On the Gains to International Trade in Risky Financial Assets *

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Abstract

We develop a dynamic analysis of international trade in risky financial assets under incomplete markets. We construct optimal portfolios, compute the benefits of expanded portfolio menus, express the equity premium puzzle in welfare terms and quantify the gains to international trade in risky assets. Our empirical implementation for 14 countries finds implausibly large gains to trade in domestic equity. This gains-to-trade puzzle is related to, but distinct from, the equity premium puzzle. It is also distinct from the home bias puzzle, because the gains do not rest on international diversification by domestic investors.

1 Introduction

This paper develops a dynamic equilibrium framework for quantifying the gains to international trade in risky financial assets. The framework handles many agents and countries, many assets, incomplete markets, and differences among agents in their exposures to asset-specific and country-specific sources of risk. We use the framework to construct optimal portfolios, express the equity premium puzzle in welfare terms and derive several measures for the gains to trade in risky assets. These measures differ in terms of data requirements and maintained assumptions about the underlying trade regime, but all of them can be implemented using national data on output and the returns on equity or other risky assets.

Our empirical implementation finds enormous gains to trade in domestic equity – equivalent to roughly 30 percent of domestic consumption for the average country in our baseline case. These gains do not rest on and do not require international portfolio diversification by domestic investors. Because the gains are implausibly large, we regard them as a puzzle for the theory. The theoretical elements that give rise to the puzzle are standard ingredients in equilibrium models of economic fluctuations and asset pricing behavior.

The gains-to-trade puzzle, as we call it, is related to the equity premium puzzle identified by Mehra and Prescott (1985) and studied by legions of researchers.[1] In particular, for plausible risk aversion levels, the huge gains to trade mainly reflect the impact of trade on the equity risk premium. However, another portion of the gains reflects the benefits of international risk sharing. The first source of the gains to trade declines with the risk aversion of domestic investors, but the risk-sharing component rises with risk aversion. We show that the gains-to-trade puzzle is not resolved by appealing to high risk aversion levels that rationalize observed equity returns. Hence, the two puzzles are related but distinct. The gains-to-trade puzzle is also distinct from "home bias" in portfolio holdings, because the large gains to trade in domestic equity do not require international diversification by domestic investors.

For at least thirty years, economists have recognized that the proportion of foreign assets held by domestic investors is much smaller than prescribed by portfolio theory. Grubel (1968) and Levy and Sarnat (1970) were among the first to identify this puzzle and to show that international diversification affords a large reduction in portfolio risk for a given level of expected returns. This home bias puzzle has been confirmed, extended and reformulated many times. Baxter and Jermann (1997), for example,

[1] Standard dynamic equilibrium models, when calibrated to a reasonable degree of risk aversion, imply an expected return premium on equity securities that is much smaller than the equity premium observed in the data – hence, the equity premium puzzle.

point out that the puzzle deepens if nonmarketable human capital is more highly correlated with domestic equity than foreign equity. Lewis (2000) finds that the gains to international diversification of equity holdings are very large in terms of equivalent consumption variations (10-50 percent of consumption for plausible preferences.)

We also find enormous gains from adding foreign equity to risky asset portfolios for investors with plausible degrees of risk aversion. In addition, we show that the theoretically optimal level of risky asset holdings is an order of magnitude larger than observed holdings. In a welfare sense, the most important portfolio puzzle is the low level of risky asset holdings, not the lack of portfolio diversification emphasized by the home bias literature.

Our work is also related to, and partly motivated by, the large body of evidence against the hypothesis of international risk sharing.[2] This evidence has inspired many efforts to quantify the gains to full international risk sharing relative to autarky or observed consumption allocations. Van Wincoop (1999) reviews fourteen recent studies along these lines. Given reasonable values for risk aversion and the risk-free interest rate, studies that allow for a nonstationary output process find sizable unrealized gains from international risk sharing.[3] Relative risk aversion near 3 implies gains on the order of 1-2 percent of GDP per year for rich countries and 5-7 percent per year for middle-income countries. These are nontrivial welfare effects, but they are dwarfed by the ones we find.

Why do we find much larger gains to trade than earlier studies, even though our theoretical benchmark involves limited rather than full international risk sharing? Unlike most earlier work, which fits the theory to data on output or consumption, we fit the theory to data on output *and* equity returns. In this way, we effectively require that (a) the per investor quantity of risk is much higher than it seems – because of limited market participation, for example – or (b) effective risk aversion is much higher than one might think – because of habit formation, for example. Previous research on the consumption-equivalent gains to international risk sharing relies entirely on consumption or output data and thereby skirts the tensions that

[2] Lewis (1999) provides an excellent synthesis of work in this general area, which includes research on the home bias puzzle. While the evidence clearly rejects the hypothesis of full risk-sharing among countries, it is less hostile to the view that high-frequency movements in national consumption are consistent with unrestricted trade in risk-free bonds. See, for example, Obstfeld (1989), Kollman (1995) and Canova and Ravn (1996).

[3] Van Wincoop draws on evidence about long term growth behavior in DeLong (1988) to argue that a random walk or first-order autoregression in growth rates is a more appropriate specification than a stationary process or one that imposes cointegration between national and world outputs. See Athanasoulis and Van Wincoop (2000) for a careful treatment of issues related to the specification of output processes.

arise when combining data on output (or consumption) with asset returns.

Lewis (2000) also finds much larger gains to trade for welfare expressions that make use of asset market data. Her analysis is related to ours, but it differs in several key respects, including the following. First, she measures the gains to full international risk sharing, whereas we explicitly treat market incompleteness and the consequent limitations on risk-sharing opportunities. Second, we show that the gains-to-trade puzzle is not resolved by choosing risk aversion to rationalize observed equity returns. Third, we show that the theory implies large gains to trade even when domestic investors do not diversify their portfolios. Fourth, we emphasize that the home bias puzzle is only one aspect of a larger portfolio puzzle.

The main points of our theoretical analysis are most easily seen in a two-period model with a single risky asset, which we take up in Section 2. The extension to multiple risky assets, also treated in Section 2, is straightforward. Section 3 develops measures for the gains to trade in the two-period setting, and Section 4 extends the theory and welfare analysis to an infinite-horizon setting. Section 5 describes the data and characterizes the behavior of output growth and equity returns for fourteen countries. Using the empirical characterization, section 6 constructs country-level measures of exposure to domestic and world equity returns, which serve as key inputs into the portfolio and welfare calculations. Section 7 calculates optimal equity positions for a representative investor in each country and provides a welfare measure of the gap between theoretically optimal and observed equity holdings. Section 8 documents the equilibrium gains-to-trade puzzle and discusses its relationship to the equity premium puzzle. Section 9 offers some concluding remarks.

2 A Two-Period Model

We begin with a two-period model of international trade in risky assets with incomplete markets. Previous theoretical work on international trade in risky assets with incomplete markets includes Helpman and Razin (1978), Svensson (1988) and Persson and Svensson (1989). More recently, Athanasoulis and Shiller (2000) devise new securities that expand international risk-sharing opportunities in a setting with incomplete markets. All of these studies treat market incompleteness as exogenous, as do we. Martin and Rey (2001) consider international trade in risky assets in a model that endogenizes market incompleteness.

2.1 The Individual's Problem

An investor h receives current income, y_0^h, and uncertain future income, $\tilde{y}_1^h$. Claims to this income stream, say labor earnings, cannot be bought and sold. However, the investor can take long or short positions in a risk-free asset with (gross) rate of return, R_f, and in a risky asset with rate of return, $\tilde{R}$. Risky asset returns and future labor income are distributed jointly normal.

The investor's period budget constraints are

$$c_0^h = y_0^h - \omega^h - \omega_f^h, \quad \text{and} \tag{1}$$

$$\tilde{c}_1^h = \tilde{y}_1^h + \omega_f^h R_f + (e^h + \omega^h)\tilde{R}, \tag{2}$$

where c_0^h and $\tilde{c}_1^h$ denote current and future consumption, e^h is an initial endowment of the risky asset, and ω_f^h and ω^h denote asset purchases. Note that ω^h is the investor's position in the risky asset net of his initial endowment, while $(e^h + \omega^h)$ is his gross position. By (2), the joint normality of $\tilde{y}_1$ and $\tilde{R}$ implies joint normality of $\tilde{c}_1$ and $\tilde{R}$.

Subject to (1) and (2), the investor chooses consumption and asset holdings at time 0 to maximize

$$U^h\left(c_0^h, \tilde{c}_1^h\right) \equiv \left(-1/A^h\right)\left[\exp(-A^h c_0^h) + \delta^h \operatorname{E}\exp(-A^h \tilde{c}_1^h)\right], \tag{3}$$

where $\delta^h \in (0,1)$ is the time discount factor, $A^h > 0$ is the investor's level of absolute risk aversion, and E indicates an expectation taken with respect to information at time 0. The Euler equations for this problem imply, using Stein's Lemma,[4] that consumption and asset returns satisfy

$$A^h \operatorname{cov}\left(\tilde{c}_1^h, \tilde{R}\right) = \operatorname{E}\left(\tilde{R}\right) - R_f \equiv ER. \tag{4}$$

2.2 Endowed and Desired Exposure

Consider a linear projection of future consumption on the risky asset return,

$$\tilde{c}_1^h = \text{constant} + \beta_c^h \tilde{R} + \tilde{\varepsilon}_c^h,$$

[4] Let $\tilde{x}$ and $\tilde{y}$ be jointly normal and $\mathrm{g}(\cdot)$ be twice differentiable. Stein's Lemma says

$$\operatorname{cov}\left(\mathrm{g}\left(\tilde{x}\right), \tilde{y}\right) = \operatorname{E}\left(\mathrm{g}'\left(\tilde{x}\right)\right)\operatorname{cov}\left(\tilde{x}, \tilde{y}\right).$$

Rubinstein (1974) provides a proof.

where β_c^h measures the sensitivity of consumption to the risky asset return, and $\tilde{\varepsilon}_c^h$ is uncorrelated with $\tilde{R}$. [5] Under the optimality condition 4,

$$\beta_c^h = \frac{ER}{A^h \operatorname{var}\left(\tilde{R}\right)}. \tag{5}$$

That is, at an optimum, the consumption sensitivity to the risky asset equals the excess return divided by the product of the return variance and the investor's absolute risk aversion. We call β_c^h in (5) the investor's "desired exposure" to the risky asset, because it reflects his optimal consumption sensitivity to the return on the risky asset. For the analysis that follows, it is important to recognize that this desired exposure is measured in units of the consumption good. For example, if $\tilde{y}_1^h$ is uncorrelated with $\tilde{R}$, then setting $\omega^h = \beta_c^h - e^h$ gives the optimal net investment in the risky asset.

Now consider the projection of the investor's future income on the risky return,

$$\tilde{y}_1^h = \text{constant} + \beta_y^h \tilde{R} + \tilde{\varepsilon}_y^h,$$

where

$$\beta_y^h = \frac{\operatorname{cov}\left(\tilde{y}_1^h, \tilde{R}\right)}{\operatorname{var}\left(\tilde{R}\right)}$$

measures the sensitivity of future income to the risky asset return. We call $\beta_y^h + e^h$ the investor's "endowed exposure" to the risky asset, because it describes how his future consumption covaries with $\tilde{R}$ when $\omega^h = 0$; i.e., when he makes no additional investment or disinvesment in the risky asset beyond his initial endowment. We will also refer to β_y^h as the implicit component of the investor's endowed exposure to the risky asset, or simply his implicit exposure.

2.3 Risky Asset Demand

The concepts of desired and endowed exposure yield a simple and intuitive expression for the investor's risky asset demand. To see this point, project both sides of equation (2) on $\tilde{R}$ and make use of (5) to obtain

$$\boxed{\omega^h = \underbrace{\frac{ER}{A^h \operatorname{var}\left(\tilde{R}\right)}}_{\substack{\text{Desired}\\ \text{Exposure}}} - \underbrace{\left(e^h + \beta_y^h\right)}_{\substack{\text{Endowed}\\ \text{Exposure}}}} \quad \text{or} \quad \boxed{\underbrace{\omega^h + e^h}_{\substack{\text{Gross Demand}\\ \text{for Risky Asset}}} = \underbrace{\frac{ER}{A^h \operatorname{var}\left(\tilde{R}\right)}}_{\substack{\text{Desired}\\ \text{Exposure}}} - \underbrace{\beta_y^h}_{\substack{\text{Implicit}\\ \text{Endowed}\\ \text{Exposure}}}} \tag{6}$$

[5]The use of linear projections is without loss of generality in view of the joint normality of $\tilde{c}_1^h$ and $\tilde{R}$. See Whittle (1983), page 10.

Equation (6) states that the investor's optimal holding of the risky asset is the difference between his desired and endowed exposures. Desired exposure rises with the excess return and declines with the investor's risk aversion, while his implicit endowed exposure is simply the slope coefficient in a regression of income on asset returns.

2.4 Asset Pricing

Now suppose that the economy contains H agents, indexed by h, who maximize the utility function (3) subject to (1) and (2). Agents differ in terms of absolute risk aversion (A^h), the covariance of income with asset returns (β_y^h), and initial endowments of the risky asset (e^h). For simplicity, assume that the risk-free asset is in perfectly elastic supply at rate of return, R_f. This assumption is not essential for our analysis, but it simplifies the determination of asset market equilibrium and the welfare calculations related to international risk sharing. This assumption also serves to isolate the incremental gains to international trade in risky assets over and above the consumption-smoothing benefits of international borrowing and lending. See Willen (1999) for a closely related analysis that endogenizes the risk-free rate of return.

Asset pricing now follows directly by equating aggregate demand and supply in the risky asset. That is, sum the risky asset demands (6) over the H investors, and make use of the market-clearing requirement of zero excess demand for the risky asset to obtain

$$\boxed{\underbrace{ER}_{\substack{\text{Excess}\\\text{Return}}} = A^* \operatorname{var}\left(\tilde{R}\right) \quad \underbrace{\left(\frac{e}{H} + \beta_y\right)}_{\substack{\textit{Per Capita}\\\text{Endowed Exposure}}},} \tag{7}$$

where $e = \sum_h e^h$ is the aggregate supply of the risky asset, $\beta_y = H^{-1} \sum_h \beta_y^h$ is the per capita implicit exposure to the risky asset, and $A^* = \left[(1/H) \sum_h (1/A^h)\right]^{-1}$ is the harmonic mean of absolute risk aversion. This equation says that the excess return (i.e., the equity premium) is proportional to absolute risk aversion, the variance of the risky asset return and the *per capita* endowed exposure to the risky asset.

2.5 Consumption

To calculate the welfare effects of trade in risky financial assets, we must first determine how the opportunity to invest in risky assets affects consumption. To do so, write the consumption Euler equation with respect to the riskless asset,

$$\exp\left(-A^h c_0^h\right) = \delta^h R_f \operatorname{E} \exp\left(-A^h \tilde{c}_1^h\right), \tag{8}$$

and take logs to obtain

$$\mathrm{E}\left(\tilde{c}_1^h\right) = c_0^h + \frac{A^h}{2}\,\mathrm{var}\left(\tilde{c}_1^h\right) + \frac{1}{A^h}\ln(R_f\delta^h), \tag{9}$$

since $\mathrm{E}\left[\exp\left(-A^h\tilde{c}_1^h\right)\right] = \exp\left[\mathrm{E}\left(-A^h\tilde{c}_1^h\right) - (1/2)\,\mathrm{var}\left(A^h\tilde{c}_1^h\right)\right]$. Next, use (1) and (2) to derive the intertemporal budget constraint,

$$c_0^h + \frac{1}{R_f}\,\mathrm{E}\left(\tilde{c}_1^h\right) = y_0^h + \frac{1}{R_f}\,\mathrm{E}\left(\tilde{y}_1^h\right) + e^h + \frac{1}{R_f}(\omega^h + e^h)ER, \tag{10}$$

and substitute for $\mathrm{E}\left(\tilde{c}_1^h\right)$ in (10) from (9) to obtain

$$c_0^h = \frac{R_f}{1+R_f}\left[\underbrace{y_0^h + \frac{\mathrm{E}\left(\tilde{y}_1^h\right)}{R_f} + e^h + \frac{(\omega^h + e^h)ER}{R_f}}_{\text{Wealth}} \quad \underbrace{-\frac{A^h}{2R_f}\,\mathrm{var}\left(\tilde{c}_1^h\right)}_{\substack{\text{Precautionary}\\ \text{Term}}} \quad -\frac{\ln\left(R_f\delta^h\right)}{A^hR_f}\right]. \tag{11}$$

Thus, consumption is proportional to wealth, duly adjusted for precautionary behavior and any difference between the subjective rate of time preference and the risk-free rate of interest. Observe that wealth includes the discounted value of excess returns on the risky asset. Since the last term in (11) plays no role in the welfare calculations below, we henceforth set $\ln\left(R_f\delta^h\right)$ to zero to simplify the expressions.

Now rewrite the precautionary term using the budget constraint (2) to obtain

$$c_0^h = \frac{R_f}{1+R_f}\left[y_0^h + \frac{\mathrm{E}\left(\tilde{y}_1^h\right)}{R_f} + e^h + \frac{(\omega^h + e^h)ER}{R_f} - \frac{A^h}{2R_f}\,\mathrm{var}\left(\tilde{y}_1^h + (e^h + \omega^h)\tilde{R}\right)\right],$$

and use the risky asset demand formula (6) to substitute for $(e^h + \omega^h)$. After some manipulations, we obtain

$$\begin{aligned}(1+R_f)c_0^h \quad &= \quad y_0^hR_f + \mathrm{E}\left(\tilde{y}_1^h + e^h\tilde{R}\right) - (1/2)A^h\,\mathrm{var}\left(\tilde{y}_1^h + e^h\tilde{R}\right) \\ &+ \quad (1/2)A^h\,\mathrm{var}\left(\tilde{R}\right)\left[\underbrace{\frac{ER}{A^h\,\mathrm{var}\left(\tilde{R}\right)}}_{\substack{\text{Desired}\\ \text{Exposure}}} \quad -\underbrace{(e^h + \beta_y^h)}_{\substack{\text{Endowed}\\ \text{Exposure}}}\right]^2.\end{aligned} \tag{12}$$

This equation shows how the opportunity to invest in the risky asset affects consumption. If an individual's optimal and endowed exposures to the risky asset are the same, then the opportunity to invest (or disinvest) in the risky asset has no effect on consumption. When they are not the same, the opportunity to invest leads to higher current consumption. In particular, the consumption increase rises with the squared difference between the desired and endowed exposures. We will show that equation (12) provides the foundation for the welfare analysis of portfolio choice menus and international trade in risky financial assets.

2.6 Welfare Analysis of Portfolio Choice Menus

Let $\left(c_0^h, \tilde{c}_1^h\right)$ and $\left(c_0^{h*}, \tilde{c}_1^{h*}\right)$ be consumption profiles for investor h under two different portfolio choice menus, Ω and Ω^*. Suppose that Ω contains the risk-free asset, while Ω^* also includes the risky asset. Our immediate objective is to calculate the utility impact, measured in equivalent consumption terms, of a change in the portfolio choice menu from Ω to Ω^*. To do so, we find the θ^h such that

$$U^h\left(c_0^h + \theta^h, \tilde{c}_1^h + \theta^h\right) = U^h\left(c_0^{h*}, \tilde{c}_1^{h*}\right). \tag{13}$$

Anticipating our application to an international setting, we call θ^h the per-period *gains from trade* in risky financial assets for investor h .

To calculate this welfare measure, use the Euler equation (8) to write utility in terms of current consumption:

$$U^h\left(c_0^h, \tilde{c}_1^h\right) \equiv -\frac{1}{A^h}\left[\exp\left(-A^h c_0^h\right) + \delta^h \operatorname{E}\exp\left(-A^h \tilde{c}_1^h\right)\right] = -\frac{1}{A^h}\left(1 + \frac{1}{R_f}\right)\exp\left(-A^h c_0^h\right). \tag{14}$$

Applying this representation of utility to both sides of (13),

$$-\frac{1}{A^h}\left(1 + \frac{1}{R_f}\right)\exp\left(-A^h[c_0^h + \theta^h]\right) = -\frac{1}{A^h}\left(1 + \frac{1}{R_f}\right)\exp\left(-A^h c_0^{h*}\right),$$

where we maintain the same risk-free rate in Ω and Ω^*. Solving for θ^h, the per-period gains from trade is simply the difference in current consumption, i.e., $\theta^h = c_0^{h*} - c_0^h$.

Now apply the consumption equation (12) to express the gains from trade in terms of endowed and desired exposures. In particular, the gains to investor h from expanding his portfolio choice menu are

$$\boxed{\theta^h = c_0^{h*} - c_0^h = \frac{A^h \operatorname{var}\left(\tilde{R}\right)}{2(1+R_f)}\left(\underbrace{\frac{ER}{A^h \operatorname{var}\left(\tilde{R}\right)}}_{\text{Desired Exposure}} - \underbrace{\left(e^h + \beta_y^h\right)}_{\text{Endowed Exposure}}\right)^2} \tag{15}$$

This expression relates the gains to trade to the investor's risk aversion, his endowed exposure to the risky asset, and the first two moments of asset returns. The bigger the gap between the desired and endowed exposures, the bigger the gains from the opportunity to invest in the risky asset.

2.7 Multiple Risky Assets

The foregoing analysis extends readily to multiple risky assets, whether foreign or domestic. An explicit treatment requires more algebra and greater notational complexity. We refer the reader to Davis and Willen (2000a) for a formal development with multiple risky assets. It is worth remarking that the two-fund separation theorem for portfolio shares fails when there are multiple risky assets, because endowed exposures to risky assets differ among investors (Davis and Willen, 2000b).

Here, we exploit a convenient shortcut to the analysis of multiple risky assets. The shortcut rests on two observations. First, we can always write an additional risky asset as a linear combination of two constructed assets – one that lies in the span of existing assets, and another that is orthogonal to existing assets. Only the orthogonal component of the new asset affects welfare. Second, given preferences of the form (3), the gains to trading the orthogonal component of the new asset are unaffected by the investor's holdings of other risky assets. Hence, we can compute the marginal welfare benefits of trading a new asset by first constructing its orthogonal component and then computing the welfare gains to the orthogonal component in isolation. To compute the total gains from two risky assets, we first compute the gains to one asset in isolation and then add the marginal benefits of trading the orthogonal component of the second asset.

3 International Trade in Risky Financial Assets

International trade in financial claims can improve risk sharing in three ways. First, trade in risk-free securities allows countries to more effectively share consumption risks in the face of transitory output shocks. Second, the opportunity to trade financial claims on risky foreign assets affords better risk sharing among investors within the domestic economy. Even if foreign assets remain in zero net supply to the domestic economy, they improve internal risk sharing by enlarging the asset span facing domestic investors. Third, international trade in risky financial claims, foreign or domestic, affords better risk sharing among countries.[6]

We place no restrictions on risk-free borrowing and lending, so this aspect of international risk sharing does not enter into our welfare calculations. We also ignore internal risk-sharing benefits by conducting the analysis in terms of representative investors for each country.[7] Nothing prevents the application of our framework to

[6] In addition, better risk sharing within or between countries reduces the precautionary demand for the risk-free asset, which raises the equilibrium risk-free rate. By specifying an exogenous risk-free rate, we shut down this secondary effect of expanded trade in risky financial assets. Our framework can be extended to encompass this effect at the cost of a substantial increase in the notational burden and the complexity of the welfare formulae. See Willen (1999) for a related analysis that endogenizes the risk-free rate.

[7] It is worth remarking that we do not assume the existence of a representative agent in each country, as in most other work on international risk sharing with incomplete markets. Rather,

gains that flow from better internal risk sharing, given suitable data on individuals or groups within countries. These internal risk-sharing benefits are distinct and, in our framework, separable from the gains to between-country risk sharing. [8]

3.1 Measuring the Gains to Trade

Our analysis of the gains to international trade in risky financial assets has two main building blocks: the equilibrium asset pricing relation (7), and the expression (12) that links consumption to desired and endowed exposures. From these building blocks, we derive analogs to (15) that apply to a change in trade regime for the risky asset. To that end, consider two regimes that differ in terms of whether the domestic risky asset is traded internationally. Assume that goods are freely traded in both regimes and that individuals can borrow and lend at an exogenous risk-free rate.

For given asset returns and endowments, equation (12) determines the representative investor's consumption in period 0. Now use (12) to compare consumption under free trade and autarky (for the domestic risky asset). Note, first, that the top line on the right side of (12) is invariant across trade regimes.[9] Second, autarky equilibrium requires that endowed exposure equals desired exposure for the representative investor. Putting these two observations together, the gains to trade for the representative investor in country h can be written

$$\boxed{\theta^h = c_0^{\mathcal{F}} - c_0^{\mathcal{A}} = \frac{A^h \operatorname{var}\left(\tilde{R}^{\mathcal{F}}\right)}{2(1+R_f)} \left[\frac{ER^{\mathcal{F}}}{A^h \operatorname{var}\left(\tilde{R}^{\mathcal{F}}\right)} - \left(\frac{e^{h\mathcal{F}}}{H} + \beta_y^{h\mathcal{F}} \right) \right]^2,} \tag{16}$$

where $\mathcal{A}$ and $\mathcal{F}$ denote autarky and free trade in the domestic risky asset, and $(e^{h\mathcal{F}}/H) + \beta_y^{h\mathcal{F}}$ is the per capita supply of the risky asset plus the per capita implicit exposure to the risky asset in country h, both valued at the free trade outcome. A^h should now be interpreted as the absolute risk aversion of the representative investor.

Section 2 establishes the existence of a representative agent with absolute risk aversion equal to the harmonic mean of individual risk aversion levels. This aggregation result does not rest on homogeneous preferences or complete markets within countries.

[8] We are unaware of any studies that isolate the gains to international trade in risky financial assets that flow from improved within-country risk sharing. See Davis and Willen (2000a,b) for applications to individual and group-level data using domestic assets in a closed-economy setting.

[9] To see this point, write the market value of the risky asset as $e = \pi\phi$, where ϕ is the number of shares in the risky asset. Combining this definition with $\tilde{R}\pi = \tilde{d}$, we see that $e\tilde{R} = \tilde{d}\phi$. This last product is simply the total payoff to the domestic supply of the risky asset, which is invariant across trade regimes.

Using the equilibrium asset pricing relation (7) under free trade, we can rewrite the per capita gains to trade in another useful form:

$$\boxed{\theta^h = \frac{A^h \operatorname{var}\left(\tilde{R}\right)}{2(1+R_f)} \left((A^W/A^h) \underbrace{\left(\frac{e^h_W}{H_W} + \beta^h_W\right)}_{\substack{\text{World Per Capita} \\ \text{Endowed Exposure} \\ \text{to Country-}h\text{ Risk}}} - \underbrace{\left(\frac{e^h}{H} + \beta^h_y\right)}_{\substack{\text{Country-}h\text{ Per Capita} \\ \text{Endowed Exposure} \\ \text{to Country-}h\text{ Risk}}} \right)^2} \tag{17}$$

where W designates a world quantity, and we have suppressed the $\mathcal{F}$ superscripts. This equation says that the gains to trade in risky financial assets for the representative investor rise with the squared difference between worldwide and own-country per capita endowed exposures to country-h risk.

Let π denote the price of the risky asset in period 0, and let $\tilde{d}$ be the asset payoff in period 1. Assuming that the payoff is invariant across trade regimes, $\tilde{R}^{\mathcal{F}}\pi^{\mathcal{F}} = \tilde{d} = \tilde{R}^{\mathcal{A}}\pi^{\mathcal{A}}$. Using this fact to express (17) in terms of autarky outcomes, we see that *the gains-to-trade measure (17) is invariant across trade regimes.* The world exposure to country-h risk is the slope coefficient in a regression of world per capita output innovations on country-h equity returns. Likewise, the own-country exposure is the slope coefficient in a regression of country-h per capita output innovations on country-h equity returns. Alternatively, using data on market capitalization, one could separately estimate the two components of own-country endowed exposure to country-h risk.

In contrast to (17), (16) rests on a maintained assumption of free trade in the domestic risky asset. Given this assumption, we can use data on asset returns and a value for risk aversion to estimate the world exposure to country-h risk. Alternatively, if we believe that the data are generated under a regime of autarky for the domestic risky asset, then we can substitute the equilibrium asset pricing relation (7) into (17) to express the gains from trade as

$$\boxed{\theta^h = c_0^{\mathcal{F}} - c_0^{\mathcal{A}} = \frac{A^h \operatorname{var}\left(\tilde{R}^{\mathcal{A}}\right)}{2(1+R_f)} \left[\frac{A^W}{A^h}\left(\frac{e^{h\mathcal{A}}_W}{H_W} + \beta^{h\mathcal{A}}_W\right) - \frac{ER^{\mathcal{A}}}{A^h \operatorname{var}\left(\tilde{R}^{\mathcal{A}}\right)} \right]^2 .} \tag{18}$$

Finally, if we are blessed enough to observe outcomes before and after a change in trade regime for the domestic risky asset, we can measure the gains to trade in terms of asset market data only. For example, when $A^W = A^h = A$ we have

$$\boxed{\theta^h = c_0^{\mathcal{F}} - c_0^{\mathcal{A}} = \frac{R_f^2}{2(1+R_f)\operatorname{var}\left(\tilde{R}^{\mathcal{A}}\right) A} \left(\frac{\pi^{\mathcal{F}} - \pi^{\mathcal{A}}}{\pi^{\mathcal{A}}}\right)^2 .} \tag{19}$$

We can also rewrite this measure in terms of the free-trade return variance using $\text{var}\left(\tilde{R}^{\mathcal{F}}\right) = (\pi^{\mathcal{A}}/\pi^{\mathcal{F}})^2 \, \text{var}\left(\tilde{R}^{\mathcal{A}}\right)$, which follows immediately from $\tilde{R}^{\mathcal{F}}\pi^{\mathcal{F}} = \tilde{R}^{\mathcal{A}}\pi^{\mathcal{A}}$.

3.2 Summary

Section 2 develops solutions for consumption, risky asset holdings, equilibrium asset returns and the welfare effects of changes in portfolio choice menus as functions of endowed and desired exposures. Using these solutions, Section 3.1 derives explicit measures for the equilibrium gains to international trade in risky financial assets. It is worth emphasizing that the formulae (16)-(19) isolate the domestic gains from moving between an autarky regime for risky assets and one in which the world freely trades the domestic risky asset. While theoretically equivalent, these measures differ in terms of the data and assumptions about trade regimes required for empirical implementation.

We can measure the gains to trade without any maintained assumption about the underlying trade regime, if we lean heavily on the equilibrium relation (7) between asset pricing and endowed exposure. Alternatively, by taking a stand on the trade regime, we can use data on asset returns in place of world or own-country exposure to the risky asset in the gains-to-trade formula. This is the approach we take in the empirical work below. If we observe outcomes before and after a change in trade regime for the risky asset, we can measure the equilibrium gains to trade entirely in terms of data on asset prices and returns and the level of risk aversion.

We can also use equation (15) to calculate the benefits of risky domestic and foreign assets for an investor that faces exogenous asset prices. We perform some calculations of this sort in Section 7 below in order to clarify the nature of the gains-to-trade puzzle that we identify and to relate our results to the voluminous literature on individual portfolio choice.

4 An Infinite-Horizon Analysis

The theoretical results in Sections 2 and 3 carry over to an infinite-horizon setting with long-lived agents. While the analysis of the infinite-horizon model is more involved, it proceeds along similar lines. We describe the infinite-horizon model and state the important theorems in this section; proofs appear in the appendix. The key assumption that facilitates our analysis of the infinite-horizon case is a nonstochastic price of risk (Sharpe ratio), defined as the excess return on the risky asset divided by its standard deviation.

From the standpoint of empirical implementation, the infinite-horizon analysis has three related payoffs. First, it allows a proper treatment of persistent income innovations and long-lived assets with stochastic dividend streams. Second, the infinite-horizon analysis clearly distinguishes intertemporal smoothing via risk-free borrowing and lending from the (incomplete) sharing of wealth shocks through a properly structured portfolio of risky assets. Third, the key welfare expressions for the gains to trade must be modified in a many-period setting.

4.1 The Model

A country is a collection of infinitely-lived investors, indexed by h. Let $\mathbf{C}^h = \left\{\tilde{c}_s^h\right\}_{s=t}^{\infty}$ be the consumption path for h, and let $\mathbf{Y}^h = \left\{\tilde{y}_s^h\right\}_{s=t}^{\infty}$ be his endowment path.

Condition 1 *Investor h has exponential utility:*

$$U_t^h\left(\mathbf{C}^h\right) = \mathrm{E}_t\left[\sum_{s=t}^{\infty}\left(\delta^h\right)^s \frac{-1}{A^h}\exp\left(-A^h\tilde{c}_s^h\right)\right].$$

Condition 2 *The endowment for investor h follows an ARIMA process with time-t innovation, $\tilde{\eta}_t^h$. The innovation has a time-invariant normal distribution with finite variance.*

Let ψ_i^h be the coefficient on the i^{th} term in the moving-average representation of the endowment process for h. That is, $\mathrm{E}_t\left(\tilde{y}_{t+i}^h\right) - \mathrm{E}_{t-1}\left(\tilde{y}_{t+i}^h\right) = \psi_i^h\tilde{\eta}_t^h$.

There are two marketable assets in the domestic economy. A risk-free one-period discount bond has unit payoff, price π_f and gross rate of return R_f, where we suppress the time subscript. As before, we treat R_f as exogenous.[10] Let

$$\mathrm{PDV}_t\left(\left\{z_s\right\}_{s=t+1}^{T}\right) = \sum_{s=t+1}^{T}\frac{1}{\Pi_{i=t+1}^{s}R_{f,i}}\mathrm{E}_t\left(\tilde{z}_s\right)$$

be an operator for the present expected value of an arbitrary time series, discounted at the risk-free rate. In this notation, $\mathrm{PDV}_t\left(\{1\}_{s=t}^{\infty}\right)$ is the present value of an annuity that pays one unit of consumption each period from t forward, and $a \equiv \left[\mathrm{PDV}_t\left(\{1\}_{s=t}^{\infty}\right)\right]^{-1} = (R_f - 1)/R_f$ is the corresponding annuitization factor. We maintain the following regularity condition.

Condition 3 *The risk-free interest rate is positive ($R_f > 1$), and the present discounted value of endowment income is finite,* $\mathrm{PDV}_t\left(\left\{\tilde{y}_s^h\right\}_{s=t}^{\infty}\right) < \infty$.

[10] Also as before, the risk-free rate can be endogenized at the cost of greater complexity in the analysis and in the welfare expressions. To obtain a nonstochastic price of risk as an equilibrium outcome with an endogenous risk-free rate, we require that the stochstic component of the aggregate endowment be a martingale.

The domestic risky asset is a claim to the dividend stream, $\left\{\tilde{d}_s\right\}_{s=t}^{\infty}$. Let $\tilde{\pi}_t$ denote the ex-dividend price of the risky asset at time t, and let $\tilde{R}_t = (\tilde{\pi}_t + \tilde{d}_t)/\tilde{\pi}_{t-1}$ be the gross one-period rate of return. Define the excess return measures, $\tilde{XR}_t = \tilde{R}_t - R_f$ and $ER_t = \mathrm{E}_t\left(\tilde{XR}_{t+1}\right)$, and let $\tilde{S}_{t+1} = \tilde{XR}_{t+1}/\operatorname{std}_t\left(\tilde{R}_{t+1}\right)$. $\mathrm{E}_t\tilde{S}_{t+1}$ is the price of risk (Sharpe ratio) at t. Also, let ϕ denote the number of shares of the risky asset, so that $\tilde{e}_t = \tilde{\pi}_t\phi$ is the market capitalization of the risky asset in units of consumption.

The following condition describes the behavior of dividends and their relationship to the endowment innovations.

Condition 4 *Dividends follow an ARIMA process with innovation path, $(\{\tilde{x}_s\}_{s=t}^{\infty})$. The dividend and endowment innovations have a time-invariant joint normal distribution with finite variance-covariance matrix:*

$$\left[\tilde{\eta}_t^1, \ldots, \tilde{\eta}_t^H, \tilde{x}_t\right]' \sim \mathbb{N}\left(0, \begin{bmatrix} \Xi & \boldsymbol{\gamma} \\ \boldsymbol{\gamma}' & \sigma_x^2 \end{bmatrix}\right).$$

Since dividends obey an ARIMA process, we have

$$\Lambda\tilde{x}_{t+1} = \mathrm{PDV}_{t+1}\left(\left\{\tilde{d}_s\right\}_{s=t+1}^{\infty}\right) - \mathrm{E}_t\left(\mathrm{PDV}_{t+1}\left(\left\{\tilde{d}_s\right\}_{s=t+1}^{\infty}\right)\right),$$

where Λ is the present value multiplier on a dividend innovation.

We rely on one additional condition to characterize individual behavior:

Condition 5 *For each h, $[\tilde{\eta}_t^h, \tilde{S}_t]$ has a time-invariant joint normal distribution with finite variance-covariance matrix.*

Since σ_x^2 is nonstochastic, it follows immediately from condition 5 that $[\tilde{\eta}_t^h, \tilde{R}_t]$ also has a joint normal distribution, conditional on information at time $t-1$. We use condition 5 to characterize individual behavior, but we also that show it holds at the equilibrium solution.

We can now describe the budget set and state the optimization problem facing individual investors. At time t, h invests $\omega_{f,t}^h$ units of consumption in the risk-free asset and ω_t^h units in the risky asset. Thus, his time-t budget set can be written

$$B_t^h = \left\{ \left\{\tilde{c}_s^h, \tilde{\omega}_{f,s}^h, \tilde{\omega}_s^h\right\}_{s=t}^{\infty} \ni \begin{array}{c} \tilde{c}_s^h + \tilde{\omega}_{f,s}^h + \tilde{\omega}_s^h = \tilde{y}_s^h + \tilde{\omega}_{f,s-1}^h R_f + \tilde{R}_s\tilde{\omega}_{s-1}^h, \\ \lim_{s\to\infty} \mathrm{E}_t[\tilde{\omega}_{f,s}^h + \tilde{\omega}_s^h]/R_f^s = 0 \end{array} \right\}$$

Investor h solves the following problem at time t:

$$\max U_t^h\left(\mathbf{C}^h\right) \text{ s.t. } \mathbf{C}^h, \boldsymbol{\Omega}^h \in B_t^h, \tag{20}$$

where $\boldsymbol{\Omega}^h = \left\{\tilde{\omega}_{f,s}^h, \tilde{\omega}_s^h\right\}_{s=t}^{\infty}$ is his portfolio path.

Given the foregoing setup, we shall solve for a rational expectations equilibrium and individual outcomes. Equilibrium requires that (i) the individual consumption and portfolio allocation paths solve (20) taking asset returns as given, (ii) the world-wide goods market clears, (iii) risky asset markets clear at all dates, and (iv) the equilibrium behavior of asset returns is consistent with investors' expectations.

The market-clearing conditions for risky assets depend on the trade regime. If the domestic risky asset is not traded internationally, then market clearing requires that own-country demand for the domestic risky asset equals the domestic supply at each date. Alternatively, if the asset is traded internationally, then the world demand for the domestic risky asset equals the world supply at each date.

4.2 Individual Optimization

Proposition 1 *Consider an investor h who can trade the risk-free asset and the domestic risky asset. Given conditions 1, 2, 3 and 5, the solution to (20) is given by:*

(i) Risky Asset: $\omega_t^h = (aA^h)^{-1} \dfrac{ER_t}{\mathrm{var}_t\left(\tilde{R}_{t+1}\right)} - \Psi^h \dfrac{\mathrm{cov}_t\left(\tilde{\eta}_{t+1}^h, \tilde{R}_{t+1}\right)}{\mathrm{var}_t\left(\tilde{R}_{t+1}\right)}$,

where $\Psi^h = \mathrm{PDV}\left(\left\{\psi_s^h\right\}_{s=0}^{\infty}\right)$ is the present value multiplier on endowment innovations for investor h.

(ii) Consumption: $c_t^h = aGW_t^h$, *where*

$$GW_t^h = \mathrm{PDV}_t\left(\left\{\tilde{y}_s^h\right\}_{s=t}^{\infty}\right) + R_f\omega_{f,t-1}^h + R_t\omega_{t-1}^h + \mathrm{PDV}_t\left(\left\{ER_{s-1}\omega_{s-1}^h\right\}_{s=t+1}^{\infty}\right)$$
$$+\frac{1}{R_f-1}\left(1/aA^h\right)\ln(R_f\delta^h) - \mathrm{PDV}_t\left(\left\{\left(aA^h/2\right)\mathrm{var}\left(\Psi^h\tilde{\eta}_s^h + \omega_{s-1}^h\tilde{R}_s\right)\right\}_{s=t+1}^{\infty}\right)$$

(iii) Risk-free Asset: $\omega_{f,t} = y_t^h + \omega_{f,s-1}^h R_f + \tilde{R}_t\omega_{t-1}^h - c_t^h - \omega_t^h$. □

Part (i) of Proposition 1 generalizes equation (6), the gross demand for the risky asset in the two-period model, to the infinite-horizon case. The first term in the expression for ω_t^h is the desired exposure to the risky asset based on information available at t. It has the same form as in the two-period model, except for the addition of the present value multiplier, a^{-1}. The second term is the implicit exposure to the risky asset; it equals the product of the present value multiplier on endowment innovations and the slope coefficient in a regression of endowment innovations on asset returns. This product can also be interpreted as the slope coefficient in a regression of the endowment present value (i.e., human capital) on the asset return.

Part (ii) says that consumption is proportional to a generalized wealth measure that includes a positive adjustment for excess returns on current and future investments in the risky asset and a negative precautionary effect that adjusts for uncertainty about future wealth (hence, future consumption). As before, we shall maintain $\ln(R_f\delta^h) = 0$ in the main text without affecting the welfare formulae below.

Using the optimal solutions in Proposition 1, we obtain

Corollary 1 *Under the conditions in Proposition 1, the value of excess returns to investor h can be written*

$$ER_t\omega_t^h = \frac{1}{aA^h}\,\mathrm{E}\left(\tilde{S}\right)^2 - \Psi^h\,\mathrm{E}\left(\tilde{S}\right)\mathrm{cov}\left(\tilde{\eta}^h, \tilde{S}\right) \text{ for all } t,$$

and generalized wealth can be written as

$$\begin{aligned} GW_t^h = {} & \mathrm{PDV}_t\left(\left\{\tilde{y}_s^h\right\}_{s=t}^{\infty}\right) + R_f\omega_{f,t-1}^h + R_t\omega_{t-1}^h - (A^h/2R_f)\,\mathrm{var}\left(\Psi^h\tilde{\eta}^h\right) \\ & + (1/2A^hR_f)\left[a^{-1}\,\mathrm{E}\left(\tilde{S}\right) - A^h\Psi^h\,\mathrm{cov}\left(\tilde{\eta}, \tilde{S}\right)\right]^2 \qquad \square \end{aligned}$$

Together, condition 5 and the first part of the corollary tell us that the value of excess returns on risky asset holdings is nonstochastic along an optimal path. It follows that the future time path of the precautionary term in the generalized wealth measure is known with certainty. This result is reflected in the expression for generalized wealth given by the second part of the corollary. Thus, there are only two sources of stochastic variation in generalized wealth (and consumption) along the optimal path: innovations to the present value of endowment income and the difference between expected and realized returns on risky asset holdings.

4.3 Asset Pricing

We now use the market clearing requirement that $\sum_{h\in H}\omega_t^h = \phi\pi_t = e_t$ to characterize asset prices. Let $\overline{\Psi\eta} = (1/H)\sum_{h\in H}\Psi^h\tilde{\eta}^h$ be the per capita mean innovation to the present value of endowment income. This quantity is a random variable with a time-invariant normal distribution by condition 4.

Proposition 2 *Given conditions 1, 2, 3 and 4, the time-t price per share of the risky asset is*

$$\pi_t = \mathrm{PDV}_t\left(\left\{\tilde{d}_s\right\}_{s=t}^{\infty}\right) - (A^*/R_f)\left(\mathrm{cov}\left(\overline{\Psi\eta}, \Lambda\tilde{x}\right) + \Lambda^2\sigma_x^2\frac{\phi}{H}\right),$$

where A^* is the harmonic mean of absolute risk aversion among investors, and ϕ is the number of shares. The time-t Sharpe ratio for the risky asset is

$$\mathrm{E}\,\tilde{S} = \frac{aA^*}{\sigma_x}\left[\Lambda\sigma_x^2\frac{\phi}{H} + \mathrm{cov}\left(\overline{\Psi\eta}, \tilde{x}\right)\right] \qquad \square$$

According to the first part of this proposition, the price of the risky asset equals the present value of future dividends plus a risk adjustment. The risk adjustment depends on the covariance of the dividend innovation and mean innovation to the endowment present value plus the variance of the asset value. If $e = 0$ (e.g., a pure financial asset) and the dividend and endowment innovations are uncorrelated, then the risk adjustment term vanishes. The second part of the proposition gives the equilibrium price of risk in terms of fundamentals.

Anticipating the empirical implementation, we recast Proposition 2 in terms of the excess return:

Corollary 2 *The equilibrium excess return on the risky asset is given by*

$$ER_t = aA^* \operatorname{var}\left(\tilde{R}_{t+1}\right) \quad \underbrace{\left[\frac{e_t}{H} + \overline{\Psi\beta}_t\right]}_{\textit{Endowed Exposure}}$$

where $\overline{\Psi\beta}_t$ *is the mean of* $\Psi^h \operatorname{cov}\left(\tilde{\eta}^h_{t+1}, \tilde{R}_{t+1}\right) / \operatorname{var}\left(\tilde{R}_{t+1}\right)$ *among investors at* t. $\square$

This corollary generalizes equation (7) to the infinite-horizon case. The corollary is useful in determining the impact of international trade in risky assets, which affects H and possibly A and $\overline{\Psi\beta}$, too.

4.4 Portfolio Choice Menus, Trade and Welfare

Definition 1 θ^h *is the uniform variation in the consumption good at each state and date that leaves* h *indifferent between consumption paths* $\mathbf{C}$ *and* $\mathbf{C}^*$*:* $U^h(\mathbf{C}+\theta) = U^h(\mathbf{C}^*)$.

The next proposition tells us how to calculate the uniform variation for a change in the asset return menu facing an individual investor.

Proposition 3 *Consider two asset return menus,* $\mathbf{R}$ *and* $\mathbf{R}^*$*, with the same value of* R_f*. Let* GW^h_t *and* GW^{h*}_t *denote the time-*t *generalized wealth levels for investor* h *under* $\mathbf{R}$ *and* $\mathbf{R}^*$*, respectively. Define* $\mathbf{C}^h$ *and* $\mathbf{C}^{h*}$ *analogously. Under conditions 1 through 4, the uniform consumption variation associated with a change from menu* $\mathbf{R}$ *to menu* $\mathbf{R}^*$ *is*

$$\theta^h_t = c^{h*}_t - c^h_t.$$

The present discounted value of the uniform variation is

$$\Theta^h = GW^{h*}_t - GW^h_t. \qquad \square$$

Our final proposition gives the equilibrium gains to international trade in a domestic risky asset for a country's representative investor. We now interpret the superscript h as indexing countries, and we use W for "world" quantities.

Proposition 4 (Equilibrium Gains to Trade) *Consider the comparison between autarky and free trade regimes for the domestic risky asset. Given conditions 1 through 4, the equilibrium consumption variation for the representative domestic investor associated with this change in trade regime is*

$$\theta^h = \frac{A^h}{2R_f}\sigma_x^2 \left[\frac{A^W}{A^h} \left(\Lambda \frac{\phi^W}{H^W} + \frac{\operatorname{cov}\left(\overline{\Psi\eta}_t^W, \tilde{x}_t\right)}{\sigma_x^2} \right) - \left(\Lambda \frac{\phi^h}{H^h} + \frac{\operatorname{cov}\left(\overline{\Psi\eta}_t^h, \tilde{x}_t\right)}{\sigma_x^2} \right) \right]^2,$$

where $\overline{\Psi\eta}^W$ is the mean per capita innovation to the present value of world endowment income, and $\overline{\Psi\eta}^h$ is the analogous quantity for country h. $\square$

The following corollary gives several expressions for the equilibrium gains to trade that are suitable for empirical work, generalizing equations (16), (17) and (18).

Corollary 3 (Equilibrium Gains to Trade in Terms of Observables) *The equilibrium gains to trade in Proposition 4 can be written as*

$$\theta^h = \frac{aA^h}{2R_f} \operatorname{var}\left(\tilde{R}_{t+1}\right) \left[\frac{A^W}{A^h} \left(\frac{e_t^W}{H^W} + \overline{\Psi\beta}_t^W \right) - \left(\frac{e_t^h}{H^h} + \overline{\Psi\beta}_t^h \right) \right]^2 \tag{21}$$

Under free trade in the risky asset, the gains to trade can be expressed as

$$\theta^h = \frac{aA^h}{2R_f} \operatorname{var}\left(\tilde{R}_{t+1}^{\mathcal{F}}\right) \left[\frac{1}{a} \frac{ER_t^{\mathcal{F}}}{A^h \operatorname{var}\left(\tilde{R}_{t+1}^{\mathcal{F}}\right)} - \left(\frac{e_t^{h\mathcal{F}}}{H^h} + \overline{\Psi\beta}_t^{h\mathcal{F}} \right) \right]^2 \tag{22}$$

Under autarky for the risky asset, the gains to trade can be expressed as

$$\theta^h = \frac{aA^h}{2R_f} \operatorname{var}\left(\tilde{R}_{t+1}^{\mathcal{A}}\right) \left[\frac{A^W}{A^h} \left(\frac{e_t^{W\mathcal{A}}}{H^W} + \overline{\Psi\beta}_t^{W\mathcal{A}} \right) - \frac{1}{a} \frac{ER_t^{\mathcal{A}}}{A^h \operatorname{var}\left(\tilde{R}_{t+1}^{\mathcal{A}}\right)} \right]^2 \tag{23}$$

5 Data Inputs and Empirical Characterization

To empirically implement the theoretical model, we must choose a country or set of countries and specify a trade regime – i.e., which agents trade which assets. The first two moments of asset returns and their covariances with endowment innovations (or regression coefficients) can be fit to historical data or chosen in some other way. We must also specify the risk-free rate of return and investor risk aversion levels.

5.1 Data Sources

We consider annual data on per capita output and equity returns for fourteen countries from 1970 to 1995. We rely on the *U.N. System of National Accounts* (SNA) for data on GDP, the International Monetary Fund's *International Financial Statistics* (IFS) for population data, and *Morgan Stanley Capital International* (MSCI) data on equity returns. The MSCI data contain total returns (including dividends) on national stock indices and a value-weighted world equity return series based on stock returns for 22 nations. Some calculations below also rely on MSCI data on stock market capitalization values. Finally, in order to calculate domestic rates of return on world equity, we use IFS data on spot exchange rates. We selected countries based on the availability of national data on equity returns back to 1970.[11]

5.2 Output Specifications

To calculate real output, we deflate nominal GDP by the ratio of nominal to real final consumption expenditures of resident households in 1990. We divide this measure of output by total population to obtain per capita output in local currency units. To obtain per capita PPP-adjusted "world output" for the 14 countries in our sample, we calculate the population-weighted mean of the country-level outputs after converting to U.S. dollars using 1990 PPP-adjusted exchange rates from the Penn World Tables. This one-time PPP adjustment does not affect our calculations of domestic rates of return on own or world equity. To facilitate easy comparisons across countries, we also express the results in PPP-adjusted 1990 U.S. dollars.

The stochastic process for output in our theoretical model implies an empirical specification in natural units rather than logs. In practice, we find similar serial correlation properties for output in logs and natural units. We initially set out to fit simple ARIMA models for each country's per capita output measures. However, except for world output at market exchange rates, we typically cannot reject the hypothesis that the first-differenced output and log output measures are serially uncorrelated in our sample period. Hence, we settled on a specification that treats each national output process as a separately estimated random walk with drift.[12]

[11] We used SNA data, because it provides output and consumption data disaggregated by broad industry categories. In Davis, Nalewaik and Willen (2000), we exploit the industry-level data to implement a version of the model with tradable and nontradable consumption goods. Since the distinction between tradable and nontradable goods did not greatly affect the character of the results, this paper sticks to the simpler approach of treating all output as tradable.

[12] We experimented with linear time trends in the random walk specifications, but only Canada showed a trend that was statistically significant at the 10 percent level.

Table 1 reports parameter estimates for these random walk specifications and p-values for the null hypothesis of no serial correlation in the differenced values. The standard deviation of annual per capita output innovations ranges from 1.8 to 3.4 percent of mean output, or from about 300 to 700 1990 U.S. dollars per person. We will use the statistics for output in natural units and a risk-free interest rate of 2.5 percent per year to construct the standard deviation of innovations to the present value of per capita endowments.

5.3 Equity Returns

Table 2 reports summary statistics for domestic real returns on own-country and world equity for each country in our sample. World equity returns average about six or seven percent per year for most countries, with a standard deviation of about 20 percent. Country differences in returns on world equity arise from real exchange rate movements. The mean return on own-country equity ranges from 5.3 percent per year for Italy to 13.9 percent for Sweden. For most countries, own-country equity returns show substantially greater volatility than world equity returns.

As the table notes, a standard test shows no significant evidence against the null hypothesis of serially uncorrelated returns. This result indicates that a version of the theoretical model with constant excess returns is not seriously at odds with the data.[13] On this basis, we proceed under the assumption of constant excess returns, which considerably simplifies the implementation of the theory. Relaxing this assumption would require a much more involved empirical analysis that explicitly treats time variation in the mean and standard deviation of excess returns. For the issues at hand, greater complexity in this respect does not have an obvious payoff.

5.4 Covariance Between Output Innovations and Returns

Unreported regressions show that domestic output innovations are contemporaneously uncorrelated with own-country equity returns in annual data. R^2 values are less than 10 percent in all cases and less than 5 percent for all but two countries. The correlation coefficient is negative for half the countries, and no country exhibits a statistically significant relationship between output innovations and contemporaneous equity returns. The corresponding regressions for log output innovations are highly similar. These results echo other studies that find little or no correlation between equity returns and contemporanous innovations in output, income and consumption.[14]

[13] More sophisticated econometric techniques and longer time series provide some evidence of time-varying expected returns on risky assets. See chapter 2 in Campbell, Lo and MacKinlay (1997).

In contrast, domestic output innovations are correlated with lagged equity returns for several countries. Table 3 reports results for regressions of domestic output innovations on contemporaneous and lagged own-country equity returns. The average R^2 value is .14, and lagged equity returns are statistically significant at the 5 percent level in Belgium, Canada, Germany, Japan, Norway and the United States. A second lag of equity returns adds little to the fit of these regressions.

Table 3 also reports a quasi-correlation statistic that we compute as $\tilde{\rho} = \left[b_0 + R_f^{-1} b_1\right]\left[\mathrm{std}(R)/\,\mathrm{std}(y)\right]$, where b_0 and b_1 are the slope coefficients on current and lagged equity returns, $R_f = 1.025$ is the gross risk-free rate of return, $\mathrm{std}(R)$ is the standard deviation of equity returns, and $\mathrm{std}(y)$ is the standard deviation of the output innovations. The quasi-correlation formula is analogous to the relationship between the Pearson correlation, ρ, and the slope coefficient in an OLS regression of y on R: $\rho = b\,\mathrm{std}(R)/\,\mathrm{std}(y)$, where b is the regression coefficient. In calibrating the theoretical model below, we use the quasi-correlation to scale the covariance between output innovations and returns in a way that accounts for the role of lagged returns.[15]

Table 4 reports regressions of domestic output innovations on contemporaneous and lagged real returns on the world equity fund. Recall that the raw returns on the world equity fund are expressed in U.S. dollars. Part A of the table considers unhedged returns in own currency units, while part B considers returns that are hedged against movements in the real exchange rate between the domestic currency and the U.S. dollar. For each country, the measure of world returns in these regressions has been othogonalized with respect to contemporaneous and lagged returns on own equity.

[14]Fama and Schwert (1977) find a near-zero correlation between aggregate labor income innovations and equity returns in the United States. Bottazzi et al. (1996) find similar results for the relationship between labor income innovations and domestic asset returns for several countries. Davis and Willen (2000) consider labor income innovations for synthetic persons defined in terms of sex, birth cohort and education. They find correlations with aggregate U.S. equity returns that rise with education but are centered near zero. Even the returns on proprietary business wealth appear to be weakly correlated with equity returns. In this regard, Heaton and Lucas (2000) report a correlation of .14 between the growth rate of U.S. proprietary business income and the stock market rate of return. It is well known from the equity premium literature that aggregate U.S. consumption growth is weakly correlated with equity returns. Campbell (1999a, Table 5) considers eleven countries and reports that the correlation between equity returns and consumption growth is even weaker outside the United States.

[15]A more standard approach would be to specify and fit a vector autoregression (VAR) in output innovations and equity returns. Given the moving average representation, we could then calculate the cumulative output response (in present value terms) to an innovation in equity returns. We eschewed this approach, because the stochastic structure of the theoretical model in Section 4 is not rich enough to accommodate this type of VAR. Generalizing the theoretical model in this direction is a useful avenue for future research.

The regressions in part A show little evidence of any correlation between domestic output innovations and unhedged returns on world equity. Indeed, the average quasi-correlation value in Part A is slightly negative. In contrast, part B indicates that domestic output innovations are positively correlated with hedged real returns on world equity. The average quasi-correlation value for hedged world equity returns in Table 4 is two-thirds as large as the value for own equity in Table 3. Based on this result, our baseline specification below treats domestic output innovations as more highly correlated with own equity returns than with (hedged) world equity returns.

6 Equity Exposure Measures

Barring observations on asset prices before and after a change in trade regime, endowed exposure is an essential input for measuring the equilibrium gains to trade. It is also essential for determining optimal portfolio positions and the individual gains from portfolio diversification. Accordingly, this section computes measures of endowed exposure to domestic and world equity. We will use these measures for the portfolio and welfare analyses in Sections 7 and 8.

Table 5 reports several measures of per capita endowed exposure to national and world equity funds for the 14 countries in our sample. Part A reports per capita endowed exposure (EE) to own-country equity, and Part B reports each country's endowed exposure to the orthogonal component of world equity. All exposure measures reflect a risk-free interest rate of 2.5 percent.

Column (2) reports EE under the assumption that equity is a claim to GDP. This assumption is belied by the empirical evidence in Section 5.4, but it is useful for two reasons. First, it delivers the largest possible value of endowed exposure, given the standard deviations of output innovations and equity returns. Second, it provides a useful point of comparison to other studies that treat equity as a claim to GDP. If equity is a claim to GDP and GDP follows a random walk, we have

$$EE(\text{GDP}) = \frac{1}{a}\frac{\text{std}(\tilde{y})}{\text{std}\,(\tilde{R})}. \tag{24}$$

To derive (24), observe that

$$\text{var}\left(\tilde{R}_t\right) = \frac{\text{var}\left(\tilde{d}_t + \tilde{\pi}_t\right)}{\pi_{t-1}^2} = \frac{\text{var}\,(\Lambda\tilde{x}_t)}{\pi_{t-1}^2} = \frac{\text{var}\,(\tilde{y}_t)}{(a\pi_{t-1})^2}.$$

The second equality follows from Condition 4 and the third from the random-walk process for output. Normalizing the number of equity shares to unity and solving for π yields (24).

Columns (3) and (4) report measures of endowed exposure calculated directly from the definition in Corollary 2:

$$EE(\text{Direct}) = \frac{e}{H} + \overline{\Psi\beta}.$$

To implement the direct measure, we calculate e/H as the sample average value of the country's per capita stock market capitalization. In calculating $\overline{\Psi\beta}$, the implicit component of endowed exposure, we do not want to double count the explicit component captured by e/H. However, national income accounts do not isolate the income flows that accrue to public companies with securitized equity.

To deal with this data limitation, observe that $EE(\text{Direct}) = EE(\text{GDP})$ when dividends are perfectly correlated with other income flows. Thus, we compute the implicit component of endowed exposure using total output, but we include an adjustment factor δ chosen to equate the direct and GDP-based exposure measures in the perfect correlation case. That is, we write

$$EE(\text{Direct}) = \frac{e}{H} + \overline{\Psi\beta} = \frac{e}{H} + \frac{\delta}{a}\frac{\text{cov}(\tilde{R},\tilde{y})}{\text{var}\left(\tilde{R}\right)} = \frac{e}{H} + \frac{\delta}{a}\frac{\text{std}(\tilde{y})\,\text{corr}(\tilde{R},\tilde{y})}{\text{std}\,(\tilde{R})}. \tag{25}$$

Evaluating this expression at $\text{corr}(\tilde{R},\tilde{y}) = 1$ and equating to $EE(\text{GDP})$ yields δ, which we report in column (1).

Given δ, we calculate (25) by setting $\text{corr}(\tilde{R},\tilde{y})$ to the quasi-correlation reported in Table 3. We use the country's fitted quasi-correlation to calculate the exposure values in column (3), and we use the average quasi-correlation value of .36 for all countries in column (4). In column (5), we calculate (25) for $\text{corr}(\tilde{R},\tilde{y}) = 0$, which reduces to the stock market capitalization value. Hence, columns (2) through (5) can be interpreted as four versions of $EE(\text{Direct})$ that differ with respect to the assumed correlation between output and equity returns. Finally, columns (6) and (7) report direct measures of endowed exposure to the orthogonal component of world equity with e/H set to 0 and δ set to 1.

The equity exposure measures in Table 5 establish three key points. First, even with a modest correlation between output innovations and returns, the endowed exposure to domestic equity is many times greater than stock market capitalization. For example, U.S. market capitalization amounts to 9 thousand per person in 1990 dollars, but a correlation between output and returns of .36 implies an endowed exposure of 43 thousand dollars per person. In other words, the effective quantity of U.S. equity is nearly five times larger than the market value of U.S. equity securities. Assuming perfect correlation, the endowed exposure of U.S. residents to U.S. equity exceeds one hundred thousand dollars per person.

Second, the optimal portfolio holdings of equity for each country's representative investor are highly sensitive to assumptions about (or estimates of) the correlation between output shocks and equity returns.[16] This point follows from the sensitivity of endowed exposure to this correlation and Proposition 1, which tells us that the optimal portfolio position equals the difference between desired and endowed exposures. We develop this point more fully in the next section, where we explicitly calculate desired exposures and optimal portfolio positions.

Third, the representative investor in each country is endowed with a fairly large exposure to the orthogonal component of the world equity fund. As reported in column (7) of Table 5, a correlation of .24 between domestic output innovations and hedged returns on world equity corresponds to an endowed exposure in the range of 17 to 45 thousand dollars per person, which is larger than per capita annual consumption. Hence, domestic investors implicitly hold a sizable long position in world equity, even if they hold little or no world equity in their financial portfolios.

7 Portfolio Puzzles

We now calculate optimal equity holdings for the representative investor in each country, *taking asset returns as given*. We first show that the theoretically optimal level of equity holdings is enormous relative to observed holdings. We then express this gap between theory and reality in welfare terms by calculating the consumption-equivalent loss implied by suboptimal equity holdings. For simplicity, we carry out these exercises for an investor who holds no foreign assets. We then revisit the home bias puzzle by calculating optimal positions in domestic and world equity and the gains from expanding the portfolio choice menu to include the world equity fund.

7.1 Optimal Equity Holdings for a Representative Investor

Consider a representative investor in each country with a portfolio choice menu that consists of the risk-free asset and domestic equity. According to Proposition 1, his desired exposure to own-country equity is

$$DE = \frac{ER}{aA\,\mathrm{var}\left(\tilde{R}\right)}. \tag{26}$$

To implement (26), we use the statistics for own equity reported in Table 2 and a value for A that corresponds to a relative risk aversion of 3 for the representative

[16] More precisely, optimal equity holdings are highly sensitive to the slope coefficient in a projection of the present value of domestic output on domestic equity returns.

investor. We approximate A (absolute risk aversion) as relative risk aversion divided by a country's sample average value of per capita real consumption. Hence, our expression for desired exposure is proportional to per capita consumption and inversely proportional to relative risk aversion.

In Table 6, Part A reports desired exposure to own-country equity calculated according to (26), and Part B reports the theoretically optimal holdings. The first column in Part B uses the fitted relationship between domestic output innovations and equity returns in the calculation of endowed exposure. The next two columns report optimal holdings for alternative values of the correlation between output innovations and returns. Part C reports each country's stock market capitalization per person. If all equity securities are held internally, then market cap is equivalent to actual per capita holdings by residents. In practice, a very large share of equity securities are internally held, as emphasized in the home bias literature, so we use market cap as a rough measure of observed holdings.

Table 6 shows that theoretically optimal equity holdings are enormous compared to observed equity holdings. Except for Italy, the optimal holdings greatly exceed stock market capitalization even in the case of perfect correlation between output innovations and equity returns. The gap between theory and reality highlighted by Table 6 is a statement about the level of risky asset holdings. In contrast, the closely related equity premium puzzle is usually cast in terms of excess returns or the price of risk. Moreover, the portfolio puzzle identfied in Table 6 is about individual behavior, not equilibrium outcomes.

There are at least two good reasons to recast the equity premium puzzle in portfolio terms. First, individual choice problems are easier to analyze than equilibrium problems. If a proposed resolution to the equity premium puzzle cannot resolve the portfolio version of the puzzle for a representative investor, there is no need to calculate a complicated equilibrium to demonstrate its failure. Second, and more important, the portfolio perspective leads directly to a welfare measure of the gap between theory and reality.

7.2 Restatement of the Portfolio Puzzle in Welfare Terms

Table 7 reports the welfare costs implied by the suboptimal equity holdings in Table 6. To calculate these costs, we make use of the expression for generalized wealth in Corollary 1. The first line in this expression is unaffected by current equity holdings. After writing the Sharpe ratio in terms of the excess return on equity and multiplying by the annuitization factor a, the second line becomes

$$\frac{aA\,\mathrm{var}\left(\tilde{R}\right)}{2R_f}\left[\frac{ER}{aA\,\mathrm{var}\left(\tilde{R}\right)}-\Psi\beta\right]^2.$$

That is, the increase in generalized wealth afforded by an optimal equity position is proportional to the squared difference between optimal and endowed exposures. The *unrealized* portion of those gains is proportional to the squared difference between optimal and actual equity exposures:

$$\frac{aA\,\mathrm{var}\left(\tilde{R}\right)}{2R_f}\left[\frac{ER}{aA\,\mathrm{var}\left(\tilde{R}\right)}-\left(\frac{e}{H}+\Psi\beta\right)\right]^2, \tag{27}$$

where we use market cap to measure actual equity holdings. By Proposition 3, equation (27) gives the consumption-equivalent welfare loss associated with a suboptimal position in domestic equity.[17]

In Part A of Table 7, we calculate (27) and divide by a country's sample average value of per capita annual consumption to express the cost as a percentage of actual consumption. The results are striking: The theory implies that a representative investor (with relative risk aversion equal to 3) suffers enormous welfare costs by failing to adopt an optimal equity position. In our preferred case, which maintains a .36 correlation between domestic output innovations and equity returns, the welfare cost in the average country amounts to 31 percent of consumption per year. Even under the extreme assumption of perfect correlation between output innovations and equity returns, the average welfare cost is 16 percent of consumption. Welfare costs of this magnitude are not easily explained by the costs of acquiring information or transacting in equity securities.

Part B of Table 7 reports the value of relative risk aversion that fits the equilibrium asset pricing condition in Corollary 2. These relative risk aversion values also rationalize observed equity holdings for a representative investor with an asset menu consisting of the risk-free security and domestic equity. The asset pricing condition implies a relative risk aversion value of 15.2 for the average country in our preferred case with a .36 correlation. Thus, as others have observed, high risk aversion levels can "resolve" the equity premium puzzle in an otherwise standard model. However,

[17] While identical in form, equations (27) and (22) are derived from different thought experiments. Equation (27) pertains to an individual agent who faces exogenous asset returns, whereas (22) pertains to the equilibrium gains to the average investor caused by a change in trade regime. In both thought experiments, observed equity returns determine desired exposure and the covariance between domestic output innovations and domestic equity returns determines endowed exposure.

as we show below in Section 8, this resolution of the equity premium puzzle does not resolve another problem for the theory.

7.3 Expanding the Choice Menu to Include World Equity

Table 8 presents results for an expanded portfolio choice menu that includes the world equity fund. To calculate optimal portfolios, we use a sequential procedure. First, we calculate the optimal position in domestic equity, considered in isolation, which we have already done in Table 6. Second, we follow the same procedure to calculate the optimal position in the orthogonalized world equity asset. This second step also gives the optimal position in the original world equity asset. Lastly, we obtain the optimal position in domestic equity as $\omega = \omega_1 - b\omega_2$, where ω_1 is the domestic equity level calculated in the first step, ω_2 is the world equity level calculated in the second step, and b is the slope coefficient in a regression of world equity returns on domestic equity returns.[18]

Table 8 also reports the consumption-equivalent welfare benefit, assuming optimal behavior, from expanding the portfolio menu to include world equity. Following Corollary 1, we calculate the increase in generalized wealth afforded by the opportunity to invest in the (orthogonalized) world equity fund. Using Proposition 3, we translate this increase in generalized wealth into a consumption-equivalent welfare gain. Throughout Table 8, we assume that domestic output innovations have a correlation of .36 with domestic equity returns and .24 with world equity returns. Other statistics for risky asset returns are fit to the sample average values in Table 2.

The optimal portfolios exhibit very large foreign equity positions and large foreign shares when relative risk aversion equals 3 (or other moderate value). The theoretically optimal position in the world equity fund exceeds one hundred thousand dollars per person in 11 of the 14 countries. The world equity fund accounts for 76 percent of total equity holdings for the representative investor in the average country.[19]

Both theoretical predictions – large foreign equity positions and large foreign shares – are very much at odds with actual portfolio behavior. Previous work on the home bias puzzle emphasizes the theory's inconsistency with well-established evidence on portfolio shares. Like Baxter and Jermann (1997), our theoretical exercise accounts for the higher correlation of output with domestic equity returns than for-

[18] See Davis and Willen (2000b) for a detailed development of this sequential approach to optimal portfolio construction in the CARA-normal framework..

[19] Since the world equity fund is a value-weighted average of domestic equity funds, the world equity shares reported in Table 2 overstate the foreign equity shares. This effect is nontrivial only for the United States.

eign equity returns. Taking this fact into account raises the share of foreign equity in the optimal portfolio, which worsens the home bias puzzle.

Table 8 also shows large welfare gains from expanding the portfolio choice menu to include world equity. The gain equals 28 percent of consumption for the representative investor in the average country, when relative risk aversion is 3. This welfare calculation is predicated upon theoretically optimal behavior before and afteI4T5O1bL3Ix5TAIns from ex

8 Equilibrium Gains to Trade in Equities

We now describe the gains-to-trade puzzle and its relationship to the equity premium puzzle. The welfare calculations in this section, unlike the ones in Section 7, account for the impact of a change in trade regime on risky asset prices. For simplicity, we consider the benefits of foreign access to domestic equity when domestic investors do not hold foreign equity. This simple case makes clear that the gains we identify do not rest on international diversification by domestic investors.

8.1 The Gains-to-Trade Puzzle

To calculate the equilibrium gains to trade in domestic equity, we apply Corollary 3. If we interpret domestic equity returns as the outcome of a trade regime with free access to domestic equity by foreign investors, then Corollary 3 instructs us to measure the per capita equilibrium gains to trade (for the domestic economy) as

$$\frac{aA^h \operatorname{var}\left(\tilde{R}\right)}{2R_f}\left[\begin{matrix}\text{Desired Exposure of Domestic} \\ \text{Residents to Own Equity}\end{matrix} - \begin{matrix}\text{Endowed Exposure of Domestic} \\ \text{Residents to Own Equity}\end{matrix}\right]^2, \tag{28}$$

where A^h is the absolute risk aversion of the representative domestic investor. This expression is numerically identical to the calculation in (27), so we can refer back to Table 7.A for the results. In our preferred case with $\operatorname{corr}(y, R) = .36$ and relative risk aversion of 3, the benefits of foreign access to domestic equity amount to 31 percent of consumption in the average country.

If, instead, we interpret domestic asset returns as the outcome of an autarky regime, then Corollary 3 instructs us to measure the equilibrium gains to trade as

$$\frac{aA^h \operatorname{var}\left(\tilde{R}\right)}{2R_f}\left[\frac{A^W}{A^h}\left(\frac{e^W}{H^W} + \overline{\Psi\beta}^W\right) - \frac{1}{a}\frac{ER}{A^h \operatorname{var}\left(\tilde{R}\right)}\right]^2 = \tag{29}$$

$$\frac{aA^h \operatorname{var}\left(\tilde{R}\right)}{2R_f}\left[\left(\frac{A^W}{A^h}\right)\begin{matrix}\text{Endowed Exposure of World} \\ \text{Investor to Domestic Equity}\end{matrix} - \begin{matrix}\text{Desired Exposure of Domestic} \\ \text{Residents to Own Equity}\end{matrix}\right]^2$$

This expression involves two new terms: A^W, the absolute risk aversion of the "average" world investor, and the world investor's endowed exposure to domestic equity.

In line with the aggregation analysis in section 2.4, we calculate absolute risk aversion for the world investor as the harmonic mean of national risk aversion levels: $A^W = N\left[\sum_h (N_h/A^h)\right]^{-1}$, where h indexes the countries in our sample, N_h is population in country h, and $N = \sum_h N_h$. In performing this calculation, we assume relative risk aversion of 3 in each country and compute the absolute risk aversion coefficients, A^h, as before. To determine the world investor's endowed exposure to domestic equity, we first measure real world output in domestic currency units by aggregating nominal output across countries at market exchange rates and deflating by the domestic price deflator. Then, following the same procedure as before, we regress per capita real world output on current and lagged domestic equity returns.

Table 9 reports the welfare results based on equation (29). In particular, column (6) reports the equilibrium gains to trade in domestic equity using a relative risk aversion of 3 to compute desired exposure. The gains amount to 32 percent of consumption for the average country, nearly the same value as we obtained in Table 7 under a different maintained assumption about the trade regime. Thus, the calculations show that the theory implies enormous domestic gains to trade in domestic equity for moderate risk aversion levels. This conclusion follows regardless of our maintained assumption about the existing trade regime for domestic equity.[20]

We interpret these enormous gains as a falsification of the theory, not as an accurate measure of the true gains to trade in domestic equity. To see how the theory fails, consider the nature of the welfare calculations. For both (28) and (29), the endowed exposure values generated by the output data and their covariances with equity returns are typically much smaller than the desired exposure values generated by the price of risk (Sharpe ratio) and a moderate risk aversion value.[21] As shown in Table 6.B, desired exposure greatly exceeds endowed exposure in most countries even under the extreme, and counterfactual, assumption of perfect correlation between output innovations and domestic equity returns.

Thus to resolve the gains-to-trade puzzle, the theory must be modified so as to increase endowed exposure per investor, reduce desired exposure per investor, or both. Limited participation models that involve a concentration of equity holdings relative to other forms of wealth tend to raise endowed exposure per investor.[22] Alternative

[20]The intermediate calculations in Table 9 highlight other noteworthy aspects of the data. First, the standard deviations of world output changes are much larger than the standard deviation of PPP-adjusted world output changes reported in Table 1. Second, the correlation between world output changes and domestic equity returns varies widely among countries, and so does the world investor's endowed exposure to domestic equity. These features of the data reflect a major role for real exchange rate movements, which drive much of the variation in world output when measured in units of the local consumption good at market exchange rates.

[21]Table 9 reports the endowed and desired exposure values used to calculate (29). Tables 5.A and 6.A report the endowed and desired exposure values used to calculate (28).

[22]We demonstrate this point in the context of our model in Davis, Nalewaik and Willen (2000), but essentially the same point has been made by other researchers who consider the equity premium in models of limited participation. See, for example, Campbell (1999b), Heaton and Lucas (1999)

preference specifications that involve greater aversion to risky consumption streams imply lower desired exposure per investor. Hence, our welfare calculations point to limited participation and higher effective risk aversion as two natural candidates for a resolution of the gains-to-trade puzzle. Models of limited participation and high risk aversion are prominent in work on the equity premium puzzle.

8.2 Connection to the Equity Premium Puzzle

To spell out the connection between the gains-to-trade and equity premium puzzles, we must specify a trade regime for domestic equity. The trade regime determines the form of the equilibrium asset pricing condition and the proper expression for the equilibrium gains to trade. For convenience, we focus on the autarky case. That is, we interpret asset market data as the outcome of a regime in which only domestic investors participate in domestic equity markets.[23] Most empirical investigations of the consumption-based asset pricing model implicitly consider the autarky case, because they relate domestic consumption growth to domestic asset returns.

Under autarky, asset market equilibrium requires that the domestic investor's endowed exposure to domestic equity equals his desired exposure. This equilibrium requirement gives rise to the asset pricing condition in Corollary 2. When we fit this condition to the data and specify a reasonable level of risk aversion, the theory delivers a much smaller excess return on equity than the data. This gap between theory and data constitutes the equity premium puzzle. Table 7.A expresses the gap in welfare terms using equation (27).

Equation (29) expresses the domestic gains to international trade in domestic equity. This welfare expression involves the endowed exposure to domestic equity for the *world* investor. *Hence, the welfare expressions (27) for the equity premium puzzle and (29) for the gains-to-trade puzzle are distinct.* The former involves the relationship between desired and endowed exposure for the domestic investor, and the latter involves the relationship of the domestic investor's desired exposure to the world investor's endowed exposure.

As a final exercise, we calculate the equilibrium gains to trade in domestic equity for risk aversion values that fit the autarky asset pricing condition in Corollary 2. This exercise has at least three motivations. First, we showed earlier that the theory implies implausibly large gains to trade, because desired exposure is so much greater

and Vissing-Jorgenson (1999). Poterba (2000), for example, provides evidence that equity holdings are much more highly concentrated than overall wealth holdings.

[23]The analysis below can be recast in similar terms when equity returns data are interpreted as the outcome of a free trade regime.

than endowed exposure. By reducing desired exposure, higher risk aversion values may generate more plausible measures of the gains to trade. Second, many proposed resolutions of the equity premium puzzle effectively boil down to high risk aversion. Third, the exercise is a crude test of over-identifying restrictions. Essentially, we are asking whether the data can fit a standard asset pricing condition while, at the same time, giving sensible values for the gains to international trade in domestic equity.

Table 9 reports the results. Column (9) gives the equilibrium gains to trade, and columns (7) and (8) report selected intermediate calculations.[24] As column (9) shows, the theory fares poorly by this test. The theoretical gains are modest for a few countries but huge for others, and the average gains are even larger than we obtained for relative risk aversion of 3.[25]

Hence, the gains-to-trade puzzle cannot be resolved by appealing to risk aversion values that rationalize the standard asset pricing condition for equity. It is worth remarking, however, that the theory fails in a different way when we constrain the risk aversion parameters to fit the asset pricing conditions. In particular, desired exposure no longer consistently exceeds endowed exposure, as seen by a comparison of columns (3) and (8) in Table 9.

9 Summary and Concluding Remarks

This paper develops a dynamic analysis of international trade in risky assets when financial markets are incomplete. The analysis can tractably handle many agents and countries, many assets and arbitrary trade regimes for risky assets. We show how to construct optimal portfolios, calculate the benefits of an expanded portfolio choice menu, express the equity premium puzzle in welfare terms and quantify the equilibrium gains to international trade in risky financial assets. We implement these calculations using data on output, equity returns and stock market capitalization for fourteen countries.

[24] See Table 7, column headed "0.36", for the relative risk aversion values implied by fitting the autarky asset pricing condition.

[25] Lewis (2000) also finds large gains to international risk sharing when she chooses risk aversion to rationalize mean equity returns. Table 5 in her study reports welfare gains that range from 7 to 90 percent of consumption, depending on country and exactly how she fits preferences to mean returns. However, her exercise differs from ours in a key respect. She calculates the gains of moving from autarky to full international risk sharing in a setting where equity is a claim to GDP. In contrast, we calculate the gains of moving from autarky to free international trade in domestic equity in a setting where equity returns are imperfectly correlated with GDP. In practice, trade in domestic equity affords quite limited possibilities for international risk sharing, as implied by the low R^2 values in our regressions of output innovations on domestic equity returns in Section 5.4.

We summarize our main findings and results as follows:

- Households are implicitly endowed with a sizable long position in domestic (and foreign) equity, even when they have zero portfolio holdings. This implicit position, which is many times larger than the market value of domestic equity, arises from the positive covariance between domestic output innovations and equity returns.

- For an investor who can borrow at the risk-free rate, and who has a reasonable degree of risk aversion, the theoretically optimal holdings of domestic and foreign equity dwarf actual holdings.

- The welfare cost of failing to adopt the theoretically optimal position in domestic equity are enormous, amounting to roughly 30 percent of consumption for the average country in our preferred specification. This figure is a consumption-equivalent welfare measure for the size of the equity premium puzzle.

- The theory also implies large optimal positions in foreign equity, much larger than actual positions. In a welfare sense, the biggest discrepancy between theory and evidence involves the level of risky asset holdings, not the portfolio shares emphasized by the home bias literature.

- The theory implies enormous domestic gains to international trade in domestic equity. This gains-to-trade puzzle holds regardless of whether we interpret equity returns data as the outcome of a free trade or an autarky regime.

- The gains-to-trade puzzle is not resolved by appealing to high risk aversion values that rationalize the standard asset pricing condition for equity.

Since the model fails so spectacularly along certain dimension, it is natural to ask whether the basic theory can be repaired so as to better fit the data and provide a reliable tool for evaluating the gains to trade in risky financial assets. We identified two natural directions for improving model performance – preference specifications that involve strong aversion to risky consumption streams and limited participation in risky asset markets. Research on the equity premium puzzle has actively pursued both directions in recent years.

Our analysis of the equilibrium gains to trade is not favorable to the first approach. In particular, the model does not deliver plausible values for the gains to trade when fit to the high risk aversion values that satisfy equity pricing conditions. This result can be interpreted as a rejection of the model's over-identifying restrictions.

At a broader level, this result highlights a serious problem with "resolutions" to asset pricing puzzles that rely on high risk aversion. Preference specifications with strong aversion to consumption risk may improve the fit of standard asset pricing conditions, but they also imply large gains to trade in risky assets. We think these gains are much too large to be believed, at least for the sample considered in this paper. The implausible magnitude of these gains argues against explanations for asset pricing anomalies that rest solely or mainly on high risk aversion.

Models with limited participation in risky asset markets offer greater promise in our view. Casual inspection of the real world reveals that households cannot borrow at the risk-free interest rate. Hence, most households lack the financial capacity or incentive to adopt the large equity positions implied by the theory, as it is formulated in this paper. Information and transaction costs further limit participation in risky asset markets.

The concentration of equity holdings (relative to other forms of wealth) helps resolve the equity premium puzzle, as others have shown. Concentrated equity holdings also mitigate the gains-to-trade puzzle by raising endowed exposure per investor, thereby bringing it closer to desired exposure and reducing the gains to trade per investor. In addition, limited participation reduces the aggregate gains to trade by precluding many households from (fully) exploiting the benefits of portfolio investments. Whether limited participation models can simultaneously fit the data on asset returns and reliably address the gains to trade in risky assets is an important question for future research.

Our analysis also points to other interesting directions for research. In this paper, we have restricted attention to simple trade regimes, small portfolio choice menus, uniform investor risk exposures within countries and uniform portfolio choice menus within countries. These restrictions can be relaxed without loss of tractability.[26] The main costs to generalizing the model in these directions are greater notational complexity and the need for richer data sets to empirically implement the model.

[26] There are no computational barriers to expanding the number of assets or countries in the framework of this paper, although large numbers of assets (or complex trade regimes) make it convenient to recast the analysis in matrix algebra terms. For a treatment of non-uniform participation in asset markets within countries, see the NBER working paper version of this study. For a treatment of heterogeneity in endowed exposures to risky financial assets within the United States, see Davis and Willen (2000a).

10 Appendix: Proofs for Section 4

Lemma 1 *Let* $\{\tilde{x}_s\}_{s=t}^{\infty}$ *be a stochastic process such that* $\mathrm{PDV}_t\left(\{\tilde{x}_s\}_{s=t}^{\infty}\right) < \infty \ \ \forall t$*, and let* $\{k_s\}_{s=t+1}^{\infty}$ *be a deterministic process such that* $\mathrm{PDV}_t\left(\{k_s\}_{s=t+1}^{\infty}\right) < \infty$*. Then*

$$\tilde{x}_t = a\,\mathrm{PDV}_t\left(\{\tilde{x}_s\}_{s=t}^{\infty}\right) - \mathrm{PDV}_t\left(\{k_s\}_{s=t+1}^{\infty}\right), \tag{30}$$

if and only if

$$\mathrm{E}_t\left(\tilde{x}_{t+1}\right) = \tilde{x}_t + k_{t+1}.$$

Proof of Lemma 1: "$\Leftarrow$" If $\mathrm{E}_t\left(\tilde{x}_{t+1}\right) = \tilde{x}_t + k_{t+1}$, then forward substitution yields $\mathrm{PDV}_t\left(\{\tilde{x}_s\}_{s=t}^{\infty}\right) = \mathrm{PDV}_t\left(\{1\}_{s=t}^{\infty}\right)\left[\tilde{x}_t + \mathrm{PDV}_t\left(\{k_s\}_{s=t+1}^{\infty}\right)\right]$. Since $a^{-1} = \mathrm{PDV}_t\left(\{1\}_{s=t}^{\infty}\right)$, $\tilde{x}_t = a\,\mathrm{PDV}_t\left(\{\tilde{x}_s\}_{s=t}^{\infty}\right) - \mathrm{PDV}_t\left(\{k_s\}_{s=t+1}^{\infty}\right)$.

"$\Rightarrow$" Equation (30) implies that

$$\frac{\tilde{x}_t}{R_f - 1} = \mathrm{PDV}_t\left(\{\tilde{x}_s\}_{s=t+1}^{\infty}\right) - \frac{R_f}{R_f - 1}\mathrm{PDV}_t\left(\{k_s\}_{s=t+1}^{\infty}\right) \tag{31}$$

First, moving forward one period, equation (31) implies

$$\frac{\tilde{x}_{t+1}}{R_f - 1} = \mathrm{PDV}_{t+1}\left(\{\tilde{x}_s\}_{s=t+2}^{\infty}\right) - \frac{R_f}{R_f - 1}\mathrm{PDV}_{t+1}\left(\{k_s\}_{s=t+2}^{\infty}\right) \tag{32}$$

By the law of iterated expectations and the definition of the PDV operator, (32) implies

$$\mathrm{PDV}_t\left(\{\tilde{x}_s\}_{s=t+2}^{\infty}\right) = \frac{\mathrm{E}_t\left(\tilde{x}_{t+1}\right)}{R_f\left(R_f - 1\right)} + \frac{R_f}{R_f - 1}\mathrm{PDV}_t\left(\{k_s\}_{s=t+2}^{\infty}\right) \tag{33}$$

Second, using (31) and the definition of the PDV operator,

$$\frac{\tilde{x}_t}{R_f - 1} = \frac{\mathrm{E}_t\left(\tilde{x}_{t+1}\right)}{R_f} + \mathrm{PDV}_t\left(\{\tilde{x}_s\}_{s=t+2}^{\infty}\right) - \frac{R_f}{R_f - 1}\mathrm{PDV}_t\left(\{k_s\}_{s=t+1}^{\infty}\right) \tag{34}$$

Combining equations (33) and (34), we get

$$\frac{\tilde{x}_t}{R_f - 1} = \frac{\mathrm{E}_t\left(\tilde{x}_{t+1}\right)}{R_f} + \frac{\mathrm{E}_t\left(\tilde{x}_{t+1}\right)}{R_f\left(R_f - 1\right)} + \frac{R_f}{R_f - 1}\mathrm{PDV}_t\left(\{k_s\}_{s=t+2}^{\infty}\right) - \frac{R_f}{R_f - 1}\mathrm{PDV}_t\left(\{k_s\}_{s=t+1}^{\infty}\right)$$

Simplifying gives $\tilde{x}_t = \mathrm{E}_t\left(\tilde{x}_{t+1}\right) - k_{t+1}$. $\square$

Proof of Proposition 1: To ease the notational burden, we suppress the index h in this proof. To show that our proposed solution solves the individual's program, we show that it satisfies the following five conditions:

§1. Period budget constraint: $\tilde{c}_s + \tilde{\omega}_{f,s} + \tilde{\omega}_s = \tilde{y}_s + \tilde{\omega}_{f,s-1}R_f + \tilde{R}_s\tilde{\omega}_{s-1}$.

§2. Transversality condition: $\lim_{s\to\infty}\mathrm{E}_t\left[\tilde{\omega}_{f,s} + \tilde{\omega}_s\right]/R_f^s = 0$

§3. Boundedness: $\mathrm{PDV}_t\left(\{\tilde{c}_s\}_{s=t}^{\infty}\right) < \infty$ and bounded value function.
§4. Euler equation for the riskless asset: $\mathrm{E}\left(\tilde{c}_{t+1}\right) - c_t = \frac{A}{2}\,\mathrm{var}\left(\tilde{c}_{t+1}\right) + \frac{1}{A}\ln(R_f\delta)$.
§5. Euler equation for the risky asset: $A\,\mathrm{cov}_t\left(\tilde{c}_{t+1}, \tilde{R}_{t+1}\right) = ER_t$.

To prove that these conditions hold, we use Lemma 1 and three additional facts:

Fact 1: $ER_t\omega_t = (1/aA)\,\mathrm{E}\left(\tilde{S}\right)^2 - \Psi\,\mathrm{E}\left(\tilde{S}\right)\mathrm{cov}\left(\tilde{\eta}_{t+1}, \tilde{S}_{t+1}\right)$ for all t.

Proof: Using the proposed solution for ω_t, we have

$$ER_t\omega_t \;=\; \frac{1}{aA}\left(\frac{ER_t}{\sigma_t\left(\tilde{R}_{t+1}\right)}\right)^2 - \Psi\,\mathrm{cov}_t\left(\tilde{\eta}_{t+1}, \frac{\tilde{R}_{t+1}}{\sigma_t\left(\tilde{R}_{t+1}\right)}\right)\frac{ER_t}{\sigma_t\left(\tilde{R}_{t+1}\right)}.$$

Fact 1 now follows by substituting $ER_t = \mathrm{E}\left(\tilde{S}\right)\sigma_t\left(\tilde{R}_{t+1}\right)$ on the right side and simplifying. □

Fact 2: $\mathrm{var}_t\left(\Psi\tilde{\eta}_{t+1} + \omega_{t+1}\tilde{R}_{t+1}\right) = \Psi^2\left[\mathrm{var}\left(\tilde{\eta}_{t+1}\right) - \mathrm{cov}\left(\tilde{\eta}_{t+1}, \tilde{S}_{t+1}\right)^2\right] + \mathrm{E}\left(\tilde{S}_{t+1}\right)^2/(aA)^2$

Proof: Using the proposed solution for ω_t, we have

$$\mathrm{var}_t\left(\Psi\tilde{\eta}_{t+1} + \omega_{t+1}\tilde{R}_{t+1}\right) =$$
$$\Psi^2\,\mathrm{var}_t\left(\tilde{\eta}_{t+1} - \frac{\mathrm{cov}_t\left(\tilde{\eta}_{t+1}, \tilde{R}_{t+1}\right)}{\mathrm{var}_t\left(\tilde{R}_{t+1}\right)}\tilde{R}_{t+1}\right) + \mathrm{var}_t\left(\frac{1}{aA}\frac{ER_t}{\mathrm{var}_t\left(\tilde{R}_{t+1}\right)}\tilde{R}_{t+1}\right)$$

Since the the first expression on the right hand side is a projection, we can re-write this equation as

$$\mathrm{var}_t\left(\Psi\tilde{\eta}_{t+1} + \omega_{t+1}\tilde{R}_{t+1}\right) =$$
$$\Psi^2\left[\mathrm{var}\left(\tilde{\eta}_{t+1}\right) - \mathrm{cov}_t\left(\tilde{\eta}_{t+1}, \frac{\tilde{R}_{t+1}}{\sigma_t\left(\tilde{R}_{t+1}\right)}\right)^2\right] + \left(\frac{1}{aA}\right)^2\left(\frac{ER_t}{\sigma_t\left(\tilde{R}_{t+1}\right)}\right)^2$$

Using $\mathrm{E}\left(\tilde{S}\right)\sigma_t\left(\tilde{R}_{t+1}\right) = ER_t$, this reduces to

$$\mathrm{var}_t\left(\Psi\tilde{\eta}_{t+1} + \omega_{t+1}\tilde{R}_{t+1}\right) = \Psi^2\left(\mathrm{var}\left(\tilde{\eta}_{t+1}\right) - \mathrm{cov}\left(\tilde{\eta}_{t+1}, \tilde{S}_{t+1}\right)^2\right) + \frac{1}{(aA)^2}\,\mathrm{E}\left(\tilde{S}\right)^2$$

□

Fact 3: $\tilde{GW}_{t+1} - \mathrm{E}_t\left(\tilde{GW}_{t+1}\right) = \Psi\tilde{\eta}_{t+1} + \left(\tilde{R}_{t+1} - \mathrm{E}_t\left(\tilde{R}_{t+1}\right)\right)\omega_t$

Proof: By Facts 1 and 2, the only stochastic components of $\tilde{GW}_{t+1}$ are $\tilde{\eta}_{t+1}$ and $\tilde{R}_{t+1}$. Thus only innovations to the first and third terms of $\tilde{GW}_{t+1}$ matter. □

To complete the proof of Proposition 1, we now establish conditions §1 to §5:

§1. Direct substitution verifies that the proposed solution satisfies the period budget constraint.

§2. Consider a finite-horizon version of the optimization problem. Let the horizon go to infinity and obtain the infinite-horizon limit to the finite-horizon intertemporal budget constraint,

$$\begin{aligned}\mathrm{PDV}_t\left(\{\tilde{c}_s\}_{s=t}^{\infty}\right) = \mathrm{PDV}_t\left(\{\tilde{y}_s\}_{s=t}^{\infty}\right) + R_f\omega_{f,t-1} + R_t\omega_{t-1}\\ + \mathrm{PDV}_t\left(\{ER_s\omega_{s-1}\}_{s=t+1}^{\infty}\right), \quad (35)\end{aligned}$$

where the proposed solution for the consumption path, $\{\tilde{c}_s\}_{s=t}^{\infty}$, is the infinite-horizon limit to the finite-horizon solution. Since our proposed solution satisfies (35), and (35) implies that the transversality condition holds, our proposed solution satisfies the transversality condition.

§3. By condition 5, $\mathrm{PDV}_t\left(\{\tilde{y}_s\}_{s=t}^{\infty}\right) < \infty$. By Fact 1, $\mathrm{PDV}_t\left(\{ER_s\omega_{s-1}\}_{s=t+1}^{\infty}\right) < \infty$. Hence, the right side of (35) is finite $\Rightarrow \mathrm{PDV}_t\left(\{\tilde{c}_s\}_{s=t}^{\infty}\right) < \infty \Rightarrow$ that the value function is bounded.

§4. By definition of the PDV operator,$[1/(R_f - 1)]\,(1/aA)\ln(R_f\delta) =$ $\mathrm{PDV}_t\left(\{(1/aA)\ln(R_f\delta)\}_{s=t+1}^{\infty}\right)$. By equation (35),

$$c_t = aGW_t = a\left[\begin{array}{c}\mathrm{PDV}_t\left(\{\tilde{c}_s\}_{s=t}^{\infty}\right) + \mathrm{PDV}_t\left(\{(1/aA)\ln(R_f\delta)\}_{s=t+1}^{\infty}\right)\\ + \mathrm{PDV}_t\left(\left\{(aA/2)\,\mathrm{var}\left(\tilde{GW}_{t+1}\right)\right\}_{s=t+1}^{\infty}\right)\end{array}\right]$$

By Fact 2, $\mathrm{var}\left(\tilde{GW}_{t+1}\right)$ is nonstochastic, so the stochastic process for consumption satisfies Lemma 1. Thus $\mathrm{E}_t\left(\tilde{c}_{t+1}\right) - c_t = (a^2A/2)\,\mathrm{var}\left(\tilde{GW}_{t+1}\right) + (1/A)\ln(R_f\delta)$. But $(a^2A/2)\,\mathrm{var}\left(\tilde{GW}_{t+1}\right) = (A/2)\,\mathrm{var}\left(\tilde{c}_{t+1}\right)$, which shows that the Euler equation for the risk-free asset holds.

§5. Using the proposed solution for consumption,

$$\mathrm{cov}\left(\tilde{c}_{t+1}, \tilde{R}_{t+1}\right) = a\,\mathrm{cov}\left(\tilde{GW}_{t+1}, \tilde{R}_{t+1}\right).$$

By Fact 3, we can rewrite the right side as

$$
\begin{aligned}
&= a\,\mathrm{cov}\left(\Psi\tilde{\eta}_{t+1},\tilde{R}_{t+1}\right)+a\,\mathrm{cov}\left(\tilde{R}_{t+1}\omega_t,\tilde{R}_{t+1}\right)\\
&= a\,\mathrm{cov}\left(\Psi\tilde{\eta}_{t+1},\tilde{R}_{t+1}\right)+\frac{1}{A}ER_{t+1}-a\,\mathrm{cov}\left(\Psi\tilde{\eta}_{t+1},\tilde{R}_{t+1}\right)
\end{aligned}
$$

thus satisfying the Euler equation with respect to the risky asset.

□

Proof of Corollary 1: Fact 1 establishes the first part of the corollary. As regards the second part, Facts 1 and 2 imply

$$
\begin{aligned}
ER_t\omega_{t-1}-\frac{aA}{2}\mathrm{var}_t\left(\Psi\tilde{\eta}_{t+1}+\omega_{t+1}\tilde{R}_{t+1}\right) &= -\frac{aA}{2}(\Psi)^2\,\mathrm{var}\left(\tilde{\eta}\right)\\
&+\frac{1}{2aA}\mathrm{E}\left(\tilde{S}\right)^2-\Psi\,\mathrm{E}\left(\tilde{S}\right)\mathrm{cov}\left(\tilde{\eta},\tilde{S}\right)+\frac{(aA)^2}{2}(\Psi)^2\,\mathrm{cov}\left(\tilde{\eta},\tilde{S}\right)^2
\end{aligned}
$$

The last three terms on the right hand side can be simplified to get

$$
\begin{aligned}
ER_t\omega_{t-1}-\frac{aA}{2}\mathrm{var}_t\left(\Psi\tilde{\eta}_{t+1}+\omega_{t+1}\tilde{R}_{t+1}\right) &= -\frac{aA}{2}\Psi^2\,\mathrm{var}\left(\tilde{\eta}\right)\\
&+\frac{1}{2aA}\left[\mathrm{E}\left(\tilde{S}\right)-\frac{aA}{2}\Psi\,\mathrm{cov}\left(\tilde{\eta},\tilde{S}\right)\right]^2
\end{aligned}
$$

Thus:

$$
\begin{aligned}
&\mathrm{PDV}_t\left(\{ER_{s-1}\omega_{s-1}\}_{s=t+1}^{\infty}\right)-\mathrm{PDV}_t\left(\left\{(aA/2)\,\mathrm{var}\left(\Psi\tilde{\eta}_s+\omega_{s-1}\tilde{R}_s\right)\right\}_{s=t+1}^{\infty}\right)\\
&\quad=-\mathrm{PDV}_t\left(\left\{\frac{aA}{2}\Psi^2\,\mathrm{var}\left(\tilde{\eta}\right)\right\}_{s=t+1}^{\infty}\right)+\\
&\quad\mathrm{PDV}_t\left(\left\{\frac{1}{2aA}\left[\mathrm{E}\left(\tilde{S}\right)-\frac{aA}{2}\Psi\,\mathrm{cov}\left(\tilde{\eta},\tilde{S}\right)\right]^2\right\}_{s=t+1}^{\infty}\right)\\
&\quad=-\frac{A}{2R_f}\Psi^2\,\mathrm{var}\left(\tilde{\eta}\right)+\frac{1}{R_f}\left(\frac{1}{2aA}\left[\mathrm{E}\left(\tilde{S}\right)-\frac{aA}{2}\Psi\,\mathrm{cov}\left(\tilde{\eta},\tilde{S}\right)\right]^2\right) \qquad (36)
\end{aligned}
$$

which yields the desired expression for generalized wealth. □

Proof of Proposition 2: We first show that the pricing relation π_t in the first part of the proposition implies Condition 5 and the Sharpe ratio given by the second part of the proposition. Then we show that the proposed solution for π_t satisfies the market-clearing condition for the risky asset.

Let $\Gamma = (A^*/R_f)\left(\operatorname{cov}\left(\overline{\Psi\tilde{\eta}}, \Lambda\tilde{x}\right) + \Lambda^2\sigma_x^2\frac{\phi}{H}\right)$. We have

$$\tilde{R}_{t+1} = \frac{\tilde{\pi}_{t+1} + \tilde{d}_{t+1}}{\pi_t} = \frac{1}{\pi_t}\left(\mathrm{PDV}_{t+1}\left(\left\{\tilde{d}_s\right\}_{s=t+1}^{\infty}\right) - \Gamma\right),$$

so that $\tilde{R}_{t+1} - \mathrm{E}_t\left(\tilde{R}_{t+1}\right) = \Lambda\tilde{x}/\pi_t$. Hence,

$$\mathrm{std}_t\left(\tilde{R}_{t+1}\right) = \frac{\Lambda\sigma_x}{\pi_t}. \tag{37}$$

Now calculate the realized excess return:

$$\begin{aligned} \tilde{XR}_{t+1} &= (R_f/\pi_t)\left[\frac{\tilde{\pi}_{t+1} + \tilde{d}_{t+1}}{R_f} - \pi_t\right] \\ &= (R_f/\pi_t)\left[\frac{\mathrm{PDV}_{t+1}\left(\left\{\tilde{d}_s\right\}_{s=t+1}^{\infty}\right) - \Gamma}{R_f} - \mathrm{PDV}_t\left(\left\{\tilde{d}_s\right\}_{s=t+1}^{\infty}\right) + \Gamma\right] \end{aligned} \tag{38}$$

Recognizing that $\{\tilde{d}_s\}$ is an ARMA process, and using the definition of the PDV operator, we have $(1/R_f)\,\mathrm{PDV}_{t+1}\left(\left\{\tilde{d}_s\right\}_{s=t+1}^{\infty}\right) - \mathrm{PDV}_t\left(\left\{\tilde{d}_s\right\}_{s=t+1}^{\infty}\right) = \Lambda\tilde{x}_{t+1}$. Thus equation (38) implies

$$\tilde{XR}_{t+1} = \frac{\Lambda\tilde{x}_{t+1}}{\pi_t} + aA^*\left[\frac{A^*}{R_f}\left(\operatorname{cov}\left(\overline{\Psi\tilde{\eta}}, \frac{\Lambda\tilde{x}}{\pi_t}\right) + \operatorname{var}\left(\frac{\Lambda\tilde{x}}{\pi_t}\right)\frac{\pi_t\phi}{H}\right)\right]. \tag{39}$$

Dividing through by equation (37) yields

$$\frac{\tilde{XR}_{t+1}}{\mathrm{std}_t\left(\tilde{R}_{t+1}\right)} = \frac{\tilde{x}_{t+1}}{\sigma_x} + aA^*\left[\operatorname{cov}\left(\overline{\Psi\tilde{\eta}}, \frac{\tilde{x}}{\sigma_x}\right) + \Lambda\frac{\phi}{H}\sigma_x\right] \tag{40}$$

Computing expectations on both sides of (40) with respect to information at time t yields the Sharpe ratio given in the proposition. Computing the time-t conditional covariance with respect to η_{t+1} on both sides of (40) establishes Condition 5.

To finish the proof, we need only show that the proposed solution for the price π_t clears the market. Since the proposed solution satisifes Condition 5, Proposition 1 gives the demand for the risky asset by h at the proposed solution:

$$\omega_t^h = (aA^h)^{-1}\frac{ER_t}{\mathrm{var}_t\left(\tilde{R}_{t+1}\right)} - \Psi^h\frac{\mathrm{cov}_t\left(\tilde{\eta}_{t+1}^h, \tilde{R}_{t+1}\right)}{\mathrm{var}_t\left(\tilde{R}_{t+1}\right)}. \tag{41}$$

Taking expectations of equation (39),

$$ER_t = aA\left[\frac{A}{R_f}\left(\operatorname{cov}\left(\overline{\Psi\tilde{\eta}}, \frac{\Lambda\tilde{x}}{\pi_t}\right) + \operatorname{var}\left(\frac{\Lambda\tilde{x}}{\pi_t}\right)\frac{\pi_t\phi}{H}\right)\right]. \tag{42}$$

Substituting equation (42) into equation (41) and taking averages across consumers gives:

$$\frac{1}{H}\sum_{h\in H}\omega_t^h = \operatorname{cov}\left(\overline{\Psi\tilde{\eta}}, \frac{\Lambda\tilde{x}}{\pi_t}\right) + \frac{\pi_t\phi}{H} - \operatorname{cov}\left(\overline{\Psi\tilde{\eta}}, \frac{\Lambda\tilde{x}}{\pi_t}\right) = \frac{\pi_t\phi}{H} = \frac{e_t}{H}$$

That is, per capita demand equals per capita supply. □

Proof of Corollary 2: Take the investor asset demand equation from Proposition 1 and sum across participants:

$$\sum_{h\in H^p}\omega_t^h = \sum_{h\in H^p}\frac{1}{aA^h}\left[(aA^h)^{-1}\frac{ER_t}{\operatorname{var}_t\left(\tilde{R}_{t+1}\right)} - \Psi^h\frac{\operatorname{cov}_t\left(\tilde{\eta}_{t+1}^h, \tilde{R}_{t+1}\right)}{\operatorname{var}_t\left(\tilde{R}_{t+1}\right)}\right] \tag{43}$$

In equilibrium, $\sum_{h\in H}\omega_t^h = e_t$, so equation (43) can be re-written as

$$e_t \operatorname{var}_t\left(\tilde{R}_{t+1}\right) = \sum_{h\in H}\left[\frac{1}{aA^h}ER_t - \Psi^h\operatorname{cov}_t\left(\tilde{\eta}_{t+1}^h, \tilde{R}_{t+1}\right)\right]. \tag{44}$$

Dividing through by H and recalling $1/A^* = \sum_{h\in H}[1/(aA^h)]$ gives the result. □

Proof of Proposition 3: The consumption Euler equation says that $\exp\left(-A^h c_t^h\right) = \delta^h R_f \operatorname{E}_t\left(\exp\left(-A^h\tilde{c}_{t+1}^h\right)\right)$, which implies

$$-\frac{\left(\delta^h\right)^\tau}{A^h}\operatorname{E}_t\left(\exp\left(-A^h\tilde{c}_{t+\tau}^h\right)\right) = -\frac{1}{A^h}\left(\frac{1}{R_f}\right)^\tau \exp\left(-A^h c_t^h\right) \quad \text{for all } \tau > 0.$$

Using this result, we have $U^h\left(C^{*h}\right) = (-1/A^h)\operatorname{PDV}_t\left(\{\mathbf{1}\}_{s=t}^\infty\right)\exp(-A^h c_0^{*h})$. Likewise, $U^h\left(C^h+\theta\right) = (-1/A^h)\operatorname{PDV}_t\left(\{\mathbf{1}\}_{s=t}^\infty\right)\exp\left(-A^h\left(c_0^h+\theta\right)\right)$. Setting $U^h(C+\theta) = U^h(C^*)$ and solving for θ proves the first part of the proposition. The second part follows from the first part and the solution for consumption in Proposition 1. □

Proof of Proposition 4: Denote outcomes under free trade in the domestic risky asset by an asterisk. By Proposition 3, the consumption-equivalent welfare difference between the free trade and autarky regimes is

$$\overline{\Theta} = \frac{1}{H}\sum_h GW_t^{h*} - \frac{1}{H}\sum_h GW_t^h,$$

where the index h runs over domestic investors. Autarky equilibrium requires that $\sum_h \omega_{t-1}^h = e_{t-1}$ and $R_t e_{t-1} = e_t$. If free trade in the domestic risky asset commences at t, then $R_t e_{t-1} = e_t^*$. Recall that, by assumption, income paths and the risk-free rate are invariant across trade regimes. Thus, using the expression for generalized wealth in Corollary 1, we can express the consumption-equivalent welfare difference between trade regimes as

$$\overline{\Theta} = \frac{e_t^* - e_t}{H} + \frac{1}{H}\sum_h (1/2A^h R_f)\left[a^{-1}\,\mathrm{E}\left(\tilde{S^*}\right) - A^h\Psi^h\,\mathrm{cov}\left(\tilde{\eta}^h, \tilde{S^*}\right)\right]^2 - \frac{1}{H}\sum_h (1/2A^h R_f)\left[a^{-1}\,\mathrm{E}\left(\tilde{S}\right) - A^h\Psi^h\,\mathrm{cov}\left(\tilde{\eta}^h, \tilde{S}\right)\right]^2 \quad (45)$$

Now consider the last two terms in equation (45) for any individual h:

$$(1/2A^h R_f)\left[a^{-1}\,\mathrm{E}\left(\tilde{S^*}\right) - A^h\Psi^h\,\mathrm{cov}\left(\tilde{\eta}^h, \tilde{S^*}\right)\right]^2 - (1/2A^h R_f)\left[a^{-1}\,\mathrm{E}\left(\tilde{S}\right) - A^h\Psi^h\,\mathrm{cov}\left(\tilde{\eta}^h, \tilde{S}\right)\right]^2 =$$
$$(1/2R_f)\left[\frac{1}{a^2A^h}\,\mathrm{E}\left(\tilde{S^*}\right)^2 - \frac{2}{a}\,\mathrm{E}\left(\tilde{S^*}\right)\mathrm{cov}\left(\Psi^h\tilde{\eta}^h, \tilde{S^*}\right) + A^h\,\mathrm{cov}\left(\Psi^h\tilde{\eta}^h, \tilde{S^*}\right)^2\right]$$
$$- (1/2R_f)\left[\frac{1}{a^2A^h}\,\mathrm{E}\left(\tilde{S}\right)^2 - \frac{2}{a}\,\mathrm{E}\left(\tilde{S}\right)\mathrm{cov}\left(\Psi^h\tilde{\eta}^h, \tilde{S}\right) + A^h\,\mathrm{cov}\left(\Psi^h\tilde{\eta}^h, \tilde{S}\right)^2\right] \quad (46)$$

By assumption, $\mathrm{cov}\left(\Psi^h\tilde{\eta}^h, \tilde{S^*}\right) = \mathrm{cov}\left(\Psi^h\tilde{\eta}^h, \tilde{S}\right)$, so

$$(1/2A^h R_f)\left[a^{-1}\,\mathrm{E}\left(\tilde{S^*}\right) - A^h\Psi^h\,\mathrm{cov}\left(\tilde{\eta}^h, \tilde{S^*}\right)\right]^2 - (1/2A^h R_f)\left[a^{-1}\,\mathrm{E}\left(\tilde{S}\right) - A^h\Psi^h\,\mathrm{cov}\left(\tilde{\eta}^h, \tilde{S}\right)\right]^2 =$$
$$(1/2R_f)\left[\frac{1}{a^2A^h}\,\mathrm{E}\left(\tilde{S^*}\right)^2 - \frac{2}{a}\,\mathrm{E}\left(\tilde{S^*}\right)\mathrm{cov}\left(\Psi^h\tilde{\eta}^h, \tilde{S}\right) - \frac{1}{a^2A^h}\,\mathrm{E}\left(\tilde{S}\right)^2 + \frac{2}{a}\,\mathrm{E}\left(\tilde{S}\right)\mathrm{cov}\left(\Psi^h\tilde{\eta}^h, \tilde{S}\right)\right] \quad (47)$$

Substituting equation (47) into equation (45) gives:

$$\overline{\Theta} = \frac{e_t^* - e_t}{H}$$
$$+(1/2R_f)\left[\frac{1}{a^2A^*}\,\mathrm{E}\left(\tilde{S^*}\right)^2 - \frac{2}{a}\,\mathrm{E}\left(\tilde{S^*}\right)\mathrm{cov}\left(\overline{\Psi\eta}, \tilde{S}\right) - \frac{1}{a^2A^*}\,\mathrm{E}\left(\tilde{S}\right)^2 + \frac{2}{a}\,\mathrm{E}\left(\tilde{S}\right)\mathrm{cov}\left(\overline{\Psi\eta}, \tilde{S}\right)\right] \quad (48)$$

But by Proposition 2

$$\mathrm{cov}\left(\overline{\Psi\eta}, \tilde{S}\right) = \frac{1}{aA^*}\left(aA^*\left[\mathrm{cov}\left(\overline{\Psi\eta}, \tilde{S}\right) + \Lambda\sigma_x\frac{\phi}{H}\right]\right) - \Lambda\sigma_x\frac{\phi}{H}$$
$$= \frac{1}{aA^*}\,\mathrm{E}\left(\tilde{S}\right) - \Lambda\sigma_x\frac{\phi}{H} \quad (49)$$

Substituting equation (49) into equation (48) and re-arranging gives:

$$\overline{\Theta} = \frac{e_t^* - e_t}{H}$$
$$+(1/2R_f)\left[\frac{1}{a^2A^*}\,\mathrm{E}\left(\tilde{S^*}\right)^2 - \frac{2}{a^2A^*}\,\mathrm{E}\left(\tilde{S^*}\right)\mathrm{E}\left(\tilde{S}\right) + \frac{1}{a^2A^*}\,\mathrm{E}\left(\tilde{S}\right)^2 + \frac{2}{a}\Lambda\sigma_x\frac{\phi}{H}\left(\mathrm{E}\left(\tilde{S^*}\right) - \mathrm{E}\left(\tilde{S}\right)\right)\right] \quad (50)$$

By Proposition 2,

$$\mathrm{E}\left(\tilde{S}\right) = \frac{aR_f}{\Lambda\sigma_x}\left[\mathrm{PDV}_t\left(\left\{\tilde{d}_s\right\}_{s=t}^{\infty}\right) - \pi_t\right],$$

which implies that

$$\left(\mathrm{E}\left(\tilde{S^*}\right) - \mathrm{E}\left(\tilde{S}\right)\right) = \frac{aR_f}{\Lambda\sigma_x}\left[\pi_t - \pi_t^*\right],$$

and

$$\frac{2}{a}\Lambda\sigma_x\frac{\phi}{H}\left(\mathrm{E}\left(\tilde{S^*}\right) - \mathrm{E}\left(\tilde{S}\right)\right) = 2R_f\frac{e_t - e_t^*}{H}. \tag{51}$$

Substituting equation (51) into equation (50) and rearranging gives:

$$\begin{aligned}\overline{\Theta} &= \frac{e_t^* - e_t}{H} + (1/2R_f)\left[\frac{1}{aA^*}\mathrm{E}\left(\tilde{S^*}\right) - \frac{1}{aA^*}\mathrm{E}\left(\tilde{S}\right)\right]^2 + (1/2R_f)2R_f\frac{e_t - e_t^*}{H} \\ &= (1/2R_f)\left[\frac{1}{aA^*}\mathrm{E}\left(\tilde{S^*}\right) - \frac{1}{aA^*}\mathrm{E}\left(\tilde{S}\right)\right]^2 \end{aligned} \tag{52}$$

Substituting in the equilibrium values for $\mathrm{E}\left(\tilde{S^*}\right)$ and $\mathrm{E}\left(\tilde{S}\right)$ from Proposition 2 into equation (52) finishes the proof. □

Proof of Corollary 3: By definition,

$$\pi_t\frac{\phi}{H} = \frac{e_t}{H}. \tag{53}$$

Also, recalling $\tilde{R}_{t+1} - \mathrm{E}_t\left(\tilde{R}_{t+1}\right) = \Lambda\tilde{x}_{t+1}/\pi_t$, it follows that

$$\left(\frac{\Lambda}{\pi_t}\right)^2\sigma_x^2 = \mathrm{var}_t\left(\tilde{R}_{t+1}\right), \quad \text{and} \tag{54}$$

$$\frac{\pi_t}{\Lambda}\frac{\mathrm{cov}\left(\overline{\Psi\eta}, \tilde{x}\right)}{\sigma_x^2} = \overline{\Psi\beta}_{t+1}^h. \tag{55}$$

Using these three equations to substitute into the gains-to-trade expression in Proposition 4 yields equation (21). To get equation (22), use the equilibrium pricing relation in Corollary 2 for the first term inside the parentheses of equation (21). To get equation (23), use the equilibrium pricing relation for the second term inside the parentheses. □

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Table 1: Random Walk Specifications for Per Capita Output
Annual Data, 1971 to 1995

	Natural Units		*Natural Logs*		
	Drift	Std. Dev.	Drift	Std. Dev.	Q-Statistic p-value
Australia	266	409	1.6	2.5	0.40
Austria	344	275	2.6	2.3	0.85
Belgium	280	313	2.1	2.4	0.37
Canada	313	569	1.9	3.3	0.30
Denmark	246	335	1.6	2.3	0.09
France	257	271	1.9	2.0	0.72
Germany, West	344	369	2.3	2.4	0.10
Italy	299	274	2.5	2.3	0.48
Japan	306	310	2.4	2.5	0.24
Netherlands	230	252	1.6	1.8	0.31
Norway	317	379	2.4	2.6	0.30
Sweden	142	467	0.9	2.9	0.14
United Kingdom	278	340	2.1	2.5	0.24
United States	300	432	1.6	2.3	0.27
World (PPP-Adjusted)	302	264	2.0	1.8	0.27

Notes:

1. The table reports the mean drift and the standard deviation of the innovations for random walk specifications fit to per capita real output. Output measures in natural units are expressed in 1990 U.S. dollars based. World output per capita is calculated as the population-weighted average of per capita country-level outputs for countries in the sample. Except for world output at market exchange rates and the U.S. output series, all output series include a 1990 PPP adjustment from the Penn World Tables.

2. The Ljung-Box Q test for autocorrelation is taken out to 6 lags. The p-value is the marginal significance level in a test of the null hypothesis of no serial correlation in the first difference of the output measure. Unreported results for per capita output in natural units are very similar.

Table 2: Summary Statistics for Domestic Real Returns on Domestic and World Equity, Percent Per Year

	Raw Returns, 1971-1995				Orthogonalized World Equity Returns, 1973-94			
	Own Equity		*World Equity*		*Unhedged*		*Hedged*	
Country	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
Australia	7.8	27.9	7.8	20.4	4.0	13.2	4.6	10.6
Austria	6.5	32.1	5.5	21.3	5.7	20.3	7.2	16.2
Belgium	10.6	24.2	6.4	21.5	1.8	14.8	3.4	13.1
Canada	6.3	17.1	9.4	17.7	7.2	12.4	6.4	12.4
Denmark	11.8	35.7	5.9	21.2	3.3	16.8	7.8	15.4
France	8.8	28.0	6.5	20.2	3.3	14.8	4.5	12.2
Germany, West	8.3	25.9	6.2	21.5	3.5	17.7	4.7	13.9
Italy	5.3	36.5	7.0	18.6	6.3	15.0	7.2	13.8
Japan	10.9	30.7	4.3	19.2	2.4	15.7	5.9	13.8
Netherlands	11.2	22.9	6.1	21.3	-2.3	9.1	2.4	12.6
Norway	12.3	49.2	6.6	20.4	6.9	20.5	8.4	17.7
Sweden	14.0	28.1	7.0	19.7	1.7	14.3	5.6	15.7
United Kingdom	11.9	33.0	7.3	21.2	1.1	15.4	3.5	13.6
United States	7.7	17.3	7.9	17.6	3.9	10.3	3.9	10.3

1. Nominal returns are converted to real returns using the domestic consumption price deflator. World equity returns in dollars are converted to local currency using contemporaneous exchange rates and then deflated.

2. P-values for Ljung-Box Q tests of the null hypothesis of no serial correlation (out to 6 lags) were greater than 0.2 for all raw asset returns.

3. The last four columns report statistics for orthogonalized world equity returns in the case of no hedging and perfect hedging of real exchange rate movements (against the U.S. dollar). World returns (asset 2) are orthogonalized with respect to current and lagged domestic returns (asset 1) as follows: First, compute the OLS regression $\tilde{R}_{2t} = \alpha + \beta_1 \tilde{R}_{1t} + \beta_2 \tilde{R}_{1,t-1} + u_t$. Second, construct the orthogonalized asset as the sum of (a) one unit of asset 2, (b) $-(\beta_1 + \beta_2)$ units of asset 1, (c) $(\beta_1 + \beta_2)$ units of the riskless asset. The orthogonalized asset has mean rate of return, $\alpha + (\beta_1 + \beta_2)R_0$, with standard deviation, $\mathrm{std}(u)$.

Table 3: Output Innovations Regressed on Domestic Equity Returns, 1972-1995

	Current Returns		Lagged Returns		Measures of Fit		
Country	Slope Coeff.	Std. Error	Slope Coeff.	Std. Error	R^2	Joint Signif.	Quasi Corr.
Australia	306	312	430	309	0.10	.31	0.49
Austria	66	179	222	179	0.07	.45	0.33
Belgium	13	260	515	260	0.16	.16	0.40
Canada	33	651	1410	653	0.18	.12	0.43
Denmark	149	206	262	206	0.08	.43	0.43
France	186	212	65	210	0.04	.68	0.26
Germany, West	-246	276	584	276	0.22	.07	0.23
Italy	-36	165	108	164	0.02	.80	0.09
Japan	206	177	575	176	0.36	.01	0.76
Netherlands	-284	227	295	221	0.15	.17	0.00
Norway	174	144	415	144	0.29	.03	0.75
Sweden	-344	347	395	347	0.09	.37	0.02
United Kingdom	208	231	323	230	0.09	.36	0.50
United States	-133	498	1080	530	0.17	.13	0.37
Mean					0.14		0.36

Notes:

1. The table reports regressions of domestic output innovations on current and lagged real returns on own equity using annual data from 1971 to 1995.

2. The second column from the right reports the marginal significance level for an F-test of the joint null hypothesis that the coefficients on current and lagged returns are zero.

3. The quasi-correlation is computed as $\tilde{\rho} = \left[b_0 + R_f^{-1} b_1\right]\left[\mathrm{std}(R)/\,\mathrm{std}(y)\right]$, where b_0 and b_1 are the slope coefficients on current and lagged equity returns, $R_f = 1.025$ is the gross risk-free rate of return, $\mathrm{std}(R)$ is the standard deviation of equity returns, and $\mathrm{std}(y)$ is the standard deviation of the output innovations. The quasi-correlation formula is analogous to the relationship between the Pearson correlation, ρ, and the slope coefficient in an OLS regression of y on R: $\rho = b\,\mathrm{std}(R)/\,\mathrm{std}(y)$, where b is the regression coefficient.

Table 4: Output Innovations Regressed on Current and Lagged World Equity Returns (Orthogonalized), 1973-1994

	A. Not Hedged Against Real Exchange Rate Risk		*B. Hedged Against Real Exchange Rate Risk*	
Country	Joint Marginal Significance	Quasi Correlation	Joint Marginal Significance	Quasi Correlation
Australia	0.58	-0.30	0.56	0.27
Austria	0.23	-0.26	0.25	0.15
Belgium	0.03	-0.66	0.57	-0.02
Canada	0.34	0.07	0.51	0.04
Denmark	0.31	0.43	0.01	0.65
France	0.24	-0.06	0.01	0.45
Germany, West	0.68	-0.23	0.63	0.24
Italy	0.30	-0.35	0.11	0.32
Japan	0.90	-0.03	0.99	0.02
Netherlands	0.13	-0.44	0.95	0.08
Norway	0.56	-0.28	0.22	-0.40
Sweden	0.59	-0.19	0.22	0.42
United Kingdom	0.26	0.43	0.02	0.69
United States	0.10	0.50	0.10	0.50
Mean		-.10		.24

Notes:

1. The table reports regressions of domestic output innovations on current and lagged real returns on the world equity mutual fund. Part (A) B considers returns that are (not) hedged against movements in the real exchange rate between the domestic currency and U.S. dollars. For each country, world returns are othogonalized with respect to domestic returns.

2. After differencing and orthogonalization, the sample period runs from 1973 to 1994. The regression results for domestic equity returns over the 1973-94 sample period are very similar to the ones reported in Table 3 for the 1972-1995 sample period, except for Japan. The quasi-correlation for Japan's domestic equity is .56 in the 1973-1994 sample period, as compared to .76 in the longer sample.

3. See note 3 in Table 3 regarding the calculation of the quasi-correlation.

Table 5: Endowed Exposure to Own-Country and World Equity
Thousands of 1990 U.S. Dollars Per Person

		A. Exposure to Own-Country Equity				B. Exposure to World Equity	
	(1)	(2)	(3)	(4)	(5)	(6) Unhedged	(7) Hedged
	Corr$(y, R) \rightarrow$	1	Table 3	.36	0	-0.1	0.24
Country	δ	EE(GDP)	EE(Direct)	EE(Direct)	Mkt. Cap	EE(Direct)	EE(Direct)
Australia	0.90	60	32	25	6	-13	38
Austria	0.97	35	12	13	1	-6	17
Belgium	0.94	53	23	21	3	-9	24
Canada	0.95	136	63	54	7	-19	45
Denmark	0.94	38	18	15	2	-8	21
France	0.94	40	12	16	2	-8	22
Germany, West	0.96	58	15	23	2	-9	26
Italy	0.96	31	4	12	1	-7	20
Japan	0.82	41	33	20	8	-8	22
Netherlands	0.91	45	4	19	4	-11	20
Norway	0.95	32	24	12	2	-8	21
Sweden	0.94	68	5	27	4	-13	29
United Kingdom	0.80	42	25	21	8	-9	25
United States	0.91	103	44	43	9	-17	41

Notes:

1. Part A: Measures of per capita endowed exposure to own-country equity based on equations (24) and (25) in the text.

Column (2) is equivalent to EE(Direct) with corr$(y, R) = 1$. Column (3) uses the country's quasi-correlation value reported in Table 3. Column (4) uses the average quasi-correlation across countries. Column (5), the 1970-1995 average value of stock market capitalization, is equivalent to EE(Direct) with corr$(y, R) = 0$.

2. Part B: Measures of per capita endowed exposure to orthogonalized world equity based on equation (25) in the text with e/H set to zero and δ set to 1. Column (6) considers returns that are not hedged against real exchange rate movements, while column (7) considers hedged returns. Correlation values are chosen to match the average quasi-correlation reported in Table 4.

3. The calculations use a risk-free interest rate of 2.5 percent per year and sample average values for the the standard deviations of output innovations and equity returns, as reported in Tables 1 and 2. δ is chosen to satisfy EE(GDP) $=$ EE(Direct), evaluated at corr$(y, R) = 1$.

Table 6: Desired Exposure to Domestic Equity and Optimal Holdings
Thousands of 1990 U.S. Dollars Per Person
Relative Risk Aversion = 3

		B. Optimal Holdings = Desired - Implicit Exposure			
	A. Desired	Correlation of y and R			C. Market
Country	Exposure	Table 3	0.36	1.0	Cap
Australia	91	64	71	31	6
Austria	40	29	28	5	1
Belgium	168	148	150	115	3
Canada	186	130	139	49	7
Denmark	81	66	68	43	2
France	91	81	77	51	2
Germany, West	96	83	75	37	2
Italy	22	20	12	-9	1
Japan	90	65	78	49	8
Netherlands	198	198	183	152	4
Norway	37	15	27	6	2
Sweden	164	162	140	95	4
United Kingdom	97	80	84	54	8
United States	293	259	260	190	9

Notes:

1. Part A reports desired exposure to own-country equity based on equation (26) in the text, assuming a risk-free interest rate of 2.5 percent per year and relative risk aversion of 3. The excess return on equity and the variance of equity returns are set to sample average values, as reported in Table 2.

2. Part B reports optimal equity holdings for a representative investor who faces a portfolio choice menu with a risk-free asset and domestic equity. Optimal holdings are calculated under alternative assumptions about the correlation between output innovations and equity returns. Optimal holdings equal desired exposure when output innovations are uncorrelated with equity returns.

3. Part C reports per capita stock market capitalization for own-country equity. These values are identical to actual per capita holdings of own-country equity in a regime with no trade in risky assets.

Table 7: Welfare Costs Implied by Suboptimal Holdings of Domestic Equity For a Representative Domestic Investor Facing Fixed Asset Returns Annual Costs as a Percentage of Consumption

	A. Annual Welfare Cost Assuming RRA = 3				B. RRA Value that Fits Asset Pricing Equilibrium Condition			
Corr(y, R) $\rightarrow$	Table 3	0.0	0.36	1.0	Table 3	0.0	0.36	1.0
Australia	10	21	13	3	8.4	46.8	10.8	4.5
Austria	5	10	5	0	9.9	130.0	9.1	3.4
Belgium	56	72	57	35	21.8	160.9	23.9	9.5
Canada	14	30	17	2	8.9	77.7	10.4	4.1
Denmark	28	43	30	13	13.6	104.0	15.9	6.3
France	25	32	23	11	22.8	123.5	17.3	6.9
Germany, West	24	32	19	5	18.7	115.8	12.7	4.9
Italy	3	4	1	1	17.2	55.0	5.6	2.2
Japan	20	42	31	15	8.2	36.1	13.8	6.6
Netherlands	92	92	79	57	141.8	141.8	31.3	13.1
Norway	3	24	12	1	4.7	71.7	9.1	3.6
Sweden	104	106	78	38	92.0	121.1	18.1	7.2
United Kingdom	29	45	33	17	11.4	34.2	14.1	6.9
United States	40	52	40	23	20.0	93.5	20.4	8.5
Unweighted Mean	32	43	31	16	28.5	93.7	15.2	6.3

Notes:

1. Part A reports the costs implied by the gap between optimal and actual exposures to domestic equity, calculated according to equation (27) and expressed as a percentage of observed consumption. Optimal exposure is calculated for an investor with relative risk aversion of 3 and a portfolio choice menu consisting of a risk-free asset and domestic equity. Actual exposure equals domestic market cap plus implicit exposure to domestic equity. Equity return statistics are set to the sample average values reported in Table 2. The risk-free interest rate is set to 2.5 percent.

2. Part B reports the value of relative risk aversion that equates optimal and actual exposures to domestic equity. Equivalently, this value fits the equilibrium asset pricing equation in Corollary 2.

Table 8: The Benefits of Portfolio Expansion
A Representative Investor in Each County Facing Fixed Asset Returns

RRA=3 Unless Noted Otherwise Country	*Optimal Holdings* Domestic Equity (Thousands)	World Equity (Thousands)	World Equity Percentage	Portfolio Expansion Benefit (Percent of Consumption)
Australia	-56	216	135	19
Austria	-10	168	107	46
Belgium	126	42	25	1
Canada	-19	314	106	49
Denmark	60	227	79	65
France	8	131	94	13
Germany, West	15	101	87	11
Italy	-49	242	126	66
Japan	30	159	84	31
Netherlands	199	-27	-16	1
Norway	29	153	84	57
Sweden	115	112	49	16
United Kingdom	65	37	36	1
United States	104	187	64	8
Unweighted Mean			76	28
Japan (RRA=13.8)	2	17	89	1.7
UK (RRA=14.1)	15	-11	-362	0.6
US (RRA=20.4)	16	-7	-90	0.1

Notes:

1. The table reports optimal domestic and world equity holdings for a representative investor in each country. Asset holdings are expressed in 1990 U.S. dollars. The rightmost column reports the annual welfare benefits, assuming optimal behavior, from expanding the portfolio menu to include the world equity fund.

2. The calculation underlying the table entries reflect the following assumptions: (a) annual risk-free interest rate of 2.5 percent, (b) means and standard deviations of risky asset returns set to the sample average values reported in Table 2, (c) a correlation of .36 between domestic output innovations and own equity re-

turns from Table 3, (d) a correlation of .24 between domestic output innovations and orthogonalized world equity returns from Table 4.b, and (e) returns on the world equity fund that are hedged against movements in the real exchange rate between the domestic currency and the U.S. dollar.

3. See the text for an explanation of how to calculate the optimal risky asset holdings and the welfare benefits from an expansion in the portfolio choice menu.

Table 9: Equilibrium Gains to Trade in Domestic Equity
Interpreting Equity Prices as Autarky Outcomes

				Relative Risk Aversion=3			*RRA Set to Fit Equilibrium Equity Pricing Condition*		
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Country	S.D. of World Output Change	Fitted Value of Corr(y, R)	World Investor Endowed Exposure	A^W/A^h	Domestic Investor Desired Exposure	Gains to Trade as % of Cons.	A^W/A^h	Domestic Investor Desired Exposure	Gains to Trade as % of Cons.
Australia	1757	0.23	59	1.01	91	2.8	1.79	32	44
Austria	1206	-0.60	-92	0.78	40	80.9	1.18	12	310
Belgium	1360	0.10	23	0.92	168	57.1	0.63	23	1
Canada	1544	0.24	89	1.08	186	7.7	1.81	63	27
Denmark	1028	0.65	77	0.84	81	1.9	0.92	18	86
France	1131	0.02	3	0.85	91	31.7	0.56	12	3
Germany, West	1130	-0.25	-45	0.83	96	64.6	0.67	15	46
Italy	1505	-0.30	-51	0.80	22	31.2	0.69	4	69
Japan	1167	0.19	30	0.77	90	28.0	1.40	33	1
Netherlands	1289	0.55	127	0.90	198	17.2	0.09	4	7
Norway	734	-0.02	-1	0.70	37	27.7	2.23	24	21
Sweden	1271	0.42	78	0.85	164	39.7	0.14	5	4
United Kingdom	1886	0.57	134	0.84	97	1.5	1.10	25	328
United States	1075	-0.02	-5	1.31	293	58.0	0.98	44	10
Mean		.13				32.1			68.5

Notes:

1. Column (1) reports the standard deviation of the first difference in per capita real world output. Real world output is computed in domestic currency units by aggregating across countries at contemporaneous market exchange rates and then deflating by the domestic price deflator. Output is expressed in 1990 U.S. dollars.

2. Column (2) reports the correlation between the change in per capita real world output and the real return on domestic equity. See note 3 of Table 3 for a description of the regression procedure used to fit this value.

3. Column (3) reports the world investor's endowed exposure to domestic equity, computed according to equation (25) and evaluated at $e = 0$ and $\delta = 1$. These endowed exposure calculations make use of column (1) for std($\tilde{y}$), column(2) for corr(y, R) and Table 2 for std($\tilde{R}$). All exposure measures are thousands of 1990 U.S. dollars.

4. Columns (4) and (7) report the ratio of absolute risk aversion for the representative world investor, A^W, to the absolute risk aversion for the representative investor in the domestic economy. A^h is computed as relative risk aversion in country h divided by per capita real consumption in country h. Table 7.B (column headed "Table 3") reports the relative risk aversion values that underlie the A^h values used in column (7).

5. Columns (5) and (8) report the desired exposure of the domestic investor to domestic equity at autarky prices and returns, calculated according to equation (26). Table 2 reports the statistics on domestic equity returns that enter into the calculation of desired exposure.

6. Columns (6) and (9) report the domestic economy equilibrium gains to trade in domestic equity based on equation (29) in the text. This welfare calculation treats equity prices and returns as the outcome of an autarky regime for domestic equity. As discussed in the text, Table 7.A gives corresponding welfare calculations that treat equity prices and returns as the outcome of a free trade regime for domestic equity.