^n = X$ for all $i < \tilde{k}$. Then, distribute $\{z_n, \ldots, z_K\}$ sequentially amongst $\{\tilde{k}, \ldots, n-1\}$ as follows:

$$\begin{aligned} \hat{x}_{\tilde{k}}^n &= \min(X, \bar{x}_{\tilde{k}}^n + \sum_{j=n}^{K} \phi^{-1}(z_j)) \\ \text{for } k > \tilde{k}, \hat{x}_k^n &= \min(X, \bar{x}_k^n + \sum_{j=n}^{K} \phi^{-1}(z_j) - \sum_{k'=\tilde{k}}^{k-1} (\hat{x}_{k'}^n - \bar{x}_{k'}^n)) \end{aligned}$$

Finally, if this procedure has not exhausted $\sum\limits_{j=n}^{K} \phi^{-1}(z_j)$, then distribute the excess sequentially to $\hat{x}_n^1, \hat{x}_n^2, \ldots$, while satisfying the constraint that none of them exceeds X. Again, define $\hat{S} \equiv g(\hat{\mathbf{x}})$.

Suppose g is a tree. Then, from convexity of ϕ, we know that

$$\phi(\hat{x}_i^n) + \phi(\hat{x}_n^i) \geq z_i \text{ for all } i = 1, \ldots, n-1 \quad (1)$$

[9]Notice that in this case $\hat{S}$ is a feasible graph since $\sum\limits_{j=1, j\neq i}^{n} x_i^j = X$ for all i.

This implies that direct benefits are at least as high in $\hat{S}$ as in g. A similar argument ensures that direct benefits are at least as high in $\hat{S}$ as in g even when the latter is not a tree.

Now, we check that indirect benefits are strictly higher in $\hat{S}$. We distinguish between two cases.

Case 1: g is a tree.

Let $D = \{ij | i \text{ and } j \text{ are not neighbours in } g\}$. For each pair i, j in D, let z_{k_i} and z_{k_j} denote the strengths of the *first* and *last* links in the path $p^*(i, j)$. Notice that while the choice of which link is first and which is last is arbitrary, the *pair* (z_{k_i}, z_{k_j}) is uniquely defined. Clearly, for all $i, j \in D$,

$$R^p(i, j) \leq z_{k_i} z_{k_j}$$

Moreover, since g is not a star, the inequality must be strict for some pair i, j. Hence, letting I denote the sum of indirect benefits in g and I' the corresponding sum in S, we have

$$I < 2 \sum_{i \neq j, i,j=1}^{n-1} z_i z_j \leq I'$$

where the last inequality follows from equation 1.

Case 2: g is not a tree.

Let g have $K > n-1$ links. Again, let D be the pairs which derive indirect benefits from each other in g, and z_{k_i}, z_{k_j} the strengths of the first and last links in $p^*(i, j)$.[10] Similarly, let I and I' denote the sum of indirect benefits in g and $\hat{S}$ respectively. Now, the number of pairs of agents deriving indirect benefits in g is strictly less than $\frac{(n-1)(n-2)}{2}$, which is the number of pairs deriving indirect benefits in $\hat{S}$. Recalling that $z_i \geq z_{i+1}$ for all $i = 1, \ldots, K$,

$$I < \sum_{i \neq j, i,j=1}^{n-1} z_i z_j \leq I'$$

This concludes the proof of Step 1.

Proof of Step 2: Notice that by construction, $\hat{x}_i^n \leq X$ for all $i = 1, \ldots, n-1$. So, either $\hat{S}$ is feasible, or $\sum \hat{x}_n^i > X$. We now show that if $\hat{S}$

[10]Since g is no longer a tree, the path $p^*(i, j)$ is not uniquely defined. The choice of best path is immaterial in what follows.

is not feasible, then we can construct a star S^* with hub n which has higher total utility compared to $\hat{S}$.

Although $\hat{S}$ is not feasible,

$$\sum_{i=1}^{n-1} \left(\hat{x}_i^n + \hat{x}_n^i\right) \leq nX \tag{2}$$

Also,

$$\hat{x}_{n-1}^n < X$$

This follows because $\hat{x}_{n-1}^n \leq \hat{x}_i^n$ for all $i < n-1$, and from equation 2.

Consider the star S where

(i) $x_i^n = \hat{x}_i^n$ for all $i \neq n-1$ and $x_n^i = \hat{x}_n^i$ for all $i \neq 1$.

(ii) $x_{n-1}^n = \min(X, \hat{x}_{n-1}^n + \hat{x}_n^1)$, and $x_n^1 = \hat{x}_{n-1}^n + \hat{x}_n^1 - x_{n-1}^n$.

Then,

$$\phi(x_{n-1}^n) + \phi(x_{n-1}^n) \geq \phi(\hat{x}_{n-1}^n) + \phi(\hat{x}_{n-1}^n) \tag{3}$$

with strict inequality holding if ϕ is strictly convex. So, the sum of direct benefits in S is at least as large as in $\hat{S}$.

We now show that the sum of indirect benefits in S is also at least as high as in $\hat{S}$. Let I and $\hat{I}$ represent the indirect benefits from S and $\hat{S}$ respectively. Since $\hat{x}_{n-1}^n < X$, $\hat{x}_n^{n-1} = 0$. Also, $\hat{x}_1^n = X$ since $\hat{x}_n^1 > 0$. Hence,

$$I - \hat{I} \geq \phi(X)[\phi(x_{n-1}^n) - \phi(\hat{x}_{n-1}^n)] + \phi(x_n^1)\phi(x_{n-1}^n) - \phi(\hat{x}_n^1)\phi(\hat{x}_{n-1}^n)$$

Now, if $x_{n-1}^n < X$, then $x_n^1 = 0$ and $\phi(x_{n-1}^n) \geq \phi(\hat{x}_{n-1}^n) + \phi(\hat{x}_n^1)$. Since $\phi(X) \geq \phi(\hat{x}_n^1)$, this implies that $I \geq \hat{I}$.

Suppose $x_{n-1}^n = X$. Then, $I \geq \hat{I}$ if

$$\phi(X)(\phi(X) + \phi(x_n^1)) \geq (\phi(X) + \phi(\hat{x}_n^1))\phi(\hat{x}_{n-1}^n) \tag{4}$$

Convexity of ϕ ensures that

$$\phi(X) + \phi(x_n^1) \geq \phi(\hat{x}_n^1) + \phi(\hat{x}_{n-1}^n)$$

Since $X \geq \hat{x}_n^1$, this ensures that equation 4 is satisfied.

Note that $\sum_{i=1}^{n-1}(x_n^i - \hat{x}_n^i) < 0$. If S is not feasible, clearly we can proceed in this way by increasing x_{n-2}^n, x_{n-3}^n, etc. until a feasible star is obtained.

This completes the proof of Step 2, as well as the first part of the theorem.

(ii) Suppose $\phi(x) = x$ for all x. Let S^* be the star with hub at n, where all arcs have strength $X + \frac{X}{n-1}$. Let S be any other star with hub n where s_{in} may not be equal to s_{jn} for $i \neq j$, but where $\sum_{i=1}^{n-1}(x_i^n + x_n^i) = nX$. It is straightforward to check that total direct benefits are maximised at both S^* and S. We now show that the sum of *indirect* benefits in S^* is greater than that in S.

Without loss of generality let s_{1n} and s_{2n} denote the weakest and strongest links in S. Consider the effect of increasing investment on s_{1n} by ε and simultaneously decreasing investment on s_{2n} by ε.

The effect on the overall value can be computed as

$$\Delta V = 2[\varepsilon(s_{2n} - s_{1n}) - \varepsilon^2]$$

Hence, for ε small enough, $\Delta V > 0$ and so local changes in the direction of equalization are profitable. But this implies that the symmetric star has higher value than the asymmetric star. ■

Remark 1 *Except for the case when investments of* i *and* j *are perfect substitutes, the theorem above does not specify how the agent at the hub allocates her resource* X*. Typically, this will depend on the degree of convexity of* ϕ*. For instance, consider the case where* $\phi(x) = \alpha x + (1-\alpha)x^2$*, for some* $\alpha \in [0,1]$*. From the second part of the theorem, the agent at the hub shold distribute* X *equally across all the nodes when* $\alpha = 1$ *at the efficient graph. This should also hold when* α *is "close" to* 1*. However, when* $\alpha = 0$ *or is sufficiently close to* 0*, the agent should "specialise" completely and invest all of* X *on one of the links.*

The assumption that ϕ is convex plays a crucial role in the proof of the theorem. Convexity ensures that *direct* benefits resulting from the investments of the peripheral nodes are maximised when they specialise completely. The star architecture is compatible with this specialised investment. The next examples illustrates the importance of convexity of ϕ in deriving the star as the unique efficient graph.

Example 1 *Let* $n = 3, X = 1$*, and* $f(x_i^j, x_j^i) = \frac{\sqrt{x_i^j + x_j^i}}{k}$*, where* $k > 0$*. Consider the symmetric complete graph* g^c*. The strength of each link is* $\frac{1}{k}$*,* *and so the total direct benefit is $\frac{6}{k}$. The symmetric star maximises total benefits amongst stars. In the symmetric star, the strength of each link is $\frac{1}{k}\sqrt{\frac{3}{2}}$. So, the total direct benefit is $\frac{4}{k}\sqrt{\frac{3}{2}}$, while the indirect benefit is $\frac{3}{k^2}$. So, g^c will be the unique efficient graph if*

$$\frac{6}{k} > \frac{4}{k}\sqrt{\frac{3}{2}} + \frac{3}{k^2}$$

One can choose a suitably large value of k to satisfy this inequality.

In this example, the two contributions are still perfect substitutes, but strength of the links is a strictly concave function of joint contributions. Strict concavity ensures that direct benefits are a strictly increasing function of the number of links, provided each link is of equal strength. Hence, the symmetric complete graph is the unique maximiser of direct benefits. The greater the degree of concavity of f, the more attractive is the complete graph, although it generates no indirect benefits.

The next example allows for *some* substitutability between x_i^j and x_j^i in building the strength of the link ij. Again, the star cannot be efficient.

Example 2 *Let $X = 1, f(x_i^j, x_j^i) = x_i^j x_j^i$. Then, the symmetric star generates a total direct benefit of 2, and indirect benefits of $(n-1)(n-2)(\frac{1}{n-1})^2 = \frac{n-2}{n-1}$. Hence, total value is less than 3. The symmetric cycle generates direct benefits of $\frac{n}{2}$. So, for $n > 5$, the cycle dominates the star.*

We now show that when "inputs" are perfect complements, no *tree* can be efficient.

Theorem 2 *Suppose Assumption 2 holds. If g is efficient, it cannot have any component with three or more nodes which is a tree.*

Proof. Suppose g is efficient and has a component with three or more nodes, where two nodes have degree one. Denote these nodes by i and j and their immediate predecessors by k and l respectively. Because the component is connected, the degrees of k and l are necessarily greater than one. But this implies that $x_k^i < X$ and $x_l^j < X$. Furthermore because $\sum_{m \in N \setminus \{i\}} x_k^m \leq X, x_k^m \leq X - x_k^i$ for all node $m \neq i$ to which k is connected. Now, this implies that the value of the indirect connection between i and j in the graph is strictly smaller than $\min\{X - x_k^i, X - x_l^j\}$). Furthermore, in an efficient graph, $x_i^k = x_k^i$ and $x_j^l = x_l^j$ so that individual i can invest $X - x_k^i$ in the direct link with j and individual j can invest $X - x_l^j$ in the link with i. But, because the value of the indirect link is smaller than $\min\{X - x_k^i, X - x_l^j\}$, the investment in the direct link strictly increases the value of the graph, yielding a contradiction.

The proof of the theorem is completed with the observation that in a tree, at least two nodes have degree one. ■

Unfortunately, we have been unable to characterize completely efficient networks when inputs are perfect complements. For low values of n, we can show that the circle is the unique efficient network structure.

Proposition 1 *Suppose Assumption 2 holds. For* $3 \leq n \leq 7$*, the circle where every link has value* $X/2$ *is the unique efficient network.*

Proof. For $n = 3$, the circle is the only connected graph which is not a tree. Now, notice that direct benefits are equal to nX and hence are maximal in the circle. For $n = 4, 5$, we show that the circle also maximizes the value of indirect benefits. Notice first that the value of an indirect connection is always bounded above by $(X/2)^2$ as the middle player must allocate X over at least two links. For $n = 4$ and $n = 5$ all indirect connections in the circle are of length 2 and have value $(X/2)^2$. Hence, the circle achieves the highest sum of indirect links and is efficient. It is easy to check that any other allocation of investments results in a lower value of indirect links, so the circle with links of equal strength is uniquely efficient.

Suppose now that $n = 6, 7$. The indirect benefit for any node in the cirle is

$$I = \frac{X^2}{2} + \frac{X^3}{4}$$

Consider any other graph g. If this graph is to "dominate" the cycle, then at least one node (say i^*) has to derive an indirect benefit exceeding I. For each k, check that the circle maximises indirect benefits from nodes at a distance of k. So, if i is to derive a larger indirect benefit in g, it must have more than two nodes at a distance of 2.[11]

[11]Since $n \leq 7$, the maximum distance between any two nodes in the circle is 3.

It is tedious to show that the maximum indirect benefit that i^* can derive occurs when i^* has two neighbours, j_1, j_2, with each neighbour of i^* having three neighbours including i^* itself. Moreover, the optimum pattern of allocation from the point of view of i^* is

$$x_i^{j_1} = x_i^{j_2} = x_{j_2}^i = x_{j_2}^i = \frac{1}{2}$$

This yields i^* a total indirect benefit of $\frac{X^2}{2} < I$. ■

For larger numbers of players, the above argument does not hold. The circle may be dominated by denser graphs, where the number of indirect connections is lower but the distance of indirect connections is lower as well. The following example shows that the circle can indeed be dominated by another network architecture (the "Peterson graph")[12] for $n = 10$.

Example 3 *Let* $n = 10$*. The circle may be dominated by the Peterson graph.*

The Peterson graph is a regular graph of degree 3 such that, for any node i, and any neighbors j and k of i, the set of direct neighbors of j and k (other than i) is disjoint. Consider the Peterson graph where every link has value $X/3$. The direct benefits are maximized, and each player has 6 indirect connections of length 2 and value $(X/3)^2$. Hence, the value of indirect connections for any player is $6X^2/9 = 2X^2/3$. In the circle, each player has 2 indirect connections of length 2, 2 indirect connections of length 3, 2 indirect connections of length 4 and 1 indirect connection of length 5, so the total value of indirect connections is given by

$$IC = \frac{X^2}{2} + \frac{X^3}{4} + \frac{X^4}{8} + \frac{X^5}{32}.$$

For small values of X, it is obvious that

$$IC < \frac{2X^2}{3}$$

so that the Peterson graph dominates the circle.

[12]See Holton and Sheehan(1993).

3.2 Efficiency under Min-Reliability

In this section, we explore the nature of efficient graphs when the reliability of a path is measured by the strength of its weakest link. First, we show that under Assumption 1, an efficient graph efficient graph must be a tree. Moreover, if $\phi(x) = x$ for all x, then all trees with equal strengths on all links are efficient.

These follow easily as a consequence of the following lemmas.

Lemma 1 *If $\phi(x) = x$ for all x, then no graph with cycles can be efficient.*

Proof. Consider a graph g with a cycle. Without loss of generality, let s_{12} be a link of minimal strength in the cycle. Since s_{12} has minimal strength, the indirect benefit that 1 and 2 can derive from each other is at least as high as the direct benefit.[13] So, deletion of the link ij does not reduce either direct or indiect benefits. On the other hand, x_1^2 and x_2^1 can be diverted to to other links in the cycle so as to increase *direct* benefits. Hence, a graph with a cycle cannot be efficient. ■

Lemma 2 *Suppose $\phi(x) = x$ for all x. Then, any tree g with links of unequal strength is dominated by the tree g with links of equal strength.*

Proof. Let g be a tree with links of unequal strength. Let g contain m links of minimal strength and M links of maximal strength. Let ij be a link of minimal strength. Let K be the set of links with maximal strength. Let g' be one of the subtrees formed by links in K, and choose some kl which is a terminal link of the subtree.

Now, consider the effect of transferring an investment of $\varepsilon > 0$ from kl to ij. This has no effect on aggregate direct benefits.

If ε is chosen sufficiently small, then the increase in strength of the link ij will increase the value of indirect connections by at least $(n - 1 - m)\varepsilon$ since ij must be the link of minimal strength in at least so many paths. On the other hand, the reduction in strength of the link kl must decrease the value of the indirect connections in at most $k - 2$ comparisons, so that the total decrease is $(k - 2)\varepsilon$. But, $n - 1 - m > M - 2$ since $M + m \leq n - 1$. Hence, there is a net increase in the overall value of indirect benefits. ■

From these two lemmas we conclude the following.

[13] Note that with min-reliability, the length of the indirect path does not matter.

Theorem 3 *If Assumption 1 holds, then an efficient graph must be a tree. Moreover, if $\phi(x) = x$ for all x, then the set of efficient graphs is the set of trees with equal strength on every link.*

Remark 2 *For any tree, it is always possible to construct links of equal strength, while satisfying individual budget constraints.Pick any node x_O as the root of the tree. Define then a binary predecessor relation corresponding to this root : $i \prec j$ if and only if $i \in P(j, x_0)$ where $P(j, x_0)$ is the unique path from j to x_0. One may also define the imediate predecessor of a node j as the node i such that $i \prec j$ and if $k \prec j$ and $k \neq i$ then $k \prec i$. Given that the network is a tree, this unique immediate predecessor is well defined. Now, to any node x in the tree, attach an integer $\kappa(x)$ which corresponds to the number of nodes which have x as a predecessor, i.e., $\kappa(x) = \#\{i, x \prec i\}$. Clearly, if x is a leaf of the tree, $\kappa(x) = 0$ and for the root, $\kappa(x_0) = n-1$. For any node x, let $I(x) = \{y_1, ..., y_m, ..., y_M\}$ denote the set of nodes which admit x as an immediate predecessor. Finally, consider the following allocation of resources for node x: invest $X(\kappa(y_m) + 1)/(n-1)$ on each point $y_m \in I(x)$ and invest the remainder, $X(n-1-\kappa(x))/(n-1)$ on the relation with the unique immediate predecessor of x. Clearly, this allocation of resources satisfies individual budget balance. Now, consider any link between x and y and assume without loss of generality that x is the immediate predecessor of y. Then the value of the link is $X(\kappa(y)+1+n-1-\kappa(y))/(n-1) = nX/(n-1)$ which is independent of x and y. Hence, this allocation of resources results in all links having equal strength.*

The nature of efficient architectures is very different under Assumption 2.

Theorem 4 *If Assumption 2 holds, the efficient graphs are the symmetric line and the cycle.*

Proof. Consider any node i, and suppose that j is a neighbour of i. The maximum indirect benefit that i can get from any node using a path involving ij is $\min(x_i^j, X - x_i^j)$, since $x_j^i + x_j^k \leq X$ for all $k \neq i, j$. Hence, for any node, the maximal indirect benefit from any other node is $\frac{X}{2}$, which is obtained by equalizing the value of every link at $X/2$. Any other architecture must involve some link of value smaller than $X/2$ and hence decrease the value.

Note that the symmetric cycle yields every node a total benefit of $(n-1)\frac{X}{2}$. Hence, these must be efficient architectures.

Clearly, no node with degree greater than 2 can attain this value. Also, no disconnected graph can be efficient. Hence, the line and cycle must be the *only* efficient graphs. ■

4 Stability

In this section, we discuss whether efficient graphs are stable. Again, we distinguish between Product Reliability and Min-Reliability.

4.1 Stability under Product Reliability

Is there any conflict between efficiency and (strong) pairwise stability?

Consider first assumption 1. Suppose that $\phi(x) = \alpha x + (1-\alpha)x^2$, with α "close" to 1. That is, ϕ is strictly convex. From what we have said in 1, the efficient graph is a star where the hub *does not* specialise completely in one of the links. But, then such a graph cannot be *Nash stable* because the agent at the hub is strictly better off by investing X in just one of the links - the hub does not derive any indirect benefits and so is better off by maximising her own direct benefit.

Now, consider the case when $\phi(x) = x$ for all x. Suppose $n = 3$. In the symmetric star, the hub (say individual 1) derives benefits of $3X$ and will never want to deviate from the star. Consider peripheral individuals 2 and 3. It is easy to check that the best joint deviation is for both 2 and 3 to set $x_2^3 = x_3^2 = X$. Then, 2 and 3 each get additional *direct* benefits of X. Howeevr, they lose indirect benefits from each other. This loss equals $\frac{9}{4}X^2$. Hence, the star is stable iff $X \geq \frac{4}{9}$.

Suppose $n > 3$. Again, let 1 be the hub. As before, individual 1 has no profitable deviation. Consider 2 and 3, and potentially mutually profitable deviations where $x_2^3 = \delta_2, x_2^1 = X - \delta_2$ and $x_3^2 = \delta_3, x_3^1 = X - \delta_3$, while other investments are in the symmetric star. Without loss of generality, let $\delta_2 \leq \delta_3$.

Individual 3 gains an additional direct benefit of δ_2 from 2, but loses the indirect benefit of $(X + \frac{X}{n-1})^2$ from 2. In addition, 2 also suffers a loss in indirect benefit from each of the other peripheral nodes. The exact loss depends on whether $p^*(3,i)$ for $i > 3$ includes 31 or $(32, 21)$.[14] Hence, the loss in indirect benefot from each of the othetr nodes $3, \ldots, n$ is at least $\delta_2(X + \frac{X}{n-1})$. So, the deviation is profitable if

$$\delta_2 > \delta_2(n-3)(X + \frac{X}{n-1}) + (X + \frac{X}{n-1})^2$$

If this inequality holds for some $\delta_2 < X$, it must also hold for $\delta_2 = X$. Substituting this value of δ_2, and simplifying, we get

$$X < \frac{(n-1)^2}{n(n^2-3n+3)}$$

Putting these observations together, we get the following.

Theorem 5 *(i) There exists f satisfying Assumption 1 such that no efficient graph is Nash stable.*

(ii) Suppose $\phi(x) = x$ for all x. Then, the symmetric star, which is the unique efficient graph, is strongly pairwise stable iff $X \geq \frac{(n-1)^2}{n(n^2-3n+3)}$.

Remark 3 : *When $n \geq 3$, $\frac{(n-1)^2}{n(n^2-3n+3)}$ is decreasing in n. Hence, as n increases, the symmetric star is strongly pairwise stable for smaller values of X. There is an obvious explanation for this. If two nodes divert investment from their links with the hub, they lose indirect benefits from other peripheral nodes. This loss increases with the number of peripheral nodes.*

The symmetric cycle with $s_{i,i+1} = \frac{X}{2}$ for all i is *not*strongly pairwise stable under Assumption 1. For suppose i and j are not neighbours in the cycle. Let the divert all their investment to the link ij. Then, both i and j gain a direct benefit of X. The loss in indirect benefit is bounded above by

$$\begin{aligned} I &= 2\left[\frac{X^2}{2} + \frac{X^3}{2} + \ldots + \frac{X^{\frac{n-1}{2}}}{2}\right] \\ &= X\left[\frac{X(1 - X^{\frac{n-3}{2}})}{1 - X}\right] \\ &< X \end{aligned}$$

[14] The latter is a possibility since we have assumed $\delta_3 \geq \delta_2$.

However, the cycle is strongly pairwise stable under Assumption 2. Since the cycle is also efficient for small values of n, this shows that efficiency can be reconciled with stability for small groups.

Theorem 6 *Let Assumption 2 hold. Then, the symmetric cycle is both Nash stable and strongly pairwise stable.*

Proof. : It is straightforward to check that the symmetric cycle is Nash stable. We just show that the symmetric cycle is strongly pairwise stable.

In the symmetric cycle, each i gets a direct benefit of X. No pattern of investment can result in higher direct benefits. So, we check whether a deviation by i and j can improve their indirect benefits.

Suppose i and j are neighbours in the cycle. Consider the effect on i of increasing investment to $\frac{X}{2}+y$ by both i and j on the link ij, and decreasing their investments on their other neighbours by y. The change in indirect benefit for i from j's other neighbour is $(\frac{X}{2}+y)(\frac{X}{2}-y)-(\frac{X}{2})^2 < 0$. A similar calculation shows that i also loses from nodes which are further away.

Suppose i and j ae not neighbours in the cycle. Let i and j mutually invest y each on the link ij and simultaneously decrease investment on their previous neighbours by $\frac{y}{2}$. It is easy to check that this is the best possible deviation.

Clearly, this can only increase indirect benefit for i if there is some k such that the distance between i and k is now lower. This means that i accesses k through j. Let k be a neighbour of j. Then, the indirect benefit for i from k is

$$I = y(\frac{X-y}{2}) = \frac{Xy}{2} - \frac{y^2}{2}$$

Now, i has reduced the strength of links with each of its previous neighbours by $\frac{y}{2}$. Also, since k is not at a distance of 2 from i in the cycle, there must be some node m, distinct from k which is at a distance of 2 from i. The loss in indirect beenfit for i from m is

$$I' = (\frac{X-y}{2})\frac{X}{2} - \frac{X^2}{4} = \frac{Xy}{2}$$

Hence, the indirect beenfit for i from k is lower than the loss in indirect benefit from m.

Repeating this argument, it can be shown that i's total indirect benefit will actually go down as a result of the deviation. ∎

4.2 Stability under Min-Reliability

We now examine the stability of efficient networks when the reliability function is R^m.

Theorem 7 *Suppose $\phi(x) = x$ for all x. Then, any tree with equal strength on all links is strongly pairwise stable.*

Proof. Every link in a symmetric tree has strength $\frac{n}{n-1}X$. Since distance between nodes does not matter under R^m, every node derives a benefit of $\frac{n}{n-1}X$ from every other node. It is easy to check that no deviation by a pair can improve both individuals' payoffs. ■

Remark 4 *In fact, it can be shown that no coalition has a profitable deviation from the symmetric tree.*

We showed earlier that under Assumption 2, the symmetric line and cycle were efficient. Again, because distances do not matter under R^m, each node derives a benefit of $\frac{X}{2}$ from every other node. It is trivial to check that no deviation by a pair (or coalition) can improve mutual payoffs. Hence, we have the following theorem.

Theorem 8 *Suppose Assumption 2 holds. Then, the symmetric line and the cycle are strongly pairwise stable.*

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