In progress

# TOWARDS A THEORY OF STORABLE VOTES

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Abstract

Motivated by the need for more flexible decision-making mechanisms in the European Union, the paper proposes a simple but novel voting scheme for deliberations taken by committees that meet regularly over time. At each meeting, committee members are allowed to store their vote for future use. The decision is then taken according to the majority of votes cast. The possibility of shifting votes intertemporally allows agents to concentrate their votes when preferences are more intense or the probability of being pivotal is higher. Although the scheme cannot achieve the first best with more than two voters, making votes storable typically leads to ex ante welfare gains. I conjecture that the result will hold in general if the number of voters is above a minimum threshold or the horizon is long enough.

## A Simple Game

A committee of $n$ individuals meets regularly to take a common decision $d$. To keep matters as simple as possible, suppose that the decision can take only two values: $d \in \{0,1\}$. We can think of $d = 0$ as maintaining the status quo, and $d = 1$ as change. Each member of the committee has preferences over the common decision, preferences indexed by a parameter $\nu$ : in period $t$, individual $i$'s utility equals $\nu_{i,t}\, d_t$ . The parameter $\nu_{i,t}$ is drawn from a distribution $F(\nu)$, defined over the support $[-1, 1]$ and that we will assume symmetric around zero. Thus whenever the realization of $\nu_{i,t}$ is negative, $i$ prefers $d_t = 0$; when $\nu_{i,t}$ is positive, $i$ prefers $d_t = 1$, and the absolute value of $\nu_{i,t}$ measures the intensity of $i$'s preferences. The distribution $F(\nu)$ is common across all committee members and all periods, and $\nu_{i,t}$ is independently distributed both across individuals and across time. The committee takes the decision every period for a total of $T$ periods, where $T$ is finite.

For concreteness, we can think of the committee as the Executive Council of the European Central Bank, meeting each month to decide whether to maintain current interest rates ($d = 0$), or to change them ($d = 1$), under the assumption that both the direction of the possible change and, more controversially, its size are known before the meeting. Each member of the Council has preferences over European monetary policy and these preferences need not be homogenous, reflecting different needs of the national economies, possibly different objectives or different individual perceptions of Europe-wide conditions. Each member's preferences are summarized by $\nu_{i,t}$. The assumption of i.i.d. shocks is not ideal in this context, but we will maintain it in this initial paper because it simplifies the analysis greatly without being fundamental to the logic of the scheme proposed.

Every period each committee member is given one vote. He can cast it in favor of the option he prefers, or store it for use at a later time. Thus in period $1$, a member can cast either $0$ or $1$ votes; if he decides to save his vote, in period $2$ he will have a total of $2$ votes at his disposal and will decide how many of these, if any, to use; and so on in all successive periods until time $T$ when the game ends. It is assumed that votes can be stored but not borrowed to avoid the difficult problems that could arise in practice if one member were to run out of votes.[1] Subject to the budget constraint that the votes cast cannot exceed the number of votes available, each member is asked to indicate his preferred decision and the number of votes he is willing to spend to support his choice. When individuals vote, they know the realization of their current $F_{i,t}$ and they know the probability distribution $F$, but cannot observe the preferences of the other members. On the other hand, since the initial allocation of votes and the history of the game are known, the number of votes that each player has at his disposal is common knowledge.

The committee selects $d$ according to which of the two alternatives has received more votes. If the votes are equal, the preferences of members who have cast zero votes are considered; if the tie is still not broken, the decision is taken with a coin toss. The tie-breaking mechanism seems plausible and has some advantages in deriving analytical solutions, but does not affect the substance of the results.[2]

The individuals' objective is to announce a policy preference and choose a number of votes each period so as to maximize the expected flow of utility over the whole time horizon. Given a discount factor *, the problem amounts to maximizing $EU^{i} = E(3_{t} *^{t} F_{i\,t}\, d_{t})$, where $E$ is the expectations operator, subject to the constraint that for each committee member the stock of available votes $k_{i\,t}$ equals the votes stored the previous period plus the allocation of $1$ new vote ($k_{i\,t} = k_{i\,t-1} - v_{i\,t-1} + 1$, where $v_{i\,t-1}$ are votes cast by $i$ at $t$-$1$). The state of the game is given by the distribution of

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[1]The constraint on borrowing is common to policy mechanisms that rely on market-type behavior (for example environmental regulation through tradable pollution licences), because it reduces the costs of mistakes and inexperience, and increases the credibility of the rules. In addition, when members are subject to appointments or elections, the inability to borrow from the future limits the extent to which current members can expropriate the power of their successors. In our model too, the constraint is imposed for purely pragmatic considerations.

[2]I have experimented with other tie-break rules - no weight on zero voters; status quo wins when votes are tied (with or without considering zero voters) - and always obtained the same qualitative results. In the application of the game to the European Central Bank, an alternative model is also possible: we could assume that the decision is between a cut ($d=-1$) or an increase ($d=1$) in interest rates, with the status quo ($d=0$) prevailing in case of ties. However this set-up would minimize the role of the status quo, while in fact maintaining the status quo is often the preferred option for most central bankers. In any case, the results of the two models are identical up to a factor of proportionality in expected payoffs.

available votes among all members; if we call $K_t$ the vector of votes held by players at time $t$, we can write the expected value of the game to individual $i$ when all players follow optimal strategies as $EV_t^{\,i}$ $(K_t\,, t)$.

The goal of the paper is to compare the storable votes scheme to the more traditional case where votes are not transferable over time, and thus each individual always casts one vote in favor of his preferred alternative. The two games are obviously identical if the time horizon reduces to a single period, but differ otherwise, and the objective is to calculate the value of the two games as the time horizon lengthens. The storable votes game requires some thought: the choice of how many votes to cast reflects not only the current intensity of preferences, relative to expected future preferences, but also the probability that a vote be pivotal, today or in the future, and thus the expectation of the other players' voting behavior over time. Formally, it is a non-stationary dynamic game, where each individual's optimal strategy will be conditioned on the realization of his preference shock, on the distribution of available votes among all players and on calendar time. For this reason, and to build intuition for the results that will follow, it is good to begin with the simplest case.

Two periods, two players.

Two players $i$ and $j$ must take decision $d$. At period $T$-$1$, they are endowed with $1$ vote each; they will both receive an additional vote at period $T$, but the game will then end. At $T$ they will both spend all available votes on their preferred alternative; thus the only problem each player must solve is what preference to announce and whether to cast $1$ or $0$ votes in its support at $T$ -$1$. Because preferences and votes are announced simultaneously and preferences shocks are i.i.d., a voter will always announce preferences truthfully: he cannot manipulate his opponent's strategy. The choice reduces to the number of votes to cast.

Consider individual $i$, and suppose for now that the realization of his preference shock $F_{iT\text{-}1}$ is positive. Thus $i$ prefers $d = 1$. If $i$ saves his vote, his expected utility is given by:

$$Eu^i{}_{T-1}(v_i = 0) = s_{i\,T-1}(3/4p_{j0} + p_{j1}/2) + d\left(p_{j0}EV^i{}_T(2{,}2) + p_{j1}EV^i{}_T(2{,}1)\right) \qquad (1)$$

where $p_{j\,v}$ is the probability that $j$ casts $v$ votes today, and $EV^i{}_T\,(s,k)$ is $i$'s expected value of the game in the next and final period, given stocks of available votes $s$ (for player $i$) and $k$ (for player $j$). If $\varepsilon_{i\,T-1}$ is positive, $i$'s expected current return is $\varepsilon_{i\,T-1}$, times the probability of obtaining the desired decision $d = 1$ when playing $0$ votes. If $j$ plays $0$ (i.e. casts $0$ votes) too, with probability ½ $d = 1$ is chosen because it is both individuals' desired option (given the symmetry of the distribution function $F(\varepsilon)$, and even if neither player is willing to spend any votes); with probability ½ $j$ prefers $d = 0$, but even in that case $i$ has a fifty per cent chance of winning the coin toss. Hence if $j$ plays $0$, the probability that $d = 1$ equals $3/4$. If instead $j$ plays $1$, then $d = 1$ is chosen only if it is $j$'s preferred option, which occurs with probability ½. Thus $i$'s expected current return must equal $\varepsilon_{i\,T-1}\,(3/4\;p_{jo} + 1/2\;p_{j1})$, the first term in equation (1). As for next period, if $i$ plays $0$ today, he will then have $2$ votes available, while $j$'s votes will depend on $j$'s current strategy.

Similarly, if individual $i$ casts his vote at $T$-$1$, his expected utility is:

$$Eu^i{}_{T-1}(v_i = 1) = \varepsilon_{iT-1}(p_{j0} + 3/4\,p_{j1}) + \delta\left(p_{j0}EV^i{}_T(1,2) + p_{j1}EV^i{}_T(1,1)\right) \qquad (2)$$

taking into account that $i$ always obtains his preferred option at $T$-$1$ if $j$ plays $0$, or with probability $3/4$ if $j$ plays $1$. Comparing (1) and (2), we obtain that $i$ will play $1$ at $T$-$1$ if and only if:

$$\varepsilon_{iT-1}(p_{j0} + p_{j1})/4 \geq \delta\left(p_{j0}\left(EV^i{}_T(2,2) - EV^i{}_T(1,2)\right) + p_{j1}\left(EV^i{}_T(2,1) - EV^i{}_T(1,1)\right)\right) \qquad (3)$$

Solving next period's expected values, we can obtain an explicit solution to the optimal strategy. In period $T$, both players cast all votes they have, and the one with most votes wins with probability $1$. Thus:

$$EV^i{}_T(2,1) = \int_{-1}^{0} s\,dF(s)(0) + \int_{0}^{1} s\,dF(s)(1) = \int_{0}^{1} s\,dF(s) \qquad (4)$$

since whenever $\varepsilon_{iT}$ is negative, $i$ will be able to impose $d = 0$, and whenever $\varepsilon_{iT}$ is positive, $d$ will equal $1$. The player with fewer votes will not be able to influence the choice of $d$, but half of the times his opponent's preferred choice matches his own. Hence:

$$EV^{i}{}_{T}(1{,}2) = 1/2\int_{-1}^{0} sdF(s) + 1/2\int_{0}^{1} sdF(s) = 0 \tag{5}$$

Finally, when the two players have the same number of votes, the value of the game at period *T* is identical to the value of the one-period non-storable votes game (with equal votes). Call the value of this latter game *W*, noticing that it is time independent and that any number of equal votes is equivalent. For any realization of $F_{iT}$, player *i* expects to obtain his preferred value of *d* three quarters of the times, either because *j* has the same preference (fifty per cent of the times), or because when they do disagree, the coin toss is in *i*'s favor (twenty five per cent of the times). That is:

$$EV^{i}{}_{T}(1{,}1) = EV^{i}{}_{T}(2{,}2) = \int_{-1}^{0} dF(s)\big(3/4(0) + 1/4\,(1)\big) + \int_{0}^{1} sdF(s)\big(3/4\,(1) + 1/4\,(0)\big)$$

or:

$$EV^{i}{}_{T}(1{,}1) = EV^{i}{}_{T}(2{,}2) = 1/2\int_{0}^{1} sdF(s) = W \tag{6}$$

Substituting (4), (5) and (6) in (3), we obtain:

$$s_{i\,T-1}(p_{j0} + p_{j1})/4 \geq d\,(p_{j0} + p_{j1})W$$

or:

$$s_{i\,T-1} \geq 4d\,W \qquad\qquad \text{if } s_{i\,T-1}>0 \tag{7}$$

It is easy to verify that if $F_{i\,T-1}$ is negative the same logic leads voter *i* to cast his vote if and only if -$F_{i\,T-1}$ $4*W.[3] We can conclude that *i*'s optimal strategy is to identify a threshold value " /

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[3]If $F_{i\,T-1}$ is negative, *i*'s expected utility from playing *1* or *0* is analogous to equations (1) and (2) above, but the negative preference shock now multiplies the corresponding probability of losing, as opposed to winning (since instantaneous utility is then different from zero only if *i* does not succeed in imposing his preference for *d = 0*). The probability of losing when casting *1* or *0* votes is the complement to *1* of the probability of winning we derived earlier, and the two expressions for expected

$4*W > 0$ such that $i$ will play $1$ whenever $*F_{i\,T-1}*$ is larger or equal to ", and play $0$ whenever $*F_{i\,T-1}*$ is smaller than ". Not surprisingly, the threshold " equals the average intensity of preferences (discounted), and is strictly smaller than $1$ as long as there is any probability mass outside the extreme values $-1$ and $1$. In the simple case where $F(F)$ is Uniform and $* = 1$, $" = 1/2$.

The conclusion was expected: if $i$'s policy preference is particularly strong today, he will be willing to sacrifice some of his possible future power to increase his chances of obtaining the desired outcome; vice versa, if his policy preference is weak, he will prefer to abstain today and increase his influence tomorrow. It was this intuition that motivated the paper. We can now verify whether it leads indeed to desirable welfare properties (at least in this simple two-period game).

Before the preference shock is realized, the expected value of the game for player $i$ equals:

$$\begin{aligned} EV_{T-1}(1,1) = & \int_0^a sdF(s)(3/4p_{j0} + 1/2p_{j1}) + \int_{-a}^0 sdF(s)(1 - 3/4p_{j0} - 1/2p_{j1}) + \\ & + \left(2F(a) - 1\right)dW\left(p_{j0} + 2p_{j1}\right) + \int_a^1 sdF(s)(p_{j0} + 3/4p_{j1}) + \\ & + \int_{-1}^{-a} sdF(s)(1 - p_{j0} - 3/4p_{j1}) + 2\left(1 - F(a)\right)dWp_{j1} \end{aligned} \tag{8}$$

The expression seems unwieldy, but is easily simplified. Because $j$ faces the identical problem, he will also condition his voting behavior on the same threshold ", and will vote $0$ with probability $2[F(")-1/2]$, and $1$ with probability $2[1-F(")]$ (where I am using, as in (8), the symmetry of the probability distribution).[4] Substituting these values for $p_{j0}$ and $p_{j1}$, we can write the expected value of the two-period game for either player as:

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utility are then immediately calculated.

[4]Notice that the equilibrium strategy is unique; in fact in this two-period example it is a dominant strategy.

$$EV_{T-1}(1{,}1) = \int_0^{\alpha} s\,dF(s)\big(F(\alpha) - 1/2\big) + \int_{\alpha}^{1} s\,dF(s)\; F(\alpha) + \delta W \qquad (9)$$

We can now establish the first result of the paper. If $W$ is the value of the one-shot non-storable votes game, call $W_{T-1}$ the value of the two-period non-storable votes game (where the finite horizon yields the unique equilibrium value $W_{T-1} = W(1+\delta)$). Then we can state:

***Proposition 1.*** *For any symmetrical distribution $F(\sigma)$, $EV_{T-1}(1,1) > W_{T-1}$ .*

***Proof***. Notice first that when $\alpha = 0$, $F(\alpha) = ½$, and when $\alpha = 1$, $F(\alpha) = 1$. In both cases, $EV_{T-1}(1,1) = W_{T-1}$. To understand the behavior of $EV_{T-1}(1,1)$ in the interval $\alpha \in (0,1)$, we can differentiate $EV_{T-1}(1,1)$ with respect to $\alpha$:

$$\frac{\partial EV_{T-1}}{\partial \alpha} = f(\alpha)\left(\int_0^1 s\,dF(s) - \alpha/2\right) \qquad (10)$$

where $f(\alpha)$ is the density $f(\sigma)$ evaluated at $\alpha$, and thus is positive. This derivative is positive at $\alpha = 0$ and has a single root; since we know $EV_{T-1}(1,1) = W_{T-1}$ at $\alpha = 0$ and at $\alpha = 1$, it follows that $EV_{T-1}(1,1) > W_{T-1}$ for all $\alpha \in (0,1)$. And since we know that $\alpha$ is strictly positive and smaller than $1$, the conclusion follows. ■

Proposition 1 states that the two-period game with two players has higher expected value when the votes can be stored. The result is not surprising and is clearly visible in expression (9): the possibility to store votes increases the likelihood that a player will win when his preference is stronger. The positive threshold $\alpha$ shifts probability mass from payoffs with relatively low value (when $|\sigma|$ is smaller than $\alpha$) to payoffs with relatively higher value (when $|\sigma|$ is larger than $\alpha$), and thus increases ex ante welfare. The analysis is simplified by assuming i.i.d. shocks and a symmetrical probability distribution, but intuitively it seems unlikely that the conclusion would be sensitive to these aspects of the problem.[5]

Notice that the proof of Proposition 1 does not rely on the equilibrium value of " - any threshold strictly between *0* and *1* would lead to welfare gains. This seems a potentially important result - if practical applications of the mechanism are seriously considered, its robustness to mistakes should be investigated further, especially in light of the difficulty of the full dynamic problem, as the next section begins to make clear.

## *T* periods, two players.

Although the logic of the problem is unchanged, the analysis becomes quickly more complicated as the number of periods increases. The difficulty is that the number of options to be considered increases very rapidly with the length of the horizon: starting from state $(k^i, s^j)$ at $t$, we need to evaluate $(k^i+1)(s^j+1)$ possible states at time $t+1$, and of course the expected value of each of these states must be solved backwards from all the possible options it itself can give rise to, and so on at all times, using as anchor the expected values of all possible different states in the terminal periods.[6] It is clear that the only possible solution method must be recursive. But here we encounter another problem: the game is non-stationary, and the equilibrium strategies depend both on the current state and on calendar time. To calculate the expected values of future states, we need to weigh them by their probability of realization, and hence by the probabilities of the players' alternative strategies in

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[5]Correlation in players' preferences will reduce the incentive to vote, but seems unlikely to affect the nature of the result. Autocorrelation of each player's preferences over time creates a more difficult problem: it could induce non-truthful voting in an effort to manipulate future voting decisions by one's opponent. In a two-period model, the final result should not be very different, but with a longer horizon the analysis would have to be modified substantially.

[6]Consider for instance the *4*-period game starting with state *(1,1)*. This initial state can give rise to *4* possible states at $t+1$ ( *(1,1), (2,1), (1,2), (2,2)*). But the value of state *(2,2)* at $t+1$, to take only one of them, reflects the expected values of the *9* possible states that at $t+2$ can result from *(2,2)* (since each of the two players at $t+1$ has the option of playing *0, 1* or *2* votes), and each of these *9* states itself reflects the values of the states it can give rise to in the last and terminal period *T* - for example state *(3,3*) at $t+2$ can give rise to *16* possible states at *T*.

equilibrium. And these change over time, even for given states. It is then a welcome surprise to be able to obtain analytical results.

Call $v_t$ the vector of strategies ($v_t^{\,i}, v_t^{\,j}$), i.e. the number of votes cast by the *2* voters at time *t*. After the realization of the preference shock $F_{i\,t}$ , player *i*'s expected value of the game at state $K_t$ is given by:

$$EV_t^{\,i}(v_t; s_{\,it}, K_t) = \max_{v_t^{\,i}} \left\{ E_{s_{\,jt}} u^i(v_t; s_{\,it}, K_t) + dE_{s_{\,jt}}[E_{s_{\,t+1}} V_{t+1}^{\,i}(K_{t+1}, s_{\,t+1})] \right\} \qquad (11)$$

where *i* takes as given player *j*'s strategy $v_t^{\,j}$. The first expectation inside the curly brackets is taken over possible realizations of $F_{j\,t}$, *j*'s preference shock (which affects *i*'s payoff because it influences *j*'s strategy). The second expectation is taken both over $F_{j\,t}$, because *j*'s current strategy affects the state of the game next period. and $F_{t+1}$ (the vector ($F_{i\,t+1}$, $F_{j\,t+1}$)), because next period preferences affect both strategies and payoffs next period. It is possible to show that the following results must hold:

***Proposition 2***: *(i) At any t, there exist pure strategies* $v_t^*$ *such that*:

$$EV_t^i(v_t^{\,i*}, v_t^{\,j*}; s_{\,it}, K_t) \geq EV_t^i(v_t^{\,i}, v_t^{\,j*}; s_{\,it}, K_t) \qquad \forall\, v_t^{\,i} \neq v_t^{\,i*}.$$

*(ii)* $v_t^{\,i*}$ *is monotonically increasing in* $*F_{i\,t}*$, *the player's intensity of preferences.*

(The proofs are in the Appendix).

In addition to establishing that an equilibrium exists, Proposition 2 states that in equilibrium a voter will never casts fewer votes when the intensity of his preferences is higher than when it is lower. This simple conclusion allows us to characterize equilibrium strategies a bit more precisely. Notice that the number of votes that a player has at his disposal is always finite, while the support of $*F_{i\,t}*$, the segment *[0,1]*, is continuous. Thus at any state of the game and time *t*, each voter must identify a series of thresholds that divide the segment *[0,1]* into a finite number of intervals. For all realizations of $*F_{i\,t}*$ in a given interval, *i* casts the same number of votes, but higher intervals must correspond to a larger number of votes. The thresholds are functions of the state of the game, calendar time and the opponent's equilibrium strategies, and although their number cannot be larger than the number of votes the player

has available, it can well be smaller -
some feasible number of votes may never be cast in equilibrium.

Proposition 2 does not state that the equilibrium is unique, and although we found it to be unique in all the numerical exercises we have run with *2* voters, we should not expect uniqueness to hold in general. In particular, it is not required for what follows. The important point is that Proposition 2 allows us to characterize each player's expected instantaneous utility in equilibrium. Consider for example the symmetrical state $(k_t, k_t)$, where both voters enter the period with identical stocks of votes. Call $Eg_t^i(k_t, k_t)$ $i$'s expected one period equilibrium utility (or payoff) before the realization of the preference shock when both players play optimal strategies, and $\alpha_{v-1}(K,t) \leq \alpha_v(K,t)$ the equilibrium thresholds such that $i$ will cast $v\text{-}1$ votes for all $|\sigma_{it}| \in [\alpha_{v-1}(K,t), \alpha_v(K,t))$. Then in a symmetrical equilibrium:

$$\begin{aligned} Eg_t^i(k_t, k_t) = & \int_0^{\alpha_1(t,k)} \sigma dF(\sigma)\big(F(\alpha_1(t,k)) - 1/2\big) + \int_{\alpha_1(t,k)}^{\alpha_2(t,k)} \sigma dF(\sigma)\big(F(\alpha_1(t,k)) + F(\alpha_2(t,k)) - 1\big) + \ldots \\ & \ldots\ldots + \int_{\alpha_{k-1}(t,k)}^{\alpha_k(t,k)} \sigma dF(\sigma)\big(F(\alpha_k(t,k)) + F(\alpha_{k-1}(t,k)) - 1\big) + \int_{\alpha_k(t,k)}^{1} \sigma dF(\sigma) F(\alpha_k(t,k)) \end{aligned} \tag{12}$$

where $0 \leq \alpha_v(k,t) \leq \alpha_{v+1}(k,t) \leq 1, \forall t, \forall v \in \{1,..,k\text{-}1\}$.[7] A more cumbersome but analogous expression describes expected one-period equilibrium payoffs in asymmetrical states.

Player $i$'s expected value of the game at state $(s_t^i, k_t^j)$ before the realization of the preference shock, is given by:

$$\begin{aligned} & EV_t^i(s_t^i, k_t^j) = \\ & \qquad Eg_t^i(s_t^i, k_t^j) + \delta EV_{t+1}^i(s_t^i - |v_t^i| + 1, k_t^j - |v_t^j| + 1) \end{aligned} \tag{13}$$

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[7]Equation (12) is analogous to (8) above. When $i$ casts $v_i$ votes, he obtains the decision he prefers with probability $1 * prob(v_j < v_i) + 3/4 * prob(v_j = v_i) + 1/2 * prob(v_j > v_i)$, or, exploiting Proposition 2, $½\,[F(\alpha_i) + F(\alpha_{i+1})]$. For each interval of $\sigma_i$ values corresponding to a given strategy, $i$'s expected return is weighted by the probability of the decision he prefers minus the probability of the decision he opposes (to account for negative realizations of $\sigma_i$ - see (8)), or $F(\alpha_i) + F(\alpha_{i+1}) - 1$.

where, with abuse of notation, we use a single expectations operator although $EV_{t+1}$ must be calculated by taking expectations over both $\digamma_t$ and $\digamma_{t+1}$. (See the Appendix for a more detailed description of how to proceed).

Expressions (12), and its analogue in asymmetrical states, and (13) allow us to establish the main result of this section:

***Proposition 3***. *For any symmetrical distribution F($\digamma$) and any $t < T$, $EV_t(1,1) > W_t$ .*

***Proof***. Intuitively, the objective is to reduce the expected value of the game at the initial period $t$ to the sum of the expected one-period equilibrium payoffs corresponding to each possible state in all future periods. Exactly as in the 2-period case, when the state is symmetric, the possibility of storing votes when preferences are weak makes expected one-period payoffs higher than in the game with non-storable votes. The problem comes in non-symmetrical states: it is the prospect of being the weaker player in these states, possibly protracted over time and absent by assumption from the game with non storable votes, that creates concerns. But notice that from any symmetrical state $(k_t, k_t)$, the probability of reaching state $(s_{t+J}^{\ i}, k_{t+J}^{\ j})$ is identical to the probability of reaching state $(k_{t+J}^{\ i}, s_{t+J}^{\ j})$. Thus when evaluating possible future states, a player will give the same weight to the two opposite asymmetrical states and in effect consider their mean expected payoff. All we require then is that this mean payoff be higher, or at least not smaller, than the expected payoff with non-storable votes. It is this observation that allows us to establish the Proposition.

The intuition is formalized in the following two results:

***Lemma 1.***

$$i) \quad Eg_t^{\ i}(k_t, k_t) \geq W \quad \forall\, t, \text{ with strict inequality at T-1.}$$
$$ii) \quad Eg_t^{\ i}(s_t^{\ i}, k_t^{\ j}) + Eg_t^{\ i}(k_t^{\ i}, s_t^{\ j}) \geq 2W \quad \forall\, t.$$

***Lemma 2.*** *Suppose the following inequalities hold at t+1*:

$$i) \quad EV_{t+1}^{\ i}(k_{t+1}, k_{t+1}) > W_{t+1}$$
$$ii) \quad EV_{t+1}^{\ i}(s_{t+1}^{\ i}, k_{t+1}^{\ j}) + EV_{t+1}^{\ i}(k_{t+1}^{\ i}, s_{t+1}^{\ j}) \geq 2W_{t+1}$$

*Then they must hold at t.*

The proofs of the two lemmas amount to manipulating expected equilibrium payoffs (expression (12) and its counterpart in asymmetrical states) and the dynamic programming equation (13). They can be found in the Appendix.

Once the two lemmas are established, Proposition 2 follows immediately. Because all votes are cast at $T$, $EV_T\,(s_T^{\,i}, k_T^{\,j}) + EV_T\,(k_T^{\,i}, s_T^{\,j}) = 2W$ ; in addition in all symmetrical equilibria at $T$-$1$, $EV_{T-1}\,(k_T, k_T) = Eg_{T-1}\,(k_T, k_T) + {*}W > W_{T-1}$ by Lemma 1. By induction, the inequalities hold at all previous times $t$, and in particular $EV_t\,(1,1) > W_t$ at any $t < T$. #

The result confirms that the intuition highlighted so clearly in the 2-period example extends to a longer horizon. Indeed we would expect the ratio $EV_t\,(1,1)/W_t$ to be higher the smaller is $t$, or the larger is $(T\text{-}t)$, the horizon remaining before the end of the game, a property that is not implied directly by the two lemmas but that was satisfied by all numerical simulations we investigated.

Notice that once again the proof of Proposition 3 makes no use of the exact values of the equilibrium thresholds, but holds for any strategies that satisfy Proposition 2. For example, some positive welfare gains would still be realized if player $i$ holding $k$ votes followed this simple rule of thumb: at any $t$, divide the interval $[0,1]$ in $k+1$ subintervals of equal size, and once $F_i$ is realized cast $v_i$ votes, where $v_i$ satisfies: $v_i\,/(k+1)\ \#\ F_i < (v_i+1)/(k+1),\ v_i\ 0\ \{0,1,..,k\}$. As commented earlier, the robustness of the welfare properties of this voting scheme to (some) strategic mistakes would deserve further study if practical applications were seriously considered.

## N players.

Because the voting mechanism proposed here is new, it was important to verify that it behaves as expected in the reasonably transparent case of *2* players. Not only its welfare properties, but the rationale behind these properties can then be seen quite clearly. But of course the real applicability and hence the real test of the scheme must come from situations involving more than *2* voters. There are two reasons to anticipate complications. First of all, as in the case of markets for votes, when the number of voters is larger than *2* there is an externality built into the scheme that must prevent the achievement of the first-best. A voter's decision to store his vote or cast additional votes changes everybody else's probability of obtaining their desired outcome and affects the value of the votes they have stored for the future. But these considerations do not enter the individual's calculations, as they instead should if full efficiency were to be achieved.[8] The implication is that any mechanism that falls short of the full Coasian bargain remains anchored in a second best world, and its welfare properties will be hard to characterize. This difficulty however does not mean that we should not expect welfare gains relative to non-storable votes - only that we cannot take these gains for granted. As remarked by Piketty (1995) with respect to interpersonal markets for votes, the possibility to account for the intensity of voters' preferences will typically overcome some of the efficiency losses due to the externality and lead to higher expected welfare than with non-tradable - or in our case non-storable - votes.[9]

The second difficulty is that the scheme's potential efficiency gains depend on the number of voters in non-trivial ways. Shifting votes from low to high realizations of the preferences' intensity parameter always shifts some of the probability of obtaining the desired decision from the former to the latter cases - but the magnitude of the change depends on the size of the committee (and, not too surprisingly on whether the number of voters is odd or even). As a simple illustration, Figure 1 reports the percentage increase in the ex ante probability of obtaining one's desired outcome from holding 2

[8]It is easy to see for example how diffuse but weak preferences might lose against a determined minority even in those cases where a static utilitarian concept (the sum of utilities) would dictate otherwise. The correct efficient yardstick is more complex because it involves the full intertemporal problem, but the principle is unchanged.

[9]Piketty's objection to tradable votes is different and stems from the more fundamental tension between market-type voting mechanisms and the optimal transmission of idiosyncratic but socially useful signals. This point is discussed in the Conclusions.

votes (when everybody else has a single one) in the one period game where all votes are cast.[10] As expected, the marginal impact of an extra vote (typically) diminishes with the size of the committee. However, when the size is small enough the extra vote is much more valuable if the number of voters is even rather than odd, and, unexpectedly, the impact increases, instead of falling, when the size of the committee goes from 3 to 5 voters.

These effects depends on the discreteness of the number of votes and voters and are independent of any potential inefficiency. In other words, even if voters chose their voting strategies cooperatively - ie even if the externality were taken into account and the thresholds set optimally - the expected welfare gains would still depend on *n*. Since in addition the severity of the externality itself changes with *n*, the number of voters affects the final welfare properties of the scheme through two separate channels which are not easy to disentangle and characterize systematically.

Figure 2 provides a clear summary of these observations. It depicts the ratio $EV_{T-1}(1,1,1,...,1)/\,W_{T-1}(n)$ in the next to last period of the game when *F(F)* is Uniform and **=1*, as function

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[10]When every voter casts *1* vote, the probability of obtaining one's preferred decision equals the probability that at least *(n-1)/2* other members agree, if *n* is odd; and the probability that either *n/2* other members agree, or that exactly *(n/2)-1* do and the tie-break is favorable, if *n* is even. Or: $1/2\left[1+(1/2)^{n-1}\binom{n-1}{\frac{n-2+I_n}{2}}\right]$ where $I_n$⁄*1* if *n* is odd, and *0* if *n* is even. When voter *i* alone controls *2* votes, he needs the agreement of at least *(n/2)+1* others if *n* is even, and of at least *(n-1)/2* or exactly *(n-3)/2* plus a favorable tie-break, if *n* is odd. The probability of *i*'s preferred outcome is then $\left[1/2+(1/2)^{n-1}\binom{n-1}{\frac{n-2}{2}}\right]$ if *n* is even, and $1/2\left[1+(1/2)^{n-1}\binom{n-1}{\frac{n-3}{2}}\frac{2n}{n-1}\right]$ if *n* is odd. It is not difficult to see that the probability of winning goes from *3/4* to *7/8* when *n* is*3,* a percentage increase of *1/6*; and from *11/16* to *13/16* when *n* is *5*, a percentage increase of *2/11*.

of the number of voters.[11] Three features are particularly noticeable: First, the ratio is larger than *1*, i.e. the welfare gains are positive, for all *n* different from *3* or *5*. Second, the ratio behaves differently for *n* odd and *n* even: especially when the number of players is small, the welfare gains from the scheme are much higher for *n* even than for *n-1* or *n+1*. However, and this is the third feature, the ratio increases with the number of players if *n* is odd and decreases if *n* is even, finally converging to a value larger than *1* for all *n* large enough. If the parameterization of *F(F)* is changed to allow larger probability mass around *0*, the graph shifts progressively upward, and eventually welfare gains become positive for all *n*, but the qualitative results are otherwise unchanged.

All considered, we are lead to conjecture that the argument for storable votes may be weakest when the number of voters is small and odd. It is therefore on the case of *n=3* that we concentrate in the remainder of the paper.

To gain some insight on possible sources of problems, we need to derive one-period equilibrium payoffs. We must proceed in three steps. First, we verify that equilibrium strategies

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[11]When votes are storable and all players can either cast *1* vote or abstain, the probability of obtaining one's preferred outcome when voting equals:

$$\sum_{x=0}^{n-1} p_1^x p_0^{n-x-1} \binom{n-1}{x} \left[ 1/2 \left( 1 + (1/2)^x \binom{x}{\frac{x-I_x}{2}} \right) \right]$$ where $I_x$/ *1* if $x$ is odd, and *0* if $x$ is even.

The corresponding probability when abstaining equals:

$$\sum_{x=0}^{n-1} p_1^x p_0^{n-x-1} \binom{n-1}{x} \left[ 1/2 \left( 1 + (1-I_x)(1/2)^{n-1} \binom{x}{\frac{x}{2}} \binom{n-x-1}{\frac{n-x-2+I_n}{2}} \right) \right]$$ where $I_n$/ *1* if *n* is odd, and *0* if *n* is even.

As argued more formally below, the probabilities $p_1(n)$ and $p_0(n)$ continue to depend on a threshold *"(n)* such that $p_1(n) = 2[1-F("(n))]$ and $p_0(n) = 2[F("(n))-1/2]$. On the basis of these equations, it is possible to derive expected payoffs and the expected value of the game. The numerical exercises model *F(F)* as a Beta distribution with curvature parameter *b* 0*{1,..,10}* (*b*=1 corresponds to the Uniform; higher *b* implies larger probability mass around *0*). However, it can be shown analytically that for any distribution *F(F)* the ratio $EV_{T-1}$ *(1,1,1,...,1)/* $W_{T-1}(n)$ converges to a number higher than *1* as the number of players becomes arbitrarily large. All calculations are available upon request or in Casella and Gelman (in progress).

continue to be characterized by Proposition 2 in the presence of more than *2* voters:

***Lemma 3***. *Suppose the number of voters is n. Proposition 2 continues to hold, and each critical threshold* "$_v^{\,i}$ *remains a function of the state (K, t).*

***Proof*** (To be added. But immediate from Proposition 2).

Second, we derive the probability of obtaining the desired outcome $d_i^*$ for player *i* casting *v* votes in an arbitrary state *K* at time *t* when facing two potential opponents *j* and *z*. For any *K* and for any *v*, there are only three alternatives that are relevant to *i*: either his vote is decisive on its own, regardless of the others' preferences; or *i* obtains the outcome he prefers if and only if he has at least *1* ally; or his vote is irrelevant to the final decision. When *i* needs at least *1* ally, with i.i.d shocks and symmetrical *F(F)*, the probability of obtaining $d_i^*$ equals *3/4* (*1* minus the probability that both opponents agree against *i*, or *1/4*); as in the case of two voters this is the relevant reference point because it corresponds to the case of non-storable votes. When *i*'s vote is decisive on its own, he obtains his favorite outcome with probability *1*; this occurs if the sum of *i*'s opponents' votes is smaller than *v*, or if it is equal to *v* and *i* wins the coin toss. Finally, *i*'s vote is irrelevant when one of the other players is sure to determine alone the final decision, i.e. when the difference between *i*'s opponents votes is larger than *v*, or when it is equal to *v* but the player casting the largest number of votes wins the coin toss. In the special case where $v = 0$, player *i* is never decisive on its own, but the tie-break rule states that he will always be consulted in case of ties among his opponents. We can then conclude:

***Lemma 4.*** *Consider an arbitrary state K. If player i, facing two potential opponents j and z, casts v votes, the probability that he obtains his preferred decision* $d_i^*$ *equals:*

$$prob(d=d_i^*)=\begin{cases} 3/4 - 1/4\, prob(\left|v_j - v_z\right| > v) - 1/8\, prob(\left|v_j - v_z\right| = v) + \\ + 1/4\, prob(v_j + v_z < v) + 1/8\, prob(v_j + v_z = v) & \text{if } v \neq 0 \\ \\ 3/4 - 1/4\, prob(\left|v_j - v_z\right| > v) & \text{if } v = 0 \end{cases}$$

*where $v_j$ and $v_z$ are the number of votes cast be j and z respectively.*

Given Lemmas 3 and 4, constructing expected one-period equilibrium payoffs is not difficult. Consider for example symmetrical state *(k,k,k)* at time *t*. Then:

$$Eg(k_t, k_t, k_t) = \sum_{v=0}^{k} \int_{\alpha_v}^{\alpha_{v+1}} s dF(s) \left[ 2\, prob(d_i^* \middle| v_i = v) - 1 \right] \tag{14}$$

where, following the two Lemmas:

$$\begin{aligned} prob(d_i^* | v_i = v) = {} & 3/4 - 2\sum_{s=1}^{k-v} \left[ F(\alpha_s) - F(\alpha_{s-1}) \right]\left[ 1 - F(\alpha_{v+s}) \right] - \\ & - \sum_{s=1}^{k-v+1} \left[ F(\alpha_s) - F(\alpha_{s-1}) \right]\left[ F(\alpha_{v+s}) - F(\alpha_{v+s-1}) \right] + \sum_{s=1}^{v} \left[ F(\alpha_s) - F(\alpha_{s-1}) \right]\left[ F(\alpha_{v-s+1}) - 1/2 \right] + \\ & + 1/2 \sum_{s=1}^{v+1} \left[ F(\alpha_s) - F(\alpha_{s-1}) \right]\left[ F(\alpha_{v-s+2}) - F(\alpha_{v-s+1}) \right] \end{aligned} \tag{15}$$

and all thresholds are function of *k* and *t*, i.e. $\alpha_v \equiv \alpha_v(k,t)$.

Equation (14) does not depend on the number of players (and has already been used in the case of *2* voters), but equation (15) clearly does. Together they yield the analytic expressions for expected payoffs.

Given Lemma 4 and equation (14), however, a different and simpler representation of expected payoffs is more transparent. For any arbitrary state, we can write:

$$\begin{aligned} & Eg(k_t, y_t, z_t) = \\ & W + \int_0^{\alpha_1} s dF(s) \left[ -\frac{1}{2} prob(\left|v_j - v_z\right| > 0) \right] + \sum_{v=1}^{k} \int_{\alpha_v}^{\alpha_{v+1}} s dF(s) [ -\frac{1}{2} prob(\left|v_j - v_z\right| > v) - \\ & - \frac{1}{4} prob(\left|v_j - v_z\right| = v) + \frac{1}{2} prob(v_j + v_z < v) + \frac{1}{4} prob(v_j + v_z) = v] \end{aligned} \tag{16}$$

where $W = 1/2\int_0^1 sdF(s)$, the expected payoff with non-storable votes (and $"_{k+1}/1$).[12]

As in the case of *2* voters, relative to non-storable votes, storable votes shift the probability of obtaining one's desired outcome away from low and towards high realizations of the preference parameter - away from instances where preferences are weaker and towards instances where they are more intense. This is clearly visible in (16): the negative terms are larger the closer to zero are the intervals over which the preference parameter is integrated, while the opposite is true for the positive terms. The fundamental mechanism that delivered welfare gains in the case of *2* voters (and that supports intuitively the idea of storable votes) remains unchanged.

What does change however is the extent of this shift - i.e. the change in the probability of being pivotal, or irrelevant. With *3* voters, gains and losses - relative to non-storable votes - require more extreme asymmetry than with *2* voters: while before a voter controlled the final outcome whenever he cast more votes than his opponent, now he must cast more votes than both his opponents combined; while before he had no say if he cast less votes than his opponent, now this only occurs if his votes and those of one of his opponent, combined, are less than the votes cast by the third voter. Two remarks follow immediately. First, the probability of these more extreme events will in general be lower than the corresponding critical probabilities in the case of *2* voters. And since it is the shift in the probability of being pivotal that underlies potential welfare gains, we should expect these gains to be lower than with *2* voters. Second, there is an asymmetry built into the critical probabilities - relative to non-storable votes, a voter increases his chances of being pivotal only when he casts at least as many votes as his two opponents, but has no say if *either one* of his two opponents is decisive on his own. In symmetrical states, the ex ante probability of the latter event must then be twice as large than the

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[12]Notice that *W* is unchanged for *n=3* and *n=2*. For arbitrary *n*,

$$W(n) = \int_0^1 sdF(s)(1/2)^{n-1}\binom{n-1}{\frac{n-2+I_n}{2}}.$$

probability of the former, implying that the negative terms in (16) are larger than the positive terms. But because the number of votes a player chooses to cast depends on the strength of his preferences, in equilibrium the negative terms multiply smaller integrals than the positive ones, and it does not follow that $Eg(k_t,k_t,k_t)$ is smaller than $W$ - indeed typically it is larger.[13] As we shall see, asymmetrical states will be more problematic.

Consider a specific example - the two-period horizon game where all voters start out with a single vote. In the first period, each player votes $0$ if his draw of $|\sigma|$ is less than a threshold $\alpha_1$, and $1$ otherwise (by Lemma 3). Equation (16) yields:

$$Eg_{T-1}(1,1,1) = W - \int_0^{\alpha_1} s\,dF(s)\,4[1-F(\alpha_1)][F(\alpha_1)-1/2] + \int_{\alpha_1}^{1} s\,dF(s)\,2[F(\alpha_1)-1/2]^2 \qquad (17)$$

which can be shown to be larger than $W$ for any $\alpha_1 \in (0,1)$ if for instance $F(\sigma)$ is Uniform. But now evaluate expected payoffs in the last period. The possible states at $T$ are: $(1,1,1)$ (if all voters cast their vote at $T-1$); $(2,2,2)$ (if no-one does); $(1,2,2)$ and its permutations (if $1$ out of $3$ voters votes at $T-1$), and $(1,1,2)$ and its permutations (if $2$ out of $3$ voters vote at $T-1$). In the last period, players always cast all their available votes. Thus the expected payoffs associated with symmetrical states at $T$ must equal $W$: $Eg_T(1,1,1) = Eg_T(2,2,2) = W$. If the state is $(1,2,2)$, each voter obtains his preferred outcome if at least one of the other voters agrees, an event that occurs with probability $3/4$, exactly as in the case of symmetrical states. Thus $Eg_T(1,2,2) = Eg_T(2,1,2) = Eg_T(2,2,1) = W$. If the state is $(1,1,2)$, however, players with a single vote only obtain their preferred choice with probability $5/8$ (if the player with $2$ votes agrees - with probability ½ - or if he disagrees, but the two smaller players agree with each other and win the coin toss - with probability $1/4\ (1/2) = 1/8$) and the player with $2$ votes with probability $7/8$ (he wins unless the $2$ smaller voters agree with each other and not with him and win the coin toss, with probability $1/4\ (1/2)=1/8$). It follows that $Eg_T(1,1,2) = \frac{1}{2} W$ and $Eg_T(2,1,1) = 3/2\ W$. This is important. In the case of $2$ voters it was possible to show that average

[13] In all numerical exercises, we found $Eg(k_t,k_t,k_t) \geq W$ as long as $b \geq 1$ when $F(\sigma)$ is Beta, i.e. unless $F(\sigma)$ is bi-modal, with maxima at the two extremes. In this latter, very polarized case preferences are mostly extreme.

expected payoffs in asymmetrical states are never lower than *W* (where, starting at time *0*, each player rationally assigns the same probability to being the weaker or the stronger player). This very strong result does not hold with *3* voters: $1/3(2\ Eg_T(1,1,2) + Eg_T(2,1,1)) = 5/6\ W < W$. We have found a simple example where the average expected payoff is lower than in the case of non-storable votes, and the reason is simple: for any one player who increases his chances to control the final outcome, relative to non-storable votes, there are *2* who decrease theirs. Ex ante, starting from a symmetrical state, any voter has only a *1* in *3* chance of eventually being the voter in control, but a *2* in *3* chance of carrying no weight.

The very strong result, and the simple proof, obtained with *2* voters will not carry over now. Whether or not it remains true that storable votes deliver ex ante welfare gains must depend on the exact value of the equilibrium thresholds, a much more difficult result to establish with any generality. In the simple two-period case we are considering here, the question is whether the higher expected payoff at *T-1* is sufficient to compensate for the possibility of a welfare loss at *T*. If $F(F)$ is Uniform, for example, and $\delta=1$, the ratio $EV_{T-1}(1,1,1)/W_{T-1}$ is reproduced in Figure 3, as function of the threshold $\alpha_1$. As the figure shows, the ratio is larger than *1* if $\alpha_1$ is larger than *.5*. But, following the same procedure described in the case of *2* players, it is easy to calculate that the equilibrium threshold in this case equals *.364*, yielding a $EV_{T-1}(1,1,1)/W_{T-1}$ ratio of *.99* (the same value shown in Figure 2 for $n=3$).[14]

However it should be clear from our discussion that the negative conclusion does not hold generally - because it depends on the exact value of the threshold, it must depend on the parameters of the model. Possible welfare losses, relative to non-storable votes, arise from those states where very asymmetrical strategies, resulting in one player's dominant power, occur with large probabilities. But the probability of reaching those states, as well as the probability of very asymmetrical strategies given

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[14] $EV_{T-1}(1,1,1) = Eg_{T-1}(1,1,1) + \delta W\left(1 - 4[F(\alpha_1) - 1/2][1 - F(\alpha_1)]^2\right)$ where $Eg_{T-1}(1,1,1)$ is given by (17). If the *3* players chose the threshold cooperatively, they would choose a higher $\alpha_1$ - by reducing the ex ante probability of casting a vote at *T-1*, they would reduce the likelihood of state *(1,1,2)* at *T*, but still allow voters to discriminate between high and low preferences enough to obtain expected welfare gains.

the state, are crucially influenced by the variance of the intensity of preferences and the length of the horizon. The lower the variance, the less likely it is that very asymmetrical states will be reached; and the longer the horizon the higher the option value of a vote, and the less likely that asymmetrical states will translate into asymmetrical strategies.

To evaluate this intuition, numerical exercises were run with different time horizons and different $F(\phi)$. The results are in Figure 4, where the expected value of the storable votes game at time $0$, relative to the value with non-storable votes, is plotted for different horizons and different distributions. The parameter $b$ summarizes the curvature of $f(\phi)$: $b=1$ corresponds to the Uniform; as $b$ increases, the relative probability mass around zero increases and the variance falls. Two regularities emerge. First, at any horizon, an increase in $b$ is associated with better ex ante welfare properties for storable relative to non-storable votes. As expected, the larger frequence of weak preferences leads players to expect higher vote savings and a lower likelihood of states dominated by a single voter. The final consequence is the improved expected performance of storable votes. Indeed, although the figure does not show it, for $b$ high enough, the ratio $EV_0/W_0$ is larger than $1$ for all $T$. Second, for $b>1$ storable votes are associated with ex ante welfare gains if the horizon $T$ is longer than a critical value $\overline{T}(b)$, where $\overline{T}$ is lower the larger is $b$. For $b=1$, the conclusion might still hold - for $T > 3$ the ratio $EV_0/W_0$ is monotonically increasing in $T$, as in the case of higher $b$'s - but the simulated horizon is not long enough to reflect it.[15]

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[15]Three further comments. First, reducing $\delta$ (increasing future discounting) raises $EV_0/W_0$, presumably because the more asymmetrical states are expected to arise at the end of the horizon and are then discounted more. But the effect is not large because it is countered in part by the reduction in vote saving, and hence the higher likelihood of these same states. Second, as the figure shows, $EV_0/W_0$ reaches a minimum at $T = 3$ for all $b$'s. Although a higher $T$ implies more vote saving, it also multiplies possible asymmetrical states - according to the numerical exercises, the second effect dominates for $T=3$, but becomes relatively less important as $T$ increases further. Finally, multiple equilibria are possible, but only in one case did we in fact find two equilibria: $b=1$ and state $(5,4,3)$ at $T-1$ (which requires $T=6$). In one equilibrium, voter $i$ never plays $1$ and player $j$ never plays $1$ or $2$. In the other equilibrium, player $j$ never plays $3$ and player $z$ never plays $1$ or $2$. Although the first equilibrium leads to slightly better welfare properties, the effect washes out almost completely in the calculation of $EV_0/W_0$. The figure uses the second equilibrium.

Better welfare properties for longer time horizons pose a commitment problem - as time passes and the end of the game approaches voters may wish to renegotiate. We ignore this aspect here - storable votes are evaluated ex ante at some constitutional stage, and we take the possibility of commitment as granted. This said, the conclusion is pleasing - the very idea of storable votes relies on the possibility of intertemporal trades and we should expect it to yield its potential benefits only if there is enough time for these trades to be possible. The lack of welfare gains with a *2*-period horizon is much less of a concern if longer horizons improve the performance of the mechanism.

Time is the crucial dimension of storable votes, where trades with oneself, present against future, substitute for trades with other agents. But how do the two different forms of trade compare in the case of votes?

### *Storable v/s Tradable Votes*

As discussed in the Introduction, a spot market for votes where transactions happen through money is outside the concerns of this paper - credit constraints and unequal wealth distribution would lead to unequal political power, an unethical outcome that this paper, interested in practical normative suggestions, sees no reason to institutionalize. Closer to our purposes however is a system where current votes are exchanged for future votes, a credit as opposed to a spot market for votes. The common perception is that trades of this type are very common in all our voting institutions - what are their welfare properties, relative to storable votes? We begin with three observations. First, notice that storable votes are easier to describe precisely than tradable votes: because they rely on individuals acting alone, storable votes require a simpler institutional background. When votes are exchanged, traders must find each other, competitors must outbid one another and future promises must be enforced. How these steps take place depends on auxiliary, but essential, institutions; different alternatives seem possible and would affect the outcome. Consider for example the question of enforcement, and this is the second observation. In our simple finite horizon model, intertemporal exchanges across voters would not be self-enforcing - debtors would be sure to renege in the last period with the usual cascading effect on trades at earlier times. We will therefore assume that tradable votes are supported by credible outside enforcement. Finally, with tradable votes, players who buy votes now are effectively borrowing against their future voting allocation. It seems appropriate then to assume that with storable votes too individuals are allowed not only to save but also to borrow. Let us simply suppose then that when votes are storable the entire stock of votes is allocated to each player at time *0*, for him to distribute over future decisions as he sees fit.

In what follows, we will consider only two simple examples that can be solved analytically. Hopefully, the differences uncovered can inform more systematic analyses in the future.

Begin with the simplest case: *2* voters and *2* periods. With storable votes, each player enters the game with *2* votes. Following the usual logical steps, it is easy to establish that voter *i* will choose to cast *2* votes if $*F_i* \$ 4*W$ and *0* votes otherwise. Although the initial number of votes is different, the outcome is then identical to the case studied earlier in the paper where each voter was endowed with *1* new vote each period. The expected value of the game is again given by equation (9).

Suppose now that votes are tradable, but not storable. At time *1*, each voter has *1* vote and three options: he can offer to sell his vote now (*S*) (and have *2* votes next period, when his opponent will have *0*, if the trade takes place); do nothing (*N*) (and have *1* vote each period, just like his opponent), or offer to buy a vote (*B*) (and have *0* votes next period, when his opponent will have *2*). Call $p_B$ the probability that a player offers to buy, $p_S$ the probability that he offers to sell, and $p_N = 1\text{-}p_B\text{-}p_S$ the probability that he does nothing. Given $F_i$ , positive for simplicity, the three alternatives lead to expected utilities:

$$
\begin{aligned}
Eu_i|S &= p_B(s_i 1/2 + d2W) + (1 - p_B)(s_i 3/4 + dW) \\
Eu_i|N &= s_i 3/4 + dW \qquad\qquad (18) \\
Eu_i|B &= p_S s_i + (1 - p_S)(s_i 3/4 + dW)
\end{aligned}
$$

It is possible of course that both voters would want to buy, or both would want to sell. With a longer number of periods, prices would emerge, but in this simple example no voter can offer more (or less) than a *1*-to-*1* exchange between a vote today and a vote tomorrow. Thus if both voters find themselves on the same side of the trade no exchange can be concluded, and this is reflected in (18). Given these equations, it is easy to see that the optimal strategy is to offer to buy a vote whenever $*F_i*$ $\$ 4*W$, and offer to sell otherwise. It follows that there is a deviation from the reference case of no-trade only when at time *1* one voter's preference intensities are above the threshold, and the other's below. But this is exactly what happens with storable votes (in all other cases, the players' strategies cancel each other). And because the threshold is the same, it is not surprising then that the expected value of the game is given once again by equation (9). With *2* players and *2* periods, the two voting mechanisms are identical.[16] Indeed, one can reasonably conjecture that with *2* voters the result will continue to hold for any arbitrary time horizon *T*.

Will this be true with *3* voters? The question is interesting because it is with more than *2* voters that the inefficiency built into market-type voting mechanisms comes into being. If the severity of the externality differs between tradable and storable votes, it is only then that such a difference will appear. Consider a *3*-voters, *2*-period game.

With storable votes each player again enters the game with *2* votes. The optimal strategy in the first period is to abstain if $*F_i* < "^S$ and cast *2* votes otherwise, where $"^S / 4*W(1-p_0^2)/(1-p_2^2)$, and $p_0=1-p_2 = 2[F("^S)-1/2]$ (the superscript *S* stands for "storable"). The expected payoff at *T-1* and the expected value of the game are given by:

[16]One important caveat. For consistency with the rest of the paper, we are maintaining the information assumption made all along: a voter makes decisions knowing his preferences but not his opponent's. In the case of tradable votes, this can result in trades between voters on the same side of an issue, albeit with different intensities. A better assumption might have voters know each other's preferred position, even as they do not know the intensity with which these positions are held. (Whether a voter will want to reveal his true position or not is an interesting question in itself). More about this in the Conclusions.

$$Eg_{T-1}^{S} = W + \int_{a^S}^{1} s\,dF(s)\frac{p_0^2}{2} - \int_{0}^{a^S} s\,dF(s)\,p_0 p_2$$
$$EV_{T-1}^{S} = Eg_{T-1}^{S} + dW(1 - p_0 p_2^2) \qquad (19)$$

with the probabilities defined above.

Consider now the case of tradable votes. The horizon continues to be *2* periods, implying that again each player can buy or sell at most *1* vote at *T-1*. As in the case of *2* voters, there are three possible alternatives: each player can offer to sell, offer to buy, or do nothing. When a transaction is proposed, it is concluded only if there is at least another voter who has made the complementary proposal. But now a new difficulty emerges: a voter willing to transact may be shut out of the market because his *2* opponents trade among themselves. Ruling out side-payments, no price and no bargaining can emerge in the *2*-period game, and we assume that if *2* willing buyers, for example, face a single seller, the successful one will be chosen with a coin toss. Taking this into account, expected utilities when taking any of the three actions can be calculated as usual. Consider for example a voter offering to sell his vote. Voter *i* offering to sell his vote succeeds in selling it if: (i) both other voters offer to buy (with probability $p_B^{\,2}$); (ii) one offers to buy and one does nothing (with probability $2p_B\,p_N$); or (iii) one offers to buy, one offers to sell and the coin toss is favorable to *i* (with probability $p_B\,p_S$). If *i* succeeds, then he is left with *0* votes today in exchange for *2* tomorrow, while his opponents have *2* and *1* votes today and *0* and *1* tomorrow. For simplicity only, suppose that $F_i$ is positive. In both period, the voter who controls *2* votes alone determines the outcome; thus *i*'s expected utility, conditional on succeeding in selling, equals $F_i/2+2*W$. If *i*'s offer is not accepted, that may be either because no-one is interested in buying (with probability $(1-p_B)^2$), in which case no transaction takes place and *i*'s expected utility equals $F_i\,3/4+*W$, or because the other *2* voters trade among themselves (with probability $p_B\,p_S$), in which case *i* carries no weight either this period or the next and his expected utility equals $F_i/2$. Thus we can write:

$$Eu_i|S = (p_B^2 + 2p_Bp_N + p_Bp_S)(s_i/2 + 2dW) + (1 - p_B)^2(s_i 3/4 + dW) + \\ + p_Bp_S s_i/2 \qquad (20)$$

The expected utilities associates with the two remaining alternatives are calculated analogously:

$$Eu_i|B = (p_S^2 + 2p_S p_N + p_B p_S)\sigma_i + (1- p_s)^2(\sigma_i 3/4 + \delta W) + p_B p_S \sigma_i / 2$$

$$Eu_i|N = p_B p_S \sigma_i + (1- 2p_B p_S)(\sigma_i 3/4 + \delta W) \qquad (20)'$$

Given these equations it is easy to establish that it is never optimal to do nothing. There is a single relevant threshold $\alpha^M$ such that voter $i$ offers to sell his vote if $|\sigma_i| < \alpha^M$, and offers to buy otherwise (the superscript $M$ stands for "market"), where once again $\alpha^M = 4\delta W$. Expected payoff at $T$-$1$ and the expected value of the game are given by:

$$Eg^M_{T-1} = W + \int_{\alpha^M}^{1} \sigma dF(\sigma)\frac{p_S^2}{2} - \int_{0}^{\alpha^M} \sigma dF(\sigma)\frac{(1+p_S)}{2}p_B$$

$$EV^M_{T-1} = Eg^M_{T-1} + \delta W(1- p_S p_B) \qquad (21)$$

where $p_S = 1 - p_B = 2[F(\alpha^M) - 1/2]$.

Comparing equations (19) and (21) is very instructive. It is particularly easy when $F(\sigma)$ is Uniform, because in that case $\alpha^S = \alpha^M = .5$, and $p_0 = p_S$, $p_2 = p_B$. The expected value of the game is unequivocally lower with tradable than with storable votes, a result that arises because expected payoffs are lower in both periods. And the reason is simple: tradable votes require two sides for a trade: with $3$ voters, the buyer guarantees himself control over the public decision in the first period, and the seller in the second. The third voter, excluded from the transaction, has no voice in either periods. With storable votes, on the other hand, each voter decides his allocation of votes on his own. If a voter decides to abstain in the first period, neither one of his opponents experiences a decline in the probability of obtaining the desired outcome in that period; not can the abstaining voter be sure of controlling the public decision in period $2$. In other words, the externality is smaller with storable than with tradable votes. The intuition seems robust: the welfare results are unchanged for all $F(\sigma)$ we have tried.[17] The probability of being rationed when votes are tradable remains positive for all finite number

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[17] As $b$ increases, $\alpha^M$ becomes larger than $\alpha^S$, implying that the ex ante probability of putting one's vote up for sale when votes are tradable increases more than the probability of abstaining.

of players, and although the game will be more complicated, the logic seems unchanged. Similarly, the emergence of prices when trading occurs over a longer horizon should be matched by an equivalent flexibility in the intertemporal program with storable votes. We cannot draw any general conclusion at this point, but again, there is no obvious reason why the welfare results should be reversed.

## APPENDIX

***Proof of Proposition 2.*** *i) Existence*. Use backward induction, exploiting the i.i.d. assumption on *F*. Then apply Milgrom and Weber (1985) on distributional strategies. (Tentative).

*ii) Monotonicity.* Suppose $F_{i\,t} > 0$. When casting $v$ votes, player $i$'s expected utility at $t$ is given by: $F_{i\,t}\, prob(d_t=1 * v_t^i = v) + * EV_{t+1}(k_t^i - v + 1, k_t^j - v_t^j + 1)$, where $prob(d_t=1 * v_t^i = v) = [prob(v_t^j < v) + 3/4\, prob(v_t^j = v) + ½\, prob(v_t^j > v)]$. Notice that $prob(d_t=1 * v_t^i = v+1) - prob(d_t=1 * v_t^i = v) = 1/4[prob(v_t^j = v) + prob(v_t^j = v+1)] \$ 0$; as expected $prob(d_t=1 * v_t^i = v)$ is monotonically increasing in $v$. Call $v'$ ($v''$) the equilibrium number of votes cast by player $i$ when $F_{i\,t} = F'$ ($F''$) (with $F' > F'' > 0$). By definition of equilibrium, the following two inequalities must hold:

$$s'\,prob(d_t = 1|v') + dEV_{t+1}(k_t^i - v' + 1, k_t^j - v^j + 1) \geq$$
$$s'\,prob(d_t = 1|v'') + dEV_{t+1}(k_t^i - v'' + 1, k_t^j - v^j + 1)$$

$$s''\,prob(d_t = 1|v'') + dEV_{t+1}(k_t^i - v'' + 1, k_t^j - v^j + 1) \geq$$
$$s''\,prob(d_t = 1|v') + dEV_{t+1}(k_t^i - v' + 1, k_t^j - v^j + 1)$$

Adding the two inequalities, we obtain:

$$(s' - s'')[prob(d_t = 1|v') - prob(d_t = 1|v'')] \geq 0$$

But with $F' > F''$ and $prob(d_t=1 * v)$ monotonically increasing in $v$, this implies $v' \$ v''$, establishing the result. The logic is identical, with the appropriate sign changes, for $F_{i\,t} < 0$ #

***Proof of Lemma 1***. Begin by proving part *i)* for symmetrical states. Recall that expected one-period equilibrium payoff is given by (12), reflecting the optimal thresholds chosen by the voters. Define a function $Q("_v, \ldots, "_k)$, representing (fictional) expected payoff when thresholds $"_1, \ldots, "_{v-1}$ are set to zero, and all other thresholds are kept at their equilibrium values (and where to simplify notation we ignore the time subscript). By construction $Q("_1, \ldots, "_k) = Eg(k,k)$. We can show that the following two conditions hold:

$a)\quad \Psi(a_k) \geq W$

$b)\quad \Psi(a_v, \ldots, a_k) \geq \Psi(a_{v+1}, \ldots, a_k)$

To establish *a)*, notice that given (12) and the definition of $\Psi(\alpha_k)$, we can write:

$$\Psi(a_k) \equiv \int_0^{a_k} s dF(s)\big(F(a_k) - 1/2\big) + \int_{a_k}^{1} s dF(s) F(a_k)$$

where $\alpha_k \in [0,1]$. At $\alpha_k = 0$ or $\alpha_k = 1$, $\Psi(\alpha_k) = W$; in addition it is easy to verify that $\partial\Psi(\alpha_k)/\partial\alpha_k$ is positive at $\alpha_k = 0$ and has a single root in the interval $\alpha_k \in (0,1]$. Hence $\Psi(\alpha_k) > W\ \forall\ \alpha_k \in (0,1)$ and $\Psi(\alpha_k) = W$ *if* $\alpha_k = 0$ *or* $1$. But from (12) we also know:

$$\Psi(a_v, \ldots, a_k) \geq \Psi(a_{v+1}, \ldots, a_k) \Leftrightarrow$$
$$\int_0^{a_v} s dF(s)\big(F(a_v) - 1/2\big) + \int_{a_v}^{a_{v+1}} s dF(s)\big(F(a_v) + F(a_{v+1}) - 1\big) \geq \int_0^{a_{v+1}} s dF(s)\big(F(a_{v+1}) - 1/2\big) \qquad \text{(A1)}$$

The left-hand side of (A1) is identical to the right-hand side if $\alpha_v = 0$ or $\alpha_v = \alpha_{v+1}$. At $\alpha_v = 0$, the left-hand side is increasing in $\alpha_v$ and again it can easily be shown that the derivative has a single root. Hence $\Psi(\alpha_v, \ldots, \alpha_k) > \Psi(\alpha_{v+1}, \ldots, \alpha_k)\ \forall\ \alpha_v \in (0, \alpha_{v+1})$ and $\Psi(\alpha_v, \ldots, \alpha_k) = \Psi(\alpha_{v+1}, \ldots, \alpha_k)$ for $\alpha_v = 0$ or $\alpha_v = \alpha_{v+1}$, and *b)* is established.

Finally, it follows that *a)* and *b)* can both hold with equality only if all thresholds are either 0 or *1* or if there exist an $\alpha_v$ and $\alpha_{v+1}$ such that $\alpha_1 = \ldots = \alpha_v = 0$ and $\alpha_{v+1} = \ldots = \alpha_k = 1$, with $v \in \{1, \ldots, k-1\}$, i.e. only if the same strategy is followed for all realizations of $F_i$. If at least one threshold is strictly between *0* and *1*, then the inequality in part *i)* of Lemma 1 is strict. We expect that to be the case in all symmetrical states, but it is particularly easy, and sufficient for our purposes, to show that this must be true at *T-1*. Suppose both players are endowed with *k* votes at *T-1*. We show that casting *v* votes for all realizations of $F_i$ cannot be an equilibrium. Given $F_{iT-1}$, which we suppose positive for simplicity, the expected utility of voter *i* casting *v* votes is given by:

$EU^i_{T-1}(v) = F_{iT-1}\,[prob(v_t^j < v) + 3/4\ prob(v_t^j = v) + ½\ prob(v_t^j > v)] + *W\,[prob(v_t^j = v) + 2prob(v_t^j > v)]$.

It is easy to establish then that:

$$EU^i_{T-1}(v+1) - EU^i_{T-1}(v) = (F_{iT-1}/4 - *W)\,[prob(v_t^j = v+1) + prob(v_t^j = v)]$$

$$EU^{i}_{T-1}(v)-EU^{i}_{T-1}(v-1)=(F_{iT-1}/4 - *W)\ [prob(v_t^j=v)+prob(v_t^j=v-1)]$$

Recall that $4*W$ θ $(0,1)$. Suppose first that $prob(v_t^j=v)>0$. Then both expressions in square brackets are non-zero and casting $v$ votes cannot be the equilibrium strategy for $F_i > 4*W$, unless $v=k$. But if $v=k$, casting $v$ votes cannot be the equilibrium strategy for $F_i < 4*W$. Thus no $v$ can be the equilibrium strategy for all $F_i$. Suppose now that $prob(v_t^j=v)=0$. Then if $prob(v_t^j=v+1)>0$, voter $i$ prefers $v$ to $v+1$ votes for $F_i < 4*W$, but $v+1$ to $v$ votes for $F_i > 4*W$ (and since the state is symmetrical, $v+1$ votes must be feasible). If $prob(v_t^j=v+1)=0$ but $prob(v_t^j=v-1)>0$, then voter $i$ must prefer $v-1$ to $v$ votes for $F_i < 4*W$. Finally suppose that $prob(v_t^j=v+1)=0$ and $prob(v_t^j=v-1)=0$. Take any $v'$ such that $prob(v_t^j=v')>0$. It is immediate to show that if $v'>v$ then player $i$ must prefer $v$ to $v'$ for $F_i < 4*W$, but $v'$ to $v$ for $F_i > 4*W$; if $v'<v$ then player $i$ must prefer $v'$ to $v$ for $F_i < 4*W$, but $v$ to $v'$ for $F_i > 4*W$. We can conclude that in all cases, no $v$ can be the equilibrium strategy for all $F_i$. Part *i)* of lemma 1 is established.#

*ii)* To establish part *ii)* of the lemma for asymmetrical states, we follow the same logic. Suppose $s > k$, and denote $\{(_1,(_2,...,(_{k+1}\}$ the equilibrium thresholds for the player holding $s$ votes at $t$, and $\{\$_1,\$_2,...,\$_k\}$ the equilibrium thresholds for the player holding $k$ votes (where, again, to simplify notation time subscripts are omitted). Notice that the player holding $s$ votes will never cast more than $k+1$. We can write:

$$Eg^i(k^i,s^j)+Eg^i(s^i,k^j)=\int_0^{b_1} s\,dF(s)\big(F(g_1)-1/2\big)+\int_{b_1}^{b_2} s\,dF(s)\big(F(g_1)+F(g_2)-1\big)+..$$

$$...+\int_{b_k}^{1} s\,dF(s)\big(F(g_k)+F(g_{k+1})-1\big)+\int_0^{g_1} s\,dF(s)\big(F(b_1)-1/2\big)+\int_{g_1}^{g_2} s\,dF(s)\big(F(b_1)+F(b_2)-1\big)+....$$

$$...+\int_{g_v}^{g_{v+1}} s\,dF(s)\big(F(b_v)+F(b_{v+1})-1\big)+.....+\int_{g_k}^{g_{k+1}} s\,dF(s)\big(F(b_k)\big)+\int_{g_{k+1}}^{1} s\,dF(s) \qquad \text{(A2)}$$

Define a function $Q(\$_1,...,\$_v,(_1,..,(_{v+1})$, representing the (fictional) sum of expected payoffs in (A2) when thresholds $\$_{v+1},...,\$_{k-1},(_{v+2},..,(_{k+1}$ are set to $1$ (which is now more convenient than setting the omitted thresholds to 0), and all other thresholds are kept at their equilibrium values. By construction, $Q(\$_1,..,\$_k,(_1,..,(_{k+1}) / Eg^i(s^i,k^j)+Eg^i(k^i,s^j)$. We can show that the following two conditions hold:

$$a)\quad \Psi(g_1) \geq 2W$$
$$b)\quad \Psi(b_1,\ldots,b_{v+1},g_1,\ldots,g_{v+2}) \geq \Psi(b_1,\ldots,b_v,g_1,\ldots,g_{v+1})$$

.

To verify *a)*, notice that given (A2) we can write:

$$\Psi(g_1) \equiv \int_0^1 sdF(s)\big(F(g_1)+1/2\big) + \int_0^{g_1} sdF(s)1/2 + \int_{g_1}^1 sdF(s)$$

Differentiating $\Psi(\gamma_1)$ with respect to $\gamma_1$, it is easy to see that $\Psi(\gamma_1) > 2W \ \forall \gamma_1 \in (0,1)$ and $\Psi(\gamma_1) = 2W$ *if* $\gamma_1 = 0$ or *1*.

To verify *b)*, notice that from (A2) we also know:

$$\Psi(b_1,\ldots,b_{v+1},g_1,\ldots,g_{v+2}) \geq \Psi(b_1,\ldots,b_v,g_1,\ldots,g_{v+1}) \Leftrightarrow$$
$$\int_{b_{v+1}}^1 sdF(s)\big(F(g_{v+2})-F(g_v)\big) + \int_{g_v}^{g_{v+2}} sdF(s)\big(F(b_{v+1})-1\big) \geq 0 \qquad \text{(A3)}$$

where $1 \geq \gamma_{v+2} \geq \gamma_v \geq 0$, and $1 \geq \beta_{v+1} \geq 0$. In addition, $\gamma_v \leq \beta_{v+1}$. (If at a given draw of the preferences parameter the individual holding $k$ votes is willing to cast $v+1$, then the individual holding $s > k$ must be willing to cast at least $v$). We can use (A3) to establish *b)*. Expression (A3) equals zero at $\gamma_{v+2} = \gamma_v$, and at $\beta_{v+1} = 1$, and it is not difficult to verify that the inequality is satisfied at $\gamma_{v+2} = 1$.[18] The derivative of (A3) with respect to $\gamma_{v+2}$ equals:

$$f(g_{v+2})\left(\int_{b_{v+1}}^1 sdF(s) - g_{v+2}\big(1-F(b_{v+1})\big)\right)$$

It has a single root, and when evaluated at $\gamma_{v+2} = \gamma_v$ equals:

$$f(g_v)\left(\int_{b_{v+1}}^1 sdF(s) - g_v\big(1-F(b_{v+1})\big)\right) \geq$$
$$f(g_v)\left(\int_{b_{v+1}}^1 sdF(s) - b_{v+1}\big(1-F(b_{v+1})\big)\right) > 0 \quad \forall\, b_{v+1} \in [0,1)$$

Thus, for $\beta_{v+1} < 1$, the derivative is positive at $\gamma_{v+2} = \gamma_v$, where condition (A3) holds with equality. Its

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[18]At $\gamma_{v+2} = 1$, (A3) becomes $\int_{b_{v+1}}^1 sdF(s)\big[1-F(g_v)\big] - \int_{g_v}^1 sdF(s)\big[1-F(b_{v+1})\big]$, non-negative for all $\beta_{v+1} \geq \gamma_v$.

single root implies that, within the range of relevant $(_{v+2}$ values, (A3) has a minimum at $(_{v+2} = 1$, where (A3) is again satisfied. It follows that (A3) is satisfied everywhere for $(_{v+2}\ 0\ [(_{v}, 1]$ and $\$_{v+1} < 1$. But we also know that it is satisfied for $\$_{v+1} = 1$. Hence *b)* is established.#

***Proof of Lemma 2.*** Once again, we proceed in two steps. First, we consider symmetrical states and show that if the conditions in Lemma 2 hold, then $EV_t^{\,i}(k_t, k_t) > W_t^{\,i}(k,k)$. Then we prove the corresponding result for asymmetrical states.

Consider state $(k_t, k_t)$. We can write:

$$EV_t^i(k_t,k_t) = Eg_t^i(k_t,k_t) + \mathbf{d}$$

$$[\,p_{i0}\Big(p_{j0}EV_{t+1}{}^i(k_t+1,k_t+1) + p_{j1}EV_{t+1}{}^i(k_t+1,k_t) + \ldots + p_{jk}EV_{t+1}{}^i(k_t+1,1)\Big) +$$

$$+\ p_{i1}\Big(p_{j0}EV_{t+1}{}^i(k_t,k_t+1) + p_{j1}EV_{t+1}{}^i(k_t,k_t) + \ldots + p_{jk}EV_{t+1}{}^i(k_t,1)\Big) + \ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots + p_{ik}\Big(p_{j0}EV_{t+1}{}^i(1,k_t+1) + p_{j1}EV_{t+1}{}^i(1,k_t) + \ldots + p_{jk}EV_{t+1}{}^i(1,1)\Big)]$$

or, more compactly:

$$EV_t^{\,i}(k_t,k_t) = Eg_t^{\,i}(k_t,k_t) + \mathbf{d}\left[\sum_{v_i=0}^{k} p_{iv_i}\left(\sum_{v_j=0}^{k} p_{jv_j} EV_{t+1}{}^i(k_t - v^i + 1, k_t - v^j + 1)\right)\right] \qquad \text{(A4)}$$

where $p_{jv}$ is the probability that $*F_{j\,t}*$ falls into the interval that corresponds to $j$'s optimal strategy $v$. But the game is symmetric, and starting from the symmetrical state $(k_t, k_t)$, $p_{iv} = p_{jv}$ for all $v$. We can thus collect terms and rewrite (A4) as:

$$\begin{aligned} EV_t^{\,i}(k_t^{\,i},k_t^{\,j}) &= Eg_t^{\,i}(k_t,k_t) + \mathbf{d} \\ &[\sum_{v=0}^{k-1}\left[p_{iv}\sum_{r=v+1}^{k} p_{jr}\Big(EV_{t+1}{}^i(k-v+1,k-r+1) + EV_{t+1}{}^i(k-r+1,k-v+1)\Big)\right] + \\ &+ \sum_{v=0}^{k} p_{iv}\,p_{jv}\,EV_{t+1}{}^i(k-v+1,k-v+1)] \end{aligned} \qquad \text{(A5)}$$

Substituting the conditions stated in Lemma 2, we then obtain:

$$EV_t^{\,i}(k_t,k_t) > Eg_t^{\,i}(k_t,k_t) + dW_{t+1}\left[2\sum_{v=0}^{k-1}\sum_{r=v+1}^{k} p_{iv}\,p_{jr} + \sum_{v=0}^{k} p_{iv}\,p_{jv}\right]$$

Once again using $p_{iv}\,p_{jr} = p_{ir}\,p_{jv}$, it is not difficult to verify that the probabilities, which span all possible equilibrium strategies, sum up to *1*. Hence :

$$EV_t^{\,i}(k_t, k_t) > Eg_t^{\,i}(k_t, k_t) + dW_{t+1}$$

But we know by Lemma 1 that $Eg_t^{\,i}(k_t, k_t) \$ W$. Hence $EV_t^{\,i}(k_t, k_t) > W_t(k, k)$, and the first part of Lemma 2 is established.

The logic of the proof is identical in the asymmetrical state $(s_t^{\,i}, k_t^{\,j})$. We can write:

$$\begin{aligned}
&EV_t^{\,i}(s_t^{\,i}, k_t^{\,j}) + EV_t^{\,i}(k_t^{\,i}, s_t^{\,j}) = Eg_t^{\,i}(s_t^{\,i}, k_t^{\,j}) + Eg_t^{\,i}(k_t^{\,i}, s_t^{\,j}) + d\\
&[p_{i0}(s_t^{\,i}, k_t^{\,j})\big(p_{j0}(s_t^{\,i}, k_t^{\,j})EV_{t+1}^{\,i}(s_t^{\,i}+1, k_t^{\,j}+1) + \ldots . + p_{jk}(s_t^{\,i}, k_t^{\,j})EV_{t+1}^{\,i}(s_t^{\,i}+1, 1)\big) + ..\\
&\ldots + p_{is}(s_t^{\,i}, k_t^{\,j})\big(p_{j0}(s_t^{\,i}, k_t^{\,j})EV_{t+1}^{\,i}(1, k_t^{\,j}+1) + \ldots . + p_{jk}(s_t^{\,i}, k_t^{\,j})EV_{t+1}^{\,i}(1,1)\big)] + d \qquad \text{(A6)}\\
&[p_{i0}(k_t^{\,i}, s_t^{\,j})\big(p_{j0}(k_t^{\,i}, s_t^{\,j})EV_{t+1}^{\,i}(k_t^{\,i}+1, s_t^{\,j}+1) + \ldots . + p_{js}(k_t^{\,i}, s_t^{\,j})EV_{t+1}^{\,i}(k_t^{\,i}+1, 1)\big) + ..\\
&\ldots + p_{ik}(k_t^{\,i}, s_t^{\,j})\big(p_{j0}(k_t^{\,i}, s_t^{\,j})EV_{t+1}^{\,i}(1, s_t^{\,j}+1) + \ldots . + p_{js}(k_t^{\,i}, s_t^{\,j})EV_{t+1}^{\,i}(1,1)\big)]
\end{aligned}$$

As always, the probability that a given strategy is chosen by either player is a function of the state; and since we are considering two different states this dependence is recognized explicitly. Using $p_{iv}(s_t^{\,i}, k_t^{\,j}) = p_{jv}(k_t^{\,i}, s_t^{\,j})$ *œv,s,k,t*, we can simplify (A6):

$$\begin{aligned}
&EV_t^{\,i}(s_t^{\,i}, k_t^{\,j}) + EV_t^{\,i}(k_t^{\,i}, s_t^{\,j}) = Eg_t^{\,i}(s_t^{\,i}, k_t^{\,j}) + Eg_t^{\,i}(k_t^{\,i}, s_t^{\,j}) + d\\
&[p_{i0}(s_t^{\,i}, k_t^{\,j})\sum_{v=0}^{k} p_{jv}(s_t^{\,i}, k_t^{\,j})\big(EV_{t+1}^{\,i}(s_t^{\,i}+1, k_t^{\,j}-v+1) + EV_{t+1}^{\,i}(k_t^{i}-v+1, s_t^{\,j}+1)\big) + .. \qquad \text{(A7)}\\
&\ldots + p_{is}(s_t^{\,i}, k_t^{\,j})\sum_{v=0}^{k} p_{jv}(s_t^{\,i}, k_t^{\,j})\big(EV_{t+1}^{\,i}(1, k_t^{\,j}-v+1) + EV_{t+1}^{\,i}(k_t^{i}-v+1, 1)\big)]
\end{aligned}$$

More compactly, we can write:

$$\begin{aligned}
&EV_t^{\,i}(s_t^{\,i}, k_t^{\,j}) + EV_t^{\,i}(k_t^{\,i}, s_t^{\,j}) = Eg_t^{\,i}(s_t^{\,i}, k_t^{\,j}) + Eg_t^{\,i}(k_t^{\,i}, s_t^{\,j}) + d\\
&\sum_{r=0}^{s} p_{ir}\sum_{v=0}^{k} p_{jv}\big(EV_{t+1}^{\,i}(s_t^{\,i}-r+1, k_t^{\,j}-v+1) + EV_{t+1}^{\,i}(k_t^{i}-v+1, s_t^{\,j}-r+1)\big) \qquad \text{(A8)}
\end{aligned}$$

(All probabilities are now conditional on the same state, and in absence of ambiguity the notation is simplified). Substituting the conditions in Lemma 2, we then derive:

$$EV_t^i(s_t^i, k_t^j) + EV_t^i(k_t^i, s_t^j) \geq Eg_t^i(s_t^i, k_t^j) + Eg_t^i(k_t^i, s_t^j) + 2dW_{t+1}\sum_{r=0}^{s}\sum_{v=0}^{k} p_{ir}p_{jv}$$

It is easy to verify that the probabilities sum up to *1*. Hence:

$$EV_t^i(s_t^i, k_t^j) + EV_t^i(k_t^i, s_t^j) \geq \left(Eg_t^i(s_t^i, k_t^j) + dW_{t+1}\right) + \\ + \left(Eg_t^i(k_t^i, s_t^j) + dW_{t+1}\right)$$

But by Lemma 1, the second part of Lemma 2 is then established.**#**

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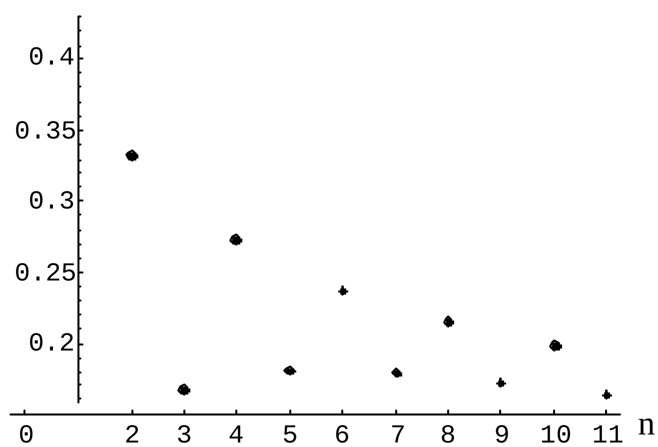
**Figure 1**

*One period game.*
*The marginal value of an extra vote*




Percentage increase in the probability of obtaining the desired outcome from holding 2 votes when everybody else has a single one, as function of the total number of voters.

**Figure 2**

*2 periods, n players*
*F(F )Uniform; *=1*

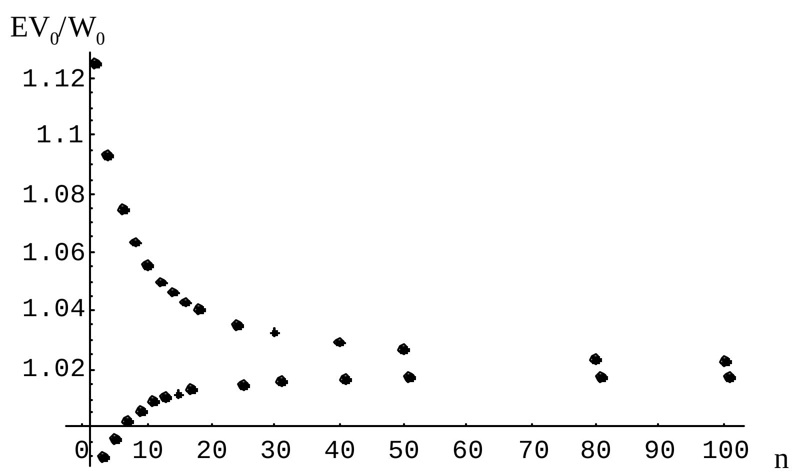


**Figure 3**

*3 players, 2 periods.*
*F(F )Uniform; *=1*

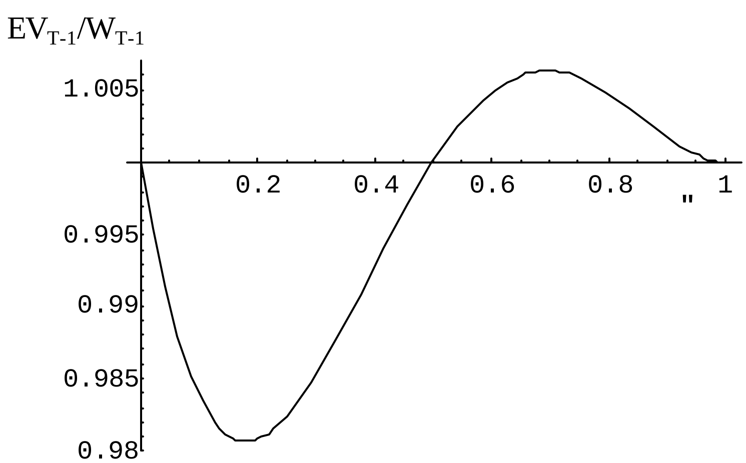
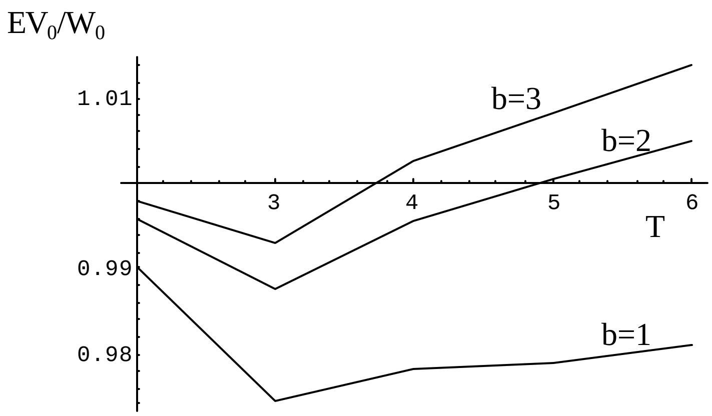


**Figure 4**

*3 players, T periods*
*F(F) Beta; *=1*




Beta distribution: $$f(s) = \frac{1}{\int_{-1}^{1}(1-s^{2})^{b-1}ds}(1-s^{2})^{b-1}$$