Impartiality and priority.
Part 2: a characterization with solidarity*

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Abstract

The ethic of 'priority' is a compromise between the extremely compensatory ethic of 'welfare equality' and the needs-blind ethic of 'income equality'. We propose an axiom of priority, and characterize resource-allocation rules that are impartial, prioritarian, and solidaristic. They comprise a class of rules which equalize across individuals some index of resources and welfare. Consequently, we provide an ethical rationalization for the many applications in which such indices have been used (e.g., the 'human development index,' 'index of primary goods,' etc.). (*JEL numbers: D63, D71.*)

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1 Introduction

In this paper, we study the ethics of resource allocation in a basic and common problem. There is a resource, available in given quantity, to be allocated among individuals, each of whom possesses a capability to transform the resource into some given valued outcome, and the achievements of individuals, with regard to that outcome, are interpersonally comparable. The data of the problem are $(N, (u_i)_{i \in N}, W)$, where N is the group of individuals in the population to be served, $W \in \mathbb{R}_+$ is the resource budget, and $u_i : \mathbb{R}_+ \to \mathbb{R}_+$ is the function which describes the capacity of individual $i \in N$ to convert the resource into the desired outcome.

In many resource allocation problems of this sort, there are two focal points of distribution: to distribute the available resource equally among all who need it, and to distribute the resource among the population so as to equalize the outcomes among them. Often, however, the 'equal-resource' allocation seems too harsh: it does not take into account the differential ability of individuals to convert the resource into the desired outcome. On the other hand, often the equal-outcome allocation seems too extreme: it may require giving the lion's share of the resource to very 'handicapped' individuals, ones with poor outcome functions, and this may appear to be unfair to those who are more fortunate. This is a familiar criticism of the Rawlsian maximin allocation.

The philosopher Derek Parfit, partly as a reaction to the extremism of Rawlsian maximin, coined the term 'priority' for the view that lies 'in between' the equal-resource and the equal-outcome view. He proposed that the right view is to give *priority* to those who are less capable of transforming resource into outcome. (Parfit did not work on a formal domain of problems, so we are paraphrasing here.) We formalize Parfit's view in the following axiom:

Priority*: Let x_i denote the amount of resources allocated to individual i. If $x_i < x_j$ then $u_i(x_i) \geq u_j(x_j)$.*

Priority says that no individual can dominate another in both resources and outcomes. In particular, individuals with a worse capability of transforming resources into outcomes are allocated more resources, a desirable feature according to Sen (1973). It is intuitively clear that priority admits a large class of possible resource allocations –just think of the possible compromises between the equal-resource and the equal-outcome allocation. Prioritarianism includes –it would seem as polar cases– those two allocations or allocation rules.

There is a second principle that we believe characterizes fairness in many problems, which we call *solidarity*. The idea is that, if an allocation rule is fair, then when new individuals join a society (e.g., through birth or immigration) then the resources allocated to all the *original* members should change in the same direction. Intuitively, if the new members bring with them a lot of resources, then everyone in the original population should gain, and if they bring with them few or no resources, then everyone in the original population should chip in some resource to help them. We formalize this axiom as follows:

***Solidarity**. Let $N \subseteq N'$. If $x = (x_i)_{i \in N}$ and $x' = (x_i)_{i \in N'} = (x^N, x^{N' \setminus N})$ are the allocation vectors for $(N, (u_i)_{i \in N}, W)$ and $(N', (u_i)_{i \in N'}, W')$ respectively, then either $x = x^N$, $x > x^N$ or $x < x^N$.*

The solidarity axiom has been used in different forms by Thomson (1983), Roemer (1986), Moulin (1987), Chun (1996), Sprumont (1996) and Fleurbaey and Maniquet (1999), among others. This axiom implies the axiom of *consistency*, an axiom that has received considerable attention in decision problems and in the theory of distributive justice (see, for instance, Young, 1994; Roemer, 1996; Moulin, 2003; Thomson, 2004; and the literature cited therein).

Finally, we believe that fairness requires *impartiality*. This means that in deciding how to allocate the resource, we ignore all attributes of persons that are irrelevant, according to our moral standard, to the problem at hand. For instance, if the problem is one of allocating scarce rescuer time to saving earthquake victims, we ignore the victim's religion and race (though perhaps not his age). In reality, impartiality is a very strong requirement. Suppose the issue is to distribute educational resources to children, who have different capacities to transform them into future earning power. One might say that impartiality requires that the allocation rule ignore the wealth of parents. However, the allocation rule that allocates such resources in the United States surely does not ignore parental wealth.

We state the impartiality axiom as:

Impartiality. *Allocation rules are defined on economic environments* $(N, (u_i)_{i \in N}, W)$.

Here are some examples where, we believe, the axioms of solidarity, priority and impartiality either apply in common practice, or, arguably, morality suggests that they should apply:

1. The resource W is parental time allocated to children, and $u_i(x_i)$ is the (predicted) success of child i if he receives parental time x_i. Solidarity says that when a new child arrives in the family, parental time to all the other children changes in the same way (decreases, here); priority says that, generally, a parent should devote more time to children who are less able, but not to the extent of rendering those children more successful than more able children.

2. Distribution of a parent's estate among children. The same ideas apply as in #1. Here, equal division of the estate is commonly done, which is consistent with solidarity and priority.

3. The resource W is the budget of educational finance; $u_i(x_i)$ is, perhaps, the predicted future wage of a child of type i if she receives x_i in educational finance. Priority says that we should devote at least as much educational resource to children who have inferior abilities to transform the resource into future earning power.

4. The Americans with Disabilities Act requires employers to spend extra resources to enable disabled workers to perform adequately on the job.

Here, $u_i(x_i)$ might be the degree of success on the job.

The reader can doubtless supply many more examples.

An example that does not satisfy priority is triage on the battlefield. The resource is physician time; $u_i(x_i)$ is the probability of survival of wounded soldier i. The army's goal is to maximize the number of soldiers who survive, and can return to battle; it will devote no resources to badly wounded soldiers for whom the function u_i is very poor. Note that fairness is not the issue here, but maximizing the effectiveness of the army.

Consider, however, the victims of an earthquake, where the resource is scarce rescuer time. u_i is the probability-of-survival function for person i where the argument is rescuer time devoted to saving i. Should the allocation of rescuer time satisfy priority, or is this a case like triage? It depends whether our objective is fairness towards individuals or to maximize the number of people saved. If it is fairness, then priority applies. It is intuitively clear that the most 'conservative' prioritarian practice would be to give all victims equal time; the most 'radical' would be to allocate the time among victims so that all have an equal probability of survival. Priority says that in no case do we allocate so much rescuer time to a more badly trapped victim that we increase his survival probability above the survival probability of a less badly trapped victim.

In this paper, we characterize, on a domain of possible problems, the set of allocation rules that jointly satisfy priority, solidarity, and impartiality. The intuitions that we have hinted at are verified: there is a large class of such rules, and the equal-resource and equal-outcome rules are polar cases in that class, on the 'conservative' and 'radical' ends. The admissible class turns out to involve *indices* of resources and outcomes. To be precise, we will show that the three axioms require us to equalize *some index* of resources and outcomes, at the highest possible level. In particular, these rules pay equal attention to resources *and* outcomes in an explicit way; they are not 'welfarist' rules that consider only the pattern of outcomes that resource allocations generate. As such, this work is a contribution to non-welfarist social-choice theory. Indeed, the non-welfarist aspect of our approach is evident in the priority axiom: for that axiom implements a special moral concern for the *amount of resource* that a person receives. (That claim is vague, but we hope the reader agrees with us.)

Using indices of resources and outcomes to measure the success of an allocation procedure is a fairly common practice. The UNDP's human development indicator is an index of a country's GDP, literacy rate, and infant mortality rate. John Rawls (1971) worked, famously, with an index of primary goods: some of those 'good' were resources, and some 'outcomes.' Amartya Sen (1980, 1992) has written of using an index of functionings as a possible measure of a person's welfare. In these examples, the social welfare supremum is thought to be the allocation of resources that equalizes the index in question, at the highest possible level. This is our characterization theorem.

In part one of the duo of papers of which this is second, we explored what is perhaps the most famous approach to implementing impartiality in ethics, the veil of ignorance. Our study led us to the view that that approach violated prioritarianism, which we take to be an ethical requirement. This paper continues that research program, in asking what allocation rules *do* satisfy prioritarianism and impartiality –and another axiom that we consider to be ethically desirable, solidarity. Further comments will follow in our conclusion.

From the viewpoint of ethics, the priority axiom implements Parfit's attempt to find a compromise between ignoring capabilities of persons, in the assignment of resources, and going (what some consider to be) overboard with regard to achieving outcome equality. As such, our characterization theorem tells us what the ethics of compromise, so viewed, require.

The rest of the paper is organized as follows. In Section 2 we present the axiomatic theory of resource allocation involving the concepts of impartiality, solidarity, and priority. In Section 3 we characterize the family of rules satisfying these three notions and in Section 4 we focus on two important rules within the family: the equal-resource rule and the equal-welfare rule. Section 5 concludes. Most of the proofs have been relegated to an Appendix.

2 The model

Let $\mathbb{N} = \{1, 2, 3, ...\}$ represent a population of all potential individuals and let $\mathcal{N}$ be a collection of finite subsets of $\mathbb{N}$. Let $N = \{1, 2, ..., n\} \in \mathcal{N}$ be a set of individuals with generic elements i and j. Individuals derive welfare from a resource, called *wealth*. We assume that $\mathbb{N} \times \mathbb{R}_+$ is endowed with a complete order. The expression $(i, W) \succsim (j, W')$ is read: "individual i equipped with wealth W enjoys a welfare level at least as high as individual j equipped with wealth W'". We assume that this order is continuous in W, and satisfies that, for any $i, j \in N$ and $W \in \mathbb{R}_+$ there is a wealth level W' such that $(i, W) \sim (j, W')$. We further assume that for any pair $i, j \in N$, $(i, 0) \sim (j, 0)$. A wealth level of zero can be thought of as inducing death, which is an equally bad outcome for all individuals. Finally, we assume that welfare is strictly increasing in wealth for every individual.

It is convenient to represent this interpersonally level comparable welfare ordering as follows. Fix a particular individual and call her individual 0. For any other individual i define a function $u_i : \mathbb{R}_+ \to \mathbb{R}_+$ where for each $W \in \mathbb{R}_+$, $u_i(W)$ is such that

$$(0, u_i(W)) \sim (i, W)$$

In other words, $u_i(W)$ is the wealth that 0 must receive in order that she enjoy the same level of welfare as individual i enjoys with wealth W.[1] We say that an individual is more able than another one if the former needs less wealth than the latter one to reach the same level of welfare. Formally,

An individual i is ***able*** *with respect to an individual j if $u_i \geq u_j$ and $u_i \neq u_j$. We also say that, in this case, individual j is* ***disabled*** *with respect to individual i.*

[1] In particular, u_0 is the identity function.

Two individuals are **comparable** if one is at least as able as the other. Obviously, there might be individuals who are not comparable.

The assumptions on $\succsim$ tell us that for all i, u_i is a continuous strictly increasing unbounded function satisfying that $u_i(0) = 0$.[2] We say that a family of functions constitutes a *dense domain* if the graphs of these functions cover the positive quadrant. We shall assume that $\{u_i : i \in \mathbb{N}\}$ constitutes a dense domain. Formally,

Dense Domain. *$\{u_i : i \in \mathbb{N}\}$ is a dense domain, i.e., for every $(a, b) \in \mathbb{R}^2_{++}$ there exists an individual $i \in \mathbb{N}$ such that $u_i(a) = b$.*

We define an **economy** e as a triple (N, u, W), where $N \in \mathcal{N}$ is the set of individuals, $u = (u_i)_{i\in N}$ is the profile of utility functions (defined as above) for individuals in N, and $W \in \mathbb{R}_+$ represents the available wealth. The family of all economies is $\mathcal{E}$.

2.1 Allocation rules

An **allocation rule** is a function F that associates to each economy $e = (N, u, W) \in \mathcal{E}$ a unique point $F(e) = (F_i(e))_{i\in N} \in \mathbb{R}^n_+$ such that $\sum_{i\in N} F_i(e) = W$. That is, an allocation rule indicates how to distribute the wealth available in an economy among its members.

Examples of rules are the following: First, the rules that assign all the available wealth to a unique individual in the economy.

Dictatorial *rule (D^j):* $D^j_i(N, u, W) = \begin{cases} W & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$.

Next, the rule that awards each agent the same amount:

Equal-Resource *rule (ER):* $ER_i(N, u, W) = \frac{W}{n}$.

An alternative to the equal-resource rule is obtained by focusing on the welfare levels individuals achieve, as opposed to what resources they receive, and choosing the vector at which these welfare levels are equal.

[2] Note that the functions u_i comprise a profile of utility functions for individuals which measure utility in a level comparable way.

Equal-Welfare *rule (EW):* $EW_i(N,u,W) = u_i^{-1}(\lambda)$, *where* $\lambda > 0$ *is chosen so that* $\sum_{i\in N} u_i^{-1}(\lambda) = W$.

Note that, for all $i \in I$, u_i^{-1} is a continuous strictly increasing unbounded function satisfying that $u_i^{-1}(0) = 0$. From here, it follows that EW is well-defined.

Another possibility is to combine these rules depending on the cardinality of the set of individuals. For instance,

Mixed *rule (M):* $M(N,u,W) = \begin{cases} EW(N,u,W) & \text{if } n = 2, \\ ER(N,u,W) & \text{otherwise.} \end{cases}$

Finally, one could also implement the idea of proportionality to construct an allocation rule. For instance,

Proportional *rule (P):* $P(N,u,W) = \lambda \cdot (u_i^{-1}(1))_{i\in N}$, *where* $\lambda > 0$ *is chosen so that* $\sum_{i\in N} u_i^{-1}(1) = \frac{W}{\lambda}$.

2.2 Axioms

We now present the axioms we want rules to satisfy. These axioms will reflect the three notions discussed above: *impartiality*, *priority* and *solidarity*.

First, it is worth mentioning that by defining rules on the class of economies $\mathcal{E}$ we are excluding much information about persons that we consider ethically irrelevant. In doing so, we are implicitly modeling *impartiality*.

We now turn to *priority*. Our axiom of *priority* says that no agent can dominate another agent both in resources and welfare.

Priority *(PR) Let* $e = (N,u,W) \in \mathcal{E}$ *and* $i, j \in N$ *such that* $F_i(e) < F_j(e)$. *Then* $u_i(F_i(e)) \geq u_j(F_j(e))$.

Note that this axiom guarantees that disabled agents receive at least as much wealth as abler ones: we discriminate positively towards the disabled. In other words, priority implies the *weak equity* axiom introduced by Sen (1973). On the other hand, the axiom also says that the obligation towards the unfortunate is limited, as a disabled person is never resourced to the extent that her welfare exceeds that of an able agent. It is also straightforward to show that priority implies a weak version of anonymity which says that individuals that are equally able are rewarded equally.

We conclude with *solidarity*. Here we rely upon a literature which has formulated various solidarity axioms in the past twenty years. Alternative versions of solidarity have been considered in different contexts like fair division (e.g., Thomson, 1983; Roemer, 1986), social choice (e.g., Chun, 1986), compensation problems (e.g., Moulin, 1987; Fleurbaey and Maniquet, 1999), bankruptcy problems (e.g., Chun, 1989), surplus-sharing (e.g., Keiding and Moulin, 1991), collective choice (e.g., Sprumont, 1996; Ehlers and Klaus, 2001), or house allocation (e.g., Ehlers and Klaus, 2004). Our notion of solidarity says that the arrival of immigrants, whether or not accompanied by changes in the available wealth, should affect all original agents in the same direction: all gain or all lose, or all receive the same as before.

Solidarity *(SL). Let* $e = (N, u, W) \in \mathcal{E}$ *and* $e' = (N', u', W') \in \mathcal{E}$, *such that* $N' \subseteq N$. *Let* $F_{N'}(e)$ *denote the projection of* $F(e)$ *onto the set of coordinates corresponding to* N'. *Then either* $F(e') = F_{N'}(e)$, $F(e') > F_{N'}(e)$ *or* $F(e') < F_{N'}(e)$.[3]

Note that solidarity implies that when a bad or good shock comes to an economy, all its members should share in the calamity or windfall. This property is usually known as *resource monotonicity* (e.g., Roemer, 1986). It is also straightforward to show that solidarity implies *consistency*, a property that says that if a sub-group of individuals secedes with the resource allocated to it under a rule, then in the smaller economy the rule allocates the resource in the same way. The reader is referred to Young (1994), Roemer (1996) or Thomson (2004) for the many applications that exist in the literature on distributive justice concerning this notion.

[3] Note that for $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $y = (y_1, ..., y_n) \in \mathbb{R}^n$, we write $x > y$ if $x_i > y_i$ for all $i = 1, ..., n$.

3 A characterization result

Among the rules introduced in Section 2.1 only the equal-resource and the equal-welfare rules satisfy solidarity and priority. The purpose of this section is to identify all the remaining existing rules satisfying these properties.

Let Φ be the class of functions composed of all functions $\varphi : \mathbb{R}^2_{++} \cup \{(0,0)\} \to \mathbb{R}_+$, continuous on its domain and non-decreasing, such that $\inf\{\varphi(x,y)\} = \varphi(0,0) = 0$ and for all $(x,y) > (z,t)$, $\varphi(x,y) > \varphi(z,t)$. Let φ be a function in the class Φ. For all $i \in \mathbb{N}$ define the function $\psi_i : \mathbb{R}_+ \to \mathbb{R}_+$ that determines the $\varphi-$value agent i achieves, depending on the wealth she receives, i.e., $\psi_i(w) = \varphi(w, u_i(w))$ for all $w \in \mathbb{R}_+$. Then, we define the corresponding *index-egalitarian* rule as the rule that equalizes the φ value across individuals in an economy.

Index-egalitarian *rule (E^φ): $E_i^\varphi(e) = \psi_i^{-1}(\lambda)$, where $\lambda > 0$ is chosen so that $\sum_{i\in N} \psi_i^{-1}(\lambda) = W$.*

Note that, for all $i \in I$, ψ_i^{-1} is a continuous strictly increasing unbounded function satisfying that $\psi_i^{-1}(0) = 0$. From here, it follows that E^φ is well defined. Note also that applied in this manner to an agent's wealth and welfare, φ can be considered to be a generalized index of wealth and welfare. So the rules just defined equalize a generalized index of wealth and welfare.[4]

All the rules within the family $\{E^\varphi\}_{\varphi\in\Phi}$ satisfy solidarity and priority. More remarkably, there is no other rule satisfying these two properties simultaneously, as the next result shows.

Theorem 1 *A rule F satisfies solidarity and priority if and only if $F \in \{E^\varphi\}_{\varphi\in\Phi}$.*

Theorem 1 shows that impartiality, solidarity and priority are equivalent to a kind of egalitarianism, where the equality in question is equality of a conception of well-being that is some general index of welfare and resources. In particular, prioritarianism, at least in conjunction with solidarity, does

[4]We are indebted in a major way to Klaus Nehring, who suggested the E_φ rules.

not preclude equality, but it modifies the equalisandum from 'welfare' to an index of welfare and resources.

Proof of Theorem 1.

It is easy to show that all the E^φ rules satisfy SL and PR. Conversely, let F be a rule that satisfies SL and PR. We show that there exists $\varphi \in \Phi$ such that $F = E^\varphi$. First, a preliminary lemma.

Lemma 1 *For $i, j \in \mathbb{N}$ and $\alpha > 0$ fixed, there exists $W \in \mathbb{R}_+$ such that $F_i(\{i,j\}, (u_i, u_j), W) = \alpha$.*

We now introduce some notation. Let $i \in \mathbb{N}$ be given and $\alpha \in \mathbb{R}_+$. Let $E(F, i, \alpha)$ be the domain of economies for which individual i obtains an amount of wealth α, under rule F. Formally:

$$E(F, i, \alpha) = \{e = (N, u, W) \in \mathcal{E} : F_i(e) = \alpha\}.$$

Let $C(F, i, \alpha)$ be the set of points in the plane which are achieved as wealth-welfare ordered pairs under the action of F on individuals who are members of economies in $E(F, i, \alpha)$. Formally:

$$C(F, i, \alpha) = \{(a, b) \in \mathbb{R}^2_+ : \exists e = (N, u, W) \in E(F, i, \alpha),\ j \in N \text{ s.t. } (a, b) = (F_j(e), u_j(F_j(e)))\}.$$

Our aim is to show that the family of curves $\{C(F, i, \alpha) : \alpha \in \mathbb{R}_+\}$ is the isoquant map of an appropriate function $\varphi \in \Phi$ and therefore to show that $F = E^\varphi$.

Lemma 2 *If $\alpha_1 \neq \alpha_2$ then $C(F, i, \alpha_1) \cap C(F, i, \alpha_2) = \emptyset$.*

Let $(a, b) \in \mathbb{R}^2_+$ be given. By the assumption of dense domain, and Lemma 1, there exists $\alpha \in \mathbb{R}_+$ such that $(a, b) \in C(F, i, \alpha)$. By Lemma 2, α is unique. Define then the function $\varphi : \mathbb{R}^2_{++} \cup \{(0, 0)\} \to \mathbb{R}_+$ by $\varphi(a, b) = \alpha$, where $\alpha \in \mathbb{R}_+$ is the unique number for which $(a, b) \in C(F, i, \alpha)$. Then,

Lemma 3 *Let φ defined as above. Then, $\varphi \in \Phi$.*

We show now that $F = E^{\varphi}$, i.e., $F(N,u,W) = E^{\varphi}(N,u,W)$ for all $(N,u,W) \in \mathcal{E}$. Fix $e = (N,u,W) \in \mathcal{E}$. Two cases are distinguished.

Case 1: $i \in N$.

Let $\lambda = F_i(e)$. Then, $(F_j(e), u_j(F_j(e))) \in C(F,i,\lambda)$ for all $j \in N$. By definition of φ, $\varphi(F_j(e), u_j(F_j(e))) = \lambda$, for all $j \in N$. Thus, $\psi_j(F_j(e)) = \lambda$ for all $j \in N$. Since $\sum_{j \in N} F_j(e) = W$, it follows that $F(e) = E^{\varphi}(e)$.

Case 2: $i \notin N$.

Pick two agents $j,k \in N \setminus \{i\}$. Let $w_j = F_j(e)$ and $w_k = F_k(e)$. By Lemma 1, there are two economies $\widehat{e} = (\{i,j\},(u_i,u_j),\widehat{W})$ and $\widetilde{e} = (\{i,k\},(u_i,u_k),\widetilde{W})$ such that $w_j = F_j(\widehat{e})$ and $w_k = F_k(\widetilde{e})$; let $\widehat{w}_i = F_i(\widehat{e})$ and $\widetilde{w}_i = F_i(\widetilde{e})$.

Claim. $C(F,j,w_j) = C(F,i,\widehat{w}_i)$ and $C(F,k,w_k) = C(F,i,\widetilde{w}_i)$.

Proof of the claim. We only show that $C(F,j,w_j) = C(F,i,\widehat{w}_i)$. The proof of $C(F,k,w_k) = C(F,i,\widetilde{w}_i)$ is identical.

Let $(a,b) \in C(F,i,\widehat{w}_i)$. Then, there exists $l \in \mathbb{N}$ such that $b = u_l(a)$ and $(\widehat{w}_i, a) = (F_i(e^2), F_l(e^2))$, where $e^2 = (\{i,l\},(u_i,u_l),\widehat{w}_i + a)$. By a similar argument to that of Lemma 1, there exists W^3 such that $F_i(e^3) = \widehat{w}_i$, where $e^3 = (\{i,j,l\},(u_i,u_j,u_l),W^3)$. Then, $\widehat{e}, e^2$ and e^3 belong to $E(F,i,\widehat{w}_i)$. Thus, by SL, $a = F_l(e^2) = F_l(e^3)$ and $w_j = F_j(\widehat{e}) = F_j(e^3)$. Consequently, $e^3 \in E(F,j,w_j)$ and $(a,b) \in C(F,j,w_j)$, showing that $C(F,i,\widehat{w}_i) \subseteq C(F,j,w_j)$.

Let $(a,b) \in C(F,j,w_j)$. Then, there exists $l \in \mathbb{N}$ such that $b = u_l(a)$ and $(w_j, a) = (F_j(e^2), F_l(e^2))$, where $e^2 = (\{j,l\},(u_i,u_l),w_j + a)$. By a similar argument to that of Lemma 1, there exists W^3 such that $F_j(e^3) = w_j$, where $e^3 = (\{i,j,l\},(u_i,u_j,u_l),W^3)$. Then, $\widehat{e}, e^2$ and e^3 belong to $E(F,j,w_j)$. Thus, by SL, $a = F_l(e^2) = F_l(e^3)$ and $\widehat{w}_i = F_i(\widehat{e}) = F_i(e^3)$. Consequently, $e^3 \in E(F,i,\widehat{w}_i)$ and $(a,b) \in C(F,i,\widehat{w}_i)$, showing that $C(F,i,\widehat{w}_i) \supseteq C(F,j,w_j)$. This proves the claim.

Note that $(w_j, u_j(w_j)) \in C(F,j,w_j) \cap C(F,k,w_k)$. Since $C(F,j,w_j) = C(F,i,\widehat{w}_i)$ and $C(F,k,w_k) = C(F,i,\widetilde{w}_i)$, then

$$(w_j, u_j(w_j)) \in C(F,i,\widehat{w}_i) \cap C(F,i,\widetilde{w}_i).$$

By Lemma 2, it follows that $\widehat{w}_i = \widetilde{w}_i = w_i$. Thus, $C(F, j, w_j) = C(F, i, w_i) = C(F, k, w_k)$. Therefore all the points $\{(F_l(e), u_l(F_l(e))) : l \in N\}$ lie on the w_i-isoquant of φ, and it follows, as in Case 1, that F coincides with E^φ on the entire domain $\mathcal{E}$. ■

The reader might note that although we chose a specific profile of utility functions to represent the interpersonal ordering $\succsim$, the class of rules characterized in Theorem 1 is independent of this choice.

4 Two important allocation rules

In this section we focus upon two rules within the family of $\{E^\varphi\}_{\varphi \in \Phi}$ rules. The **equal-resource** (ER) rule is the E^{φ_1} rule, where $\varphi_1(x, y) = x$. The **equal-welfare** (EW) rule is the E^{φ_2} rule, where $\varphi_2(x, y) = y$. The ER rule equalizes the wealth of individuals in all economies, whereas the EW rule equalizes the welfare of individuals in all economies. These two rules are the extreme prioritarian rules for the most able and the least able agents in an economy. More precisely, ER is the best (worst) prioritarian rule for the ablest (disablest) agent, whereas EW is the best (worst) prioritarian rule for the disablest (ablest) agent.

Proposition 1 *Let $e = (N, u, W) \in \mathcal{E}$. Let i (j) be the ablest (disablest) individual in N. Then, for all rules F satisfying priority, we have:*
(i) $ER_i(e) \geq F_i(e) \geq EW_i(e)$
(ii) $ER_j(e) \leq F_j(e) \leq EW_j(e)$

Proof.

Let F be a rule satisfying PR. Let $e = (N, u, W) \in \mathcal{E}$ and let i (j) be the ablest (disablest) individual in N. We shall show (i). The proof of (ii) is analogous.

Suppose, contrary to the claim, that $ER_i(e) < F_i(e)$. Then, there exists $k \in N$ such that $ER_k(e) > F_k(e)$. Since $ER_k(e) = ER_i(e)$, it follows that $F_i(e) > F_k(e)$. Then, by PR, $u_i(F_i(e)) \leq u_k(F_k(e))$. On the other hand, since i is the ablest individual in N, it follows that $u_i \geq u_k$. Thus, by monotonicity, $u_i(F_i(e)) > u_i(F_k(e)) \geq u_k(F_k(e))$, which is a contradiction.

Similarly, if $F_i(e) < EW_i(e)$, there exists $k \in N$ such that $EW_k(e) < F_k(e)$. Since u_i and u_k are strictly increasing, it follows that $u_i(F_i(e)) < u_i(EW_i(e)) = u_k(EW_k(e)) < u_k(F_k(e))$. Thus, by PR, $F_i(e) \geq F_k(e)$. Now, since $u_i \geq u_k$, it follows, by monotonicity, that $u_i(F_i(e)) \geq u_i(F_k(e)) \geq u_k(F_k(e))$, which is a contradiction. ■

In particular, Proposition 1 shows that, for all $\varphi \in \Phi$,

$$ER_i(e) \geq E_i^{\varphi}(e) \geq EW_i(e) \text{ and } ER_j(e) \leq E_j^{\varphi}(e) \leq EW_j(e),$$

where i and j are, respectively, the ablest and disablest individuals in e.

We can define a duality relationship between the members of the $\{E^{\varphi}\}_{\varphi \in \Phi}$ family as follows. For each $\varphi \in \Phi$, let φ^* be defined as $\varphi^*(x, y) = \varphi(y, x)$. Then, $\varphi^* \in \Phi$. We define the **dual** rule of E^{φ} as E^{φ^*}. E^{φ} and E^{φ^*} are symmetric with respect to the treatment of wealth and welfare. Note that ER and EW are dual rules.

5 Recapitulation

We have presented a characterization, in simple environments, of allocation rules that satisfy impartiality, solidarity and priority. In particular, our result resolves the tension that exists between impartiality and priority, that we highlighted in part one of the duo of papers of which this is the second, when one uses the veil of ignorance as the tool to enforce impartiality. Indeed, we get something more: impartiality and priority, together with solidarity, imply a kind of egalitarianism, where the *index* of wealth and welfare that is equalized according to justice is not determined without further assumptions. Two classical distribution rules are polar (and even 'dual') in the class of index-egalitarian rules –the equal-resource and equal-welfare allocation rules.

To characterize the family of 'index-egalitarian' rules, we had to use not only impartiality and priority, but something more, solidarity. This may be a disappointment to political philosophers often called 'left-liberal', a class which includes such writers as John Rawls (1971), Brian Barry (1995), and perhaps Thomas Scanlon (1998). For it has been a goal of that school to deduce egalitarianism from axioms which would attract almost universal assent –like impartiality and perhaps priority. The title of Barry's book, *Justice as Impartiality*, even suggests that impartiality alone implies something close to egalitarianism, as that is the kind of justice that he describes therein. The solidarity axiom, which we have needed for our characterization, seems to be itself quite radical, in that it insists upon a strong kind of cooperation among citizens. We have therefore not derived equality from axioms which would attract almost universal assent.

We conclude with some remarks linking this paper with Part 1, in which we showed that the veil of ignorance, viewed as an allocation rule, was non-prioritarian. Strictly speaking, the present paper is incomparable to that one, because the domains of the resource-allocation rules are different. In Part 1, we included the risk preferences of the individuals as information in the problem, as well as the level of resource and the interpersonal welfare ordering. We would have liked to have characterized our index-egalitarian rules on that domain, but that would be a much more difficult task.

Regarding priority: What we, the ethical observers, consider a bad outcome –having a society in which the disabled are less abundantly resourced than the able– does not coincide with the bad outcome to the individual behind the veil of ignorance who faces the birth lottery; the worst outcome for her may be being born *able* without *sufficient* resources to fully exploit that ability. This is, of course, why the veil sometimes (often) allocates less wealth to the disabled than to the able, as we showed in Part 1. We have not *defended* our axiom of priority: it is, after all, an axiom. To do so would require a fully philosophical inquiry.

Appendix: Proofs of the lemmata

We assume throughout this appendix that F is an allocation rule satisfying SL and PR.

Proof of Lemma 1

Let $i, j \in \mathbb{N}$ and $\alpha > 0$ be given and denote $e = (\{i,j\}, (u_i, u_j), W)$ for each $W > 0$. Since $F_i(e) \leq W$, it follows that, for W sufficiently small, $F_i(e) < \alpha$.

Suppose that $F_i(e) < \alpha$ for all e. Then, $F_j(e) > W - \alpha$ for all e and therefore $\lim_{W\to\infty} F_j(e) = \infty$. In particular, for all e such that $W > 2\alpha$, $F_j(e) > \alpha > F_i(e)$. Thus, by PR, $u_i(F_i(e)) \geq u_j(F_j(e))$. Since u_i is increasing, $u_i(\alpha) \geq u_j(F_j(e))$ for all e such that $W > 2\alpha$. However, since u_j is unbounded, $\lim_{W\to\infty} u_j(F_j(e)) = \infty$, a contradiction.

Thus, there exist W_1 and W_2 such that $F_i(e_1) < \alpha$ and $F_i(e_2) > \alpha$ for $e_1 = (\{i,j\}, (u_i, u_j), W_1)$ and $e_2 = (\{i,j\}, (u_i, u_j), W_2)$. Consider the following two sets:

$$\Omega^< = \{W \in (0, +\infty) : F_i(e) < \alpha\} \text{ and } \Omega^> = \{W \in (0, +\infty) : F_i(e) > \alpha\}.$$

Then, $W_1 \in \Omega^<$ and $W_2 \in \Omega^>$. Thus,

$$\Omega^< \neq \emptyset \text{ and } \Omega^> \neq \emptyset. \tag{1}$$

Furthermore, it is obvious that

$$\Omega^< \cap \Omega^> = \emptyset. \tag{2}$$

We show now that

$$\Omega^< \text{and } \Omega^> \text{are open sets.} \tag{3}$$

Claim. $\Omega^<$ and $\Omega^>$ are open sets.

Proof of the claim. Let $W \in \Omega^<$ and $\overline{\alpha} = F_i(e) < \alpha$. Let $\varepsilon = \frac{\alpha - \overline{\alpha}}{2}$. We show that $(W - \varepsilon, W + \varepsilon) \subset \Omega^<$. By SL, $(W - \varepsilon, W) \subset \Omega^<$. Suppose, by contradiction, that there exists $W^* \in (W, W + \varepsilon)$ such that $W^* \notin \Omega^<$, i.e., $F_i(e^*) \geq \alpha$, for $e^* = (\{i,j\}, (u_i, u_j), W^*)$. By SL, $F_j(e^*) > F_j(e) = W - \overline{\alpha}$. Then, $W^* = F_j(e^*) + F_i(e^*) > W - \overline{\alpha} + \alpha = W + 2\varepsilon$, which contradicts that $W^* \in (W, W + \varepsilon)$. This shows that $\Omega^<$ is an open set.

Let $W \in \Omega^{>}$ and $\overline{\alpha} = F_i(e) > \alpha$. Let $\varepsilon = \frac{\overline{\alpha}-\alpha}{2}$. We show that $(W - \varepsilon, W + \varepsilon) \subset \Omega^{>}$. By SL, $(W, W + \varepsilon) \subset \Omega^{>}$. Suppose, by contradiction, that there exists $W^* \in (W - \varepsilon, W)$ such that $W^* \notin \Omega^{>}$, i.e., $F_i(e^*) \leq \alpha$, for $e^* = (\{i,j\}, (u_i, u_j), W^*)$. By SL, $F_j(e^*) < F_j(e) = W - \overline{\alpha}$. Then, $W^* = F_j(e^*) + F_i(e^*) < W - \overline{\alpha} + \alpha = W - 2\varepsilon$, which contradicts that $W^* \in (W - \varepsilon, W)$. This shows that $\Omega^{>}$ is an open set, which proves the claim.

Now, if, contrary to the statement, $F_i(e) \neq \alpha$, for all $W \in \mathbb{R}_{++}$, then

$$\mathbb{R}_{++} \subset \Omega^{>} \cup \Omega^{<}. \tag{4}$$

Finally, (1), (2), (3) and (4) together say that $(0, +\infty)$ is not connected, which is a contradiction. ■

Proof of Lemma 2

We show first that any $C(F, i, \alpha)$ is downward sloping, i.e., if (a, b), $(a', b') \in C(F, i, \alpha)$ and $a' > a$ then $b' \leq b$. Suppose, to the contrary, that $b' > b$. By definition, there exist $e = (N, u, W)$ and $e' = (N', u', W') \in E(F, i, \alpha)$ and $j \in N$, $k \in N'$ such that $(a, b) = (F_j(e), u_j(F_j(e)))$, $(a', b') = (F_k(e'), u_k(F_k(e')))$. As well, there is a wealth W^* such that $e^* = (\{i,j,k\}, (u_i, u_j, u_k), W^*) \in E(F, i, \alpha)$ (same argument as Lemma 1). By SL, we know that $F_j(e^*) = a$, $F_k(e^*) = a'$ and so $F_j(e^*) < F_k(e^*)$. Thus, by PR, $u_j(F_j(e^*)) \geq u_k(F_k(e^*))$. However, we also know that, by hypothesis, $b = u_j(F_j(e^*)) < u_k(F_k(e^*)) = b'$, a contradiction.

We show now that $\{C(F, i, \alpha) : \alpha \in \mathbb{R}_+\}$ is a collection of disjoint sets. Let $\alpha_1 > \alpha_2$. Suppose $(a, b) \in C(F, i, \alpha_1) \cap C(F, i, \alpha_2)$. Let $e_1 = (N_1, u_1, \alpha_1) \in E(F, i, \alpha_1)$, $e_2 = (N_2, u_2, \alpha_2) \in E(F, i, \alpha_2)$ and $j \in N_1$, $k \in N_2$ such that $(a, b) = (F_j(e_1), u_j(F_j(e_1))) = (F_k(e_2), u_k(F_k(e_2)))$. Consider the economies $\widehat{e}_1 = (\{i,j\}, (u_i, u_j), a + \alpha_1)$ and $\widehat{e}_2 = (\{i,k\}, (u_i, u_k), a + \alpha_2)$. SL implies that $F_i(\widehat{e}_1) = \alpha_1$ and $F_i(\widehat{e}_2) = \alpha_2$. By a similar argument to that of Lemma 1, there is a $W > a + \alpha_2$ such that $\widetilde{e}_2 = (\{i,k\}, (u_i, u_k), W) \in E(F, i, \alpha_1)$. By SL, applied to $\widehat{e}_2$ and $\widetilde{e}_2$ we know that $F_k(\widetilde{e}_2) > F_k(\widehat{e}_2) = a$. Therefore,

$(a,b) < (F_k(\widetilde{e}_2), u_k(F_k(\widetilde{e}_2))) \in C(F,i,\alpha_1)$. This contradicts the fact that $C(F,i,\alpha_1)$ is downward sloping. ■

We need additional machinery to prove Lemma 3.

Of two sets A and B in the plane we say that B lies above A if

1. For all $(a_1,a_2) \in A$ there exists $(b_1,b_2) \in B$ such that $(a_1,a_2) < (b_1,b_2)$.
2. There is no $(a_1,a_2) \in A$ and $(b_1,b_2) \in B$ such that $(b_1,b_2) < (a_1,a_2)$.

Clearly, if B lies above A, then A does not lie above B.

Claim. *If $\alpha_1 > \alpha_2$ then $C(F,i,\alpha_1)$ lies above $C(F,i,\alpha_2)$.*

Proof of the claim. Let $(a,b) \in C(F,i,\alpha_2)$, and let $e = (\{i,j\},(u_i,u_j),W) \in E(F,i,\alpha_2)$ such that $F_j(e) = a$, $u_j(F_j(e)) = b$. Since $F_i(e) = \alpha_2$, and by a similar argument to that of the proof of 1, increasing the wealth from its value in e, we eventually find a wealth value W^* such that $e^* = (\{i,j\},(u_i,u_j),W^*) \in E(F,i,\alpha_1)$. Let $(a',b') = (F_j(e^*), u_j(F_j(e^*)))$. Then, $(a',b') \in C(F,i,\alpha_1)$. Furthermore, since F satisfies SL and u_j is strictly increasing, $(a',b') > (a,b)$.

Conversely, let $(a,b) \in C(F,i,\alpha_1)$. Suppose there were a point $(a'',b'') \in C(F,i,\alpha_2)$ such that $(a'',b'') > (a,b)$. We know that there is a point (α_1,d_1) in $C(F,i,\alpha_1)$–because $(F_i(e),u_i(F_i(e))) = (\alpha_1,d_1)$ for any e in $E(F,i,\alpha_1)$– and in like manner there is a point $(\alpha_2,d_2) \in C(F,i,\alpha_2)$, and $d_1 = u_i(\alpha_1) > u_i(\alpha_2) = d_2$ (because both points are associated with agent i). Thus we have

$$\begin{aligned} C(F,i,\alpha_1) &\ni (\alpha_1,d_1) > (\alpha_2,d_2) \in C(F,i,\alpha_2), \\ C(F,i,\alpha_2) &\ni (a'',b'') > (a,b) \in C(F,i,\alpha_1) \end{aligned} \tag{5}$$

Without loss of generality, assume that $\alpha_2 < a''$. Then, it follows that $\alpha_2 < \alpha_1 < a < a''$.[5]

Let $\Lambda(C(F,i,\alpha))$ be the **support** of the curve $C(F,i,\alpha)$, i.e., the wealth values for which there exist welfare levels such that the pairs are achieved under the action of F. Formally:

$$\Lambda(C(F,i,\alpha)) = \{a \in \mathbb{R}_+ : \exists\, b \in \mathbb{R}_+ \text{ s.t. } (a,b) \in C(F,i,\alpha)\}.$$

[5] If $\alpha_2 > a''$, then $a < a'' < \alpha_2 < \alpha_1$, and the ensuing argument would be analogous.

Similarly, let $\Gamma(C(F,i,\alpha))$ be the **range** of the curve $C(F,i,\alpha)$, i.e., the welfare levels for which there exist wealth values such that the pairs are achieved under the action of F. Formally,

$$\Gamma(C(F,i,\alpha)) = \{b \in \mathbb{R}_+ : \exists\ a \in \mathbb{R}_+ \text{ s.t. } (a,b) \in C(F,i,\alpha)\}.$$

For $k = 1,2$, let $\gamma^k : \Lambda(C(F,i,\alpha_k)) \to \mathbb{R}_+$ be the mapping whose graph coincides with the curve $C(F,i,\alpha)$, i.e., $Gr(\gamma) = C(F,i,\alpha)$.[6] Then, by (5) and the fact that $C(F,i,\alpha_k)$ are downward sloping, we have

$$\begin{array}{l} \max\{\gamma^2(\alpha_1)\} \leq \min\{\gamma^2(\alpha_2)\} \leq \max\{\gamma^2(\alpha_2)\} < \min\{\gamma^1(\alpha_1)\} \text{ and} \\ \min\{\gamma^2(a)\} \geq \max\{\gamma^2(a'')\} \geq \min\{\gamma^2(a'')\} > \max\{\gamma^1(a)\}. \end{array} \tag{6}$$

Assume we know that all $C(F,i,\alpha)$ sets are connected. Then, both $Gr(\gamma^1) = C(F,i,\alpha_1)$ and $Gr(\gamma^2) = C(F,i,\alpha_2)$ are connected sets. So are their supports and therefore

$$(\alpha_1, a) \subset \Lambda(C(F,i,\alpha_1)) \cap \Lambda(C(F,i,\alpha_2)).$$

Since $\max\{\gamma^2(\alpha_1)\} < \min\{\gamma^1(\alpha_1)\}$ and $\min\{\gamma^2(a)\} > \max\{\gamma^1(a)\}$, it follows that there exists $x \in (\alpha_1, a)$ such that $\gamma^1(x) \cap \gamma^2(x) \neq \emptyset$, which means that $C(F,i,\alpha_1) \cap C(F,i,\alpha_2) \neq \emptyset$. This contradicts Lemma 2.

Thus, it just remains to show that a $C(F,i,\alpha)$ set is connected. Since $C(F,i,\alpha)$ is downward sloping, it follows that if it is not connected then either $\Lambda(C(F,i,\alpha))$ is not connected or $\Gamma(C(F,i,\alpha))$ is not connected.

Case 1: $\Lambda(C(F,i,\alpha))$ is not connected.

Let $a, b \in \Lambda(C(F,i,\alpha))$ such that $a < b$ and $\lambda \cdot a + (1-\lambda)b \notin \Lambda(C(F,i,\alpha))$ for all $\lambda \in (0,1)$. Since $C(F,i,\alpha)$ is downward sloping, it follows that $\min\{\gamma(a)\} \geq \max\{\gamma(b)\}$. Fix $\widehat{\lambda} \in (0,1)$ and let $\widehat{x} = \widehat{\lambda} \cdot a + (1-\widehat{\lambda})b$. Consider $\theta = \frac{\min\{\gamma(a)\}}{\widehat{x}}$ and let $u_j(x) = \theta \cdot x$, for all $x \in \mathbb{R}_+$. Then,

$$\max\{\gamma(b)\} \leq u_j(\widehat{x}) = \min\{\gamma(a)\}.$$

[6] Note that γ may well be a multi-valued function.

We know, by Lemma 1, that there exists $w \in \mathbb{R}_+$ such that $(w, u_j(w)) \in C(F, i, \alpha)$. Thus, $u_j(w) \cap \gamma(w) \neq \emptyset$.

Now, since $C(F, i, \alpha)$ is downward sloping, we have that

$$y < u_j(\widehat{x}) \text{ for all } y \in \gamma(x) \text{ such that } x \in \Lambda(C(F, i, \alpha)) \text{ and } x > b,$$

and

$$y > u_j(\widehat{x}) \text{ for all } y \in \gamma(x) \text{ such that } x \in \Lambda(C(F, i, \alpha)) \text{ and } x < a.$$

Since u_j is strictly increasing, it follows that $u_j(x)$ and $\gamma(x)$ do not cross, which is a contradiction.

Case 2: $\Gamma(C(F, i, \alpha))$ is not connected.

Let $a, b \in \Gamma(C(F, i, \alpha))$ such that $a < b$ and $\lambda \cdot a + (1-\lambda)b \notin \Gamma(C(F, i, \alpha))$ for all $\lambda \in (0, 1)$. Assume there exists $\overline{x}$ such that $\{a, b\} \subset \gamma(\overline{x})$.[7] Fix $\widehat{\lambda} \in (0, 1)$ and let $\theta = \frac{\widehat{\lambda} \cdot a + (1-\widehat{\lambda})b}{\overline{x}}$. Consider $u_j(x) = \theta \cdot x$, for all $x \in \mathbb{R}_+$. Then, $u_j \in \mathcal{U}$ and $u_j(\widehat{x}) = \widehat{\lambda} \cdot a + (1-\widehat{\lambda})b$. We know, by Lemma 1, that there exists $w \in \mathbb{R}_+$ such that $(w, u_j(w)) \in C(F, i, \alpha)$. Thus, $u_j(w) \cap \gamma(w) \neq \emptyset$.

Now, since $C(F, i, \alpha)$ is downward sloping, it follows that

$$y < u_j(\widehat{x}) \text{ for all } y \in \gamma(x) \text{ such that } x \in \Lambda(C(F, i, \alpha)) \text{ and } x > \widehat{x},$$

and

$$y > u_j(\widehat{x}) \text{ for all } y \in \gamma(x) \text{ such that } x \in \Lambda(C(F, i, \alpha)) \text{ and } x < \widehat{x}.$$

Since u_j is strictly increasing, it follows that $u_j(x)$ and $\gamma(x)$ do not cross, which is a contradiction. This completes the proof of the claim.

Proof of Lemma 3

Since F satisfies PR, it is straightforward to show that $\varphi(0, 0) = 0$ and $\varphi(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2_+$.

$$\varphi(0, 0) = 0 \leq \varphi(x, y) \text{ for all } (x, y) \in \mathbb{R}^2_+. \tag{7}$$

[7] If this were not the case, then $\Lambda(C(F, i, \alpha))$ would not be connected, and the proof of Case 1 would be valid to conclude.

Let $x, x', y \in \mathbb{R}_{++}$ such that $x < x'$. If $\varphi(x,y) > \varphi(x',y)$ then, by the above claim, $C(F,i,\varphi(x,y))$ lies above $C(F,i,\varphi(x',y))$. In such a case, since $(x',y) \in C(F,i,\varphi(x',y))$, there exists $(z,t) \in C(F,i,\varphi(x,y))$ such that $(x',y) < (z,t)$. Then, $(z,t) > (x,y)$. This contradicts Lemma 2. Thus,

$$\varphi(x,y) \leq \varphi(x',y) \text{ for all } x, x', y \in \mathbb{R}_{++} \text{ such that } x < x'. \tag{8}$$

Similarly,

$$\varphi(x,y) \leq \varphi(x',y) \text{ for all } x, x', y \in \mathbb{R}_{++} \text{ such that } y < y'. \tag{9}$$

Finally, let $(x,y),(z,t) \in \mathbb{R}^2_+$ such that $(x,y) > (z,t)$. By downward sloppiness, $\varphi(x,y) \neq \varphi(z,t)$. If $\varphi(x,y) < \varphi(z,t)$ then, by the above claim, $C(F,i,\varphi(z,t))$ lies above $C(F,i,\varphi(x,y))$. We have, however, that $(x,y) \in C(F,i,\varphi(x,y))$, $(z,t) \in C(F,i,\varphi(z,t))$ and $(x,y) > (z,t)$, which represents a contradiction. Thus,

$$\varphi(x,y) > \varphi(z,t) \text{ for all } (x,y),(z,t) \in \mathbb{R}^2_+ \text{ such that } (x,y) > (z,t). \tag{10}$$

We show now that φ is continuous on $\mathbb{R}^2_+$. To do so, let $\{(a_n,b_n)\}_n$ be a sequence in $\mathbb{R}^2_+$ converging to $(a,b) \in \mathbb{R}^2_+$. We must show that $\{\alpha_n\}_n = \{\varphi(a_n,b_n)\}_n$ converges to $\alpha = \varphi(a,b)$. Of the three sets: $\Omega^> = \{n \in \mathbb{N} : \alpha_n > \alpha\}$, $\Omega^< = \{n \in \mathbb{N} : \alpha_n < \alpha\}$, $\Omega^= = \{n \in \mathbb{N} : \alpha_n = \alpha\}$, at least one is infinite. If $\Omega^=$ is the only infinite set, then the claim is obviously true. So suppose this is not the case; let $\Omega^<$ be infinite. (The proof if $\Omega^>$ is infinite is the same.) The claim is true unless $\Omega^<$ has a limit point $\overline{\alpha} < \alpha$. Therefore, suppose that this were the case. Denote by $\{\alpha_k\}$ a subsequence of $\Omega^<$ that converges to $\overline{\alpha}$. Consider the curve $C(F,i,\frac{\alpha+\overline{\alpha}}{2})$. There is a ball, B, about $(a,b) \in C(F,i,\alpha)$ which, by the above claim, lies above this curve because $\alpha > \frac{\alpha+\overline{\alpha}}{2}$. But for large k, $(a_{\alpha_k}, b_{\alpha_k}) \in B$. This is impossible, since for large k, all points in $C(F,i,\alpha_k)$ lie below $C(F,i,\frac{\alpha+\overline{\alpha}}{2})$. Thus, φ is continuous on $\mathbb{R}^2_+$. This, together with (7), (8), (9) and (10), shows that $\varphi \in \Phi$, as desired. ■

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