

# A Learning Model of Dividend Smoothing

PETER M. DEMARZO AND YULIY SANNIKOV\*

January 18, 2008

PRELIMINARY DRAFT

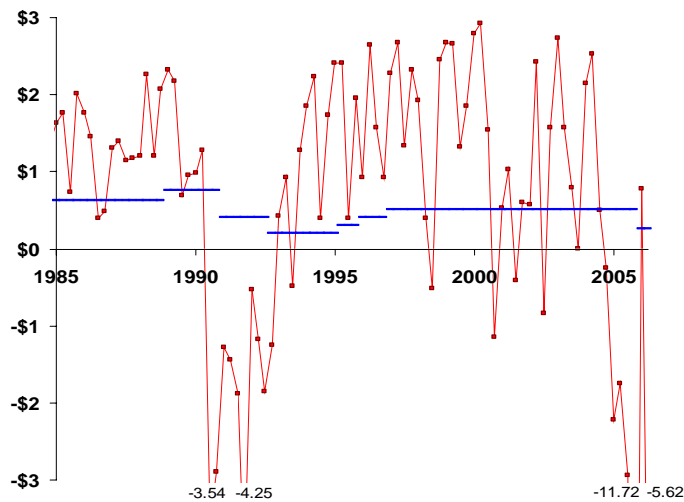
## Abstract

We derive the optimal dynamic contract in a continuous-time principal-agent setting, in which both investors and the agent learn about the firm's profitability over time. We show that the optimal contract can be implemented through the firm's payout policy. The firm accumulates cash until it reaches a target balance that depends on the agent's perceived productivity. Once this target balance is reached, the firm starts paying dividends equal to its expected future earnings, while any temporary shocks to earnings either add to or deplete the firm's cash reserves. The firm is liquidated if it depletes its cash reserves. We also show that once the firm initiates dividends, this liquidation policy is first-best, despite the agency problem.

---

\* Stanford University and UC Berkeley/NYU. This research is based on work supported in part by the NBER and the National Science Foundation under grant No. 0452686.

One of the important puzzles in corporate finance is the smoothness of corporate dividends relative to earnings and cash flows. In an early empirical study, Lintner (1956) developed a model of dividend policy in which he proposed that firms adjust their dividends slowly to maintain a target long-run payout ratio. One motivation for this partial adjustment, suggested by Miller and Modigliani (1961), is that managers base their dividend decisions on their perception of the permanent component of earnings, and avoid adjusting dividends based on temporary or cyclical fluctuations. Such a policies lead to dividend payouts that have much lower volatility than earnings over short time horizons. As an example, Figure 1 shows dividends and earnings for General Motors from 1985 until 2006.



**Figure 1: Quarterly dividends and earnings for General Motors.**

In an early empirical study, Fama and Blacomin (1968) confirm the smoothness of dividend payouts and demonstrate the dependence of dividends on lagged earnings surprises. Numerous more recent studies (see, e.g. Allen and Michaely (2003) and Brav et. al. (2005)) continue to find that firm’s smooth their dividends (and their payouts more generally).<sup>1</sup> Furthermore, dividends tend to be paid by mature firms, and dividend changes tend to result in significant stock price reactions in the same direction,

<sup>1</sup> When discussing payout policy we will for simplicity often refer to dividends alone, but our discussion should be interpreted to include share repurchases as well. Similarly, when discussing debt or leverage policies, we are referring to the firm’s *net* debt, which includes cash reserves.

suggesting that investors view dividends as important indicators of the firm's future cash flows.

In this paper we develop a dynamic contracting model of dividend smoothing. We consider a natural principal-agent setting in which the agent can reduce effort in the firm and engage in outside activities that generate private benefits. Both the principal (outside investors) and the agent are risk neutral, but the agent is wealth-constrained. We depart from the standard principal-agent setting by assuming both investors and the agent learn over time about the firm's expected future profitability based on its current cash flows. A contract provides the agent with incentives by specifying the agent's compensation and whether the firm will continue or be forced to shut down as a function of the firm's history of reported earnings.

After solving for the optimal contract, we show that it can be implemented through the firm's payout policy and a capital structure in which the agent holds a share of the firm's equity. When the firm is young, it makes no payouts and accumulates cash until it reaches a target level of financial slack that is positively related to the agent's perceived productivity. Once this target balance is reached, the firm initiates dividend payments. From that point on, the firm pays dividends at a rate equal to its *expected* future earnings. The firm absorbs any temporary shocks to earnings by increasing or decreasing its cash reserves, and it may also borrow. However, when the firm's debt reaches the liquidation value of its assets, the debt holders liquidate the firm.

This payout policy captures well the stylized facts associated with observed payout policies cited above. Immature firms do not pay dividends, but instead retain their earnings to invest, repay debt, and build cash reserves. For these firms the value of internal funds is high, as they risk running out of cash and being prematurely liquidated. But once the firm has sufficient financial slack, dividends are then paid. The level of dividends is based on the firm's estimate of the permanent component of its earnings, resulting in dividend payments that are much smoother than earnings themselves. Because dividend changes reflect permanent changes to profitability, they are persistent and have substantial implications for firm value.

Despite the broad empirical evidence that firms smooth their dividends, normative theoretical models of dividend smoothing have proved rather elusive. Modigliani and Miller (1961) showed that dividend policy is irrelevant if capital markets are perfect and investment policy is held constant, so one could argue that observed dividend policy is one of many “neutral” variations that firms could adopt. Such a view is difficult to reconcile with the stock price reaction to dividend changes discussed above; it is also contrary to the large body of evidence that market imperfections are economically significant and important drivers of corporate financial policy more generally.

One key difficulty with providing a theoretical model in which firm’s *optimally* smooth their payouts relative to earnings is that, given a fixed investment policy, it necessarily implies that the firm’s optimal leverage or cash position (its net debt) must be correspondingly “non-smooth.” This observation is hard to reconcile with the standard trade-off theory of capital structure and payout policy, which predicts that firms will maintain a target level of leverage/financial slack that balances the tax benefits of leverage (equivalently, the tax disadvantage of retaining cash) with potential costs of financial distress. In such a model, temporary cash flow shocks should be passed through to the firm’s payouts as it tries to maintain its leverage target.

Myers’ (1984) description of the pecking order hypothesis does include the prediction that variations in net cash flow will be absorbed largely by adjustments to the firm’s debt. However, this conclusion is based on the assumption that firms’ dividends are sticky in the short run, and no theoretical justification is provided for this assumption.<sup>2</sup>

We take a different approach in this paper. First, we use an optimal mechanism design approach to identify the real variables of interest in our model: the optimal timing of the liquidation decision, and the payoffs of the agent and investors after any history. In spirit of Modigliani and Miller (1961), in our model there may be many optimal dividend policies that can implement this optimal mechanism if there is no cost to raising equity capital in the event that the firm runs out of cash in the future.

---

<sup>2</sup> Taken to its logical extreme, the adverse selection argument in favor of debt given by Myers and Majluf (1984) would suggest that the firm should never pay dividends in order to avoid future finance costs. If a firm were to pay dividends (for some other unknown reason), it would presumably be balancing the marginal benefit of dividends with the marginal value of financial slack. In this case one would expect dividends and financial slack to move together.

Yet, in practice there are both institutional and informational costs to raising equity. Investment banks charge a 7% fee on public offerings. Moreover, new equity could be under-priced due to adverse selection. The pecking-order theory of Myers (1984) and the adverse selection arguments of Myers and Majluf (1984) imply that firms should raise funds using securities that are least sensitive to the firm's private information, i.e. debt. These arguments suggest that firms should abstain from paying dividends in order to avoid future finance costs, at least as long as the risk of running out of cash and triggering inefficient liquidation exists.

With these costs in mind, we identify the unique implementation of the optimal mechanism that provides the fastest possible payout rate subject to the constraint that the firm will not need to raise external capital in the future. Thus, in our model firms build up a target level of internal funds to ensure that they never need to liquidate inefficiently due to financial constraints. This constraint alone cannot explain dividend smoothing, as once the target is reached we would expect all excess cash flows to be paid out as dividends. The key driver of dividend smoothing in our model is the fact that there is learning about the firm's profitability based on the current level of the firm's cash flows. When cash flows are high, the firm's perceived profitability increases. This raises the cost of liquidating the firm (we are liquidating a more profitable enterprise), and therefore raises the optimal level of financial slack. Thus, a portion of the firm's high current cash flow will optimally be used to increase its cash reserves, resulting in a smoothed dividend policy.

The firm pays out dividends *exactly* at the rate of expected earnings in our model in order to have just enough cash to avoid inefficient liquidation, given current profitability. Then the firm's cash balances fall after bad news about future profitability due to negative earnings surprises, and rise after good news, due to positive earnings surprises. These changes in financial slack are optimal when there is an agency problem given the new information earnings provide regarding the firm's profitability. Thus, dividend smoothing in our implementation of the optimal contract can be explained by two reasons (1) financial slack is needed to avoid inefficient liquidation and (2) the optimal level of financial slack varies with the shocks to the firm's earnings.

Our model implies that firms smooth dividends by absorbing their cash flow variations through variations in their leverage or available financial slack. This conclusion is confirmed by empirical evidence. Indeed, as documented by Fazzari, Hubbard, and Petersen (1988) and others, financially unconstrained dividend-paying firms do appear to employ investment policies that are less sensitive to shocks to the firm's earnings or cash flows. On the other hand, there is evidence that firms' cash and leverage positions are strongly influenced by past profitability, even when firms are financially unconstrained (see, e.g., Fama and French (2002)).

In our model, the firm gains financial strength over time to mitigate the inefficiencies connected with moral hazard. This is a common prediction of a diverse range of dynamic contracting model, such as Albuquerque and Hopenhayn (2001), Atkeson and Cole (2005), DeMarzo and Sannikov (2006), Biais, Mariotti, Plantin and Rochet (2007) and Biais, Mariotti, Rochet and Villeneuve (2007). The key ingredient of our setting that drives the smooth dividend policy is the combination of moral hazard and learning about the firm's profitability. That is, it is important in our model that the firm's cash flows carry information both about the agent's effort and the agent's skill.

In the next section of the paper, we describe the continuous-time principal-agent problem with learning about the firm's profitability. Then in Section 2, we present the solution to this problem when moral hazard is absent, which is based purely on option-value considerations. This solution is important to understand the optimal long-term contract with moral hazard, which we derive in Section 3. Finally, in Section 3, we show how this optimal contract can be implemented in terms of the firm's payout policy.

## **1. Model**

In this section we present a continuous-time formulation of the firm and the principal-agent problem that arises between the manager running the firm and outside investors. In our model, risk-neutral outside investors hire a risk-neutral agent to run a firm. Both the

agent and investors discount the future at rate  $r > 0$ . Investors have unlimited wealth, whereas the agent has no initial wealth and must consume non-negatively.<sup>3</sup>

When the agent puts full effort  $a_t = 1$ , the firm generates cash flows at an expected rate equal to  $\delta_t$ . However, the agent may also reduce his effort, divert his own or the firm's resources for personal benefit, and make corporate decisions out of private interest. When the agent's effort/activity level to generate value for the firm is  $a_t \leq 1$ , the firm's the cumulative cash flow process  $X_t$  is defined by

$$dX_t = a_t \delta_t dt + \sigma dZ_t \quad (1)$$

where  $\sigma$  is the volatility of cash flows and  $Z$  is a standard Brownian motion. At the same time, the agent's private benefit is  $\lambda(1 - a_t)$ , where the parameter  $\lambda < 1$  reflects the fact that it is more efficient to use the agent's and the firm's resources to make profit than for the agent's private benefit.

The firm's profitability evolves over time due to changing market conditions as well as the evolution of the manager's talent. The realization of the firm's current cash flow is informative about these changes in profitability. We model this by assuming that  $\delta_t$  starts at  $\delta_0 > 0$  and evolves according to

$$d\delta_t = v(dX_t - a_t \delta_t dt) = v\sigma dZ_t \quad (2)$$

as long as  $\delta_t > 0$ . If  $\delta_t$  ever reaches 0, it is absorbed there.<sup>4</sup>

Because of the learning about the firm's profitability as specified in (2), in our contractual environment the agent may have private information not only about his effort,

---

<sup>3</sup> The assumption that the agent has no initial wealth is without loss of generality; equivalently, we can assume the agent has already invested any initial wealth in the firm. The agent's limited liability prevents a general solution to the moral hazard problem in which the firm is simply sold to the agent.

<sup>4</sup> According to equation (2), cash flows  $X_t$  and firm profitability  $\delta_t$  are driven by the same Brownian motion. This equation arises as the steady state of a filtering problem in which  $\delta_t$  is the current estimate of the firm's true underlying profitability  $\delta_t^*$ , and cash flows carry information about this profitability. However, imposing the additional constraint that  $\delta_t^*$  (and therefore  $\delta_t$ ) remains non-negative – which is needed for our interpretation of effort -- gives rise to technical issues, since we lose normality and inferences become very messy. Instead of dealing with distracting technical complexities, we choose to work with a simpler “reduced form” (2), which is motivated by the filtering problem.

but also about the firm's profitability. Indeed, if the principal expects the agent to choose effort  $a_t$ , the principal will update his belief  $\hat{\delta}_t$  about firm profitability according to

$$d\hat{\delta}_t = v(dX_t - a_t\hat{\delta}_t dt), \quad \hat{\delta}_0 = \delta_0. \quad (3)$$

Thus, if the agent chooses a different effort strategy  $\hat{a}_t \neq a_t$ , the principal's belief  $\hat{\delta}_t$  will be incorrect.

The agent can quit the firm at any time. In that event the agent devotes his full attention and resources, but not those of the firm, to outside activities (so that  $a_t = 0$ ), earning private benefits at expected rate  $\lambda\delta_t$ . Thus, the value of the agent's payoff in the event that he quits the firm is represented by

$$R(\delta_t) = \frac{\lambda\delta_t}{r}. \quad (4)$$

The firm requires external capital of  $K \geq 0$  to be started. The investors contribute this capital and in exchange receive the cash flows generated by the firm less any compensation paid to the agent. The agent's compensation is determined by a long-term contract. This contract specifies, based on the history of the firm's cash flows, non-negative compensation for the agent while the firm operates, as well as a termination time when the firm is liquidated. Formally, a contract is a pair  $(C, \tau)$ , where  $C$  is a non-decreasing  $X$ -measurable process that represents the agent's cumulative compensation (i.e.,  $dC_t \geq 0$  is the agent's compensation at time  $t$ ) and  $\tau$  is an  $X$ -measurable stopping time. In the event that the firm is liquidated, the agent engages in his outside option and so receives the payoff specified in (4). The investors receive a liquidation value from the firm's assets that may depend on the firm's productivity,

$$L(\delta_\tau) = L_0 + \frac{\kappa\delta_\tau}{r}, \quad (5)$$



where  $\lambda + \kappa < 1$ .<sup>5</sup>

A contract  $(C, \tau)$  together with an  $X$ -measurable effort recommendation  $a$  is optimal given an expected payoff of  $W_0$  for the agent if it maximizes the principal's profit

$$E \left[ \int_0^\tau e^{-rt} (a_t \delta_t dt - dC_t) + e^{-r\tau} L(\delta_\tau) \right] \quad (6)$$

subject to

$$W_0 = E \left[ \int_0^\tau e^{-rt} (\lambda(1 - a_t) \delta_t dt + dC_t) + e^{-r\tau} R(\delta_\tau) \right] \text{ given strategy } a \quad (7)$$

and

$$W_0 \geq E \left[ \int_0^\tau e^{-rt} (\lambda(1 - \tilde{a}_t) \delta_t dt + dC_t) + e^{-r\tau} R(\delta_\tau) \right] \text{ for any other strategy } \tilde{a} \quad (8)$$

By varying  $W_0 > R(\delta_0)$ , we can use this solution to consider different divisions of bargaining power between the agent and the investors. For example, if the agent enjoys all the bargaining power due to competition between investors, then the agent will receive the maximal value of  $W_0$  subject to the constraint that the investors' payoff be at least equal to their initial investment,  $K$ . We say that the effort recommendation  $a$  is *incentive-compatible* with respect to the contract  $(C, \tau)$  if it satisfies (7) and (8) for some  $W_0$ .

REMARKS. For simplicity, we specify the contract assuming that the agent's compensation and the termination time  $\tau$  are determined by the cash flow process, ruling out public randomization. This assumption is without loss of generality, as we will later verify that public randomization would not improve the contract.

---

<sup>5</sup> In principle, the investors' belief  $\hat{\delta}_t$  about the firm's productivity might also affect the liquidation value of assets. However, because in equilibrium  $\hat{\delta}_t = \delta_t$ , without loss of generality we specify  $L$  as a function of  $\delta_t$ .

## 2. The First-Best Solution.

Before solving for the optimal contract, we derive the first-best solution as a benchmark. In the first-best, the principal can control the agent's effort, and so we can ignore the incentive constraints (8). Then it is optimal to let the agent take effort  $a_t = 1$  until liquidation, since it is cheaper to provide the agent with a flow of utility by paying him than by letting him divert attention to private activities. Then the total cost of providing the agent with a payoff of  $W_0$  is

$$E \left[ \int_0^\tau e^{-rt} dC_t \right] = W_0 - E \left[ e^{-r\tau} R(\delta_\tau) \right],$$

and the principal's payoff is

$$E \left[ \int_0^\tau e^{-rt} \delta_t dt + e^{-r\tau} (L(\delta_\tau) + R(\delta_\tau)) \right] - W_0.$$

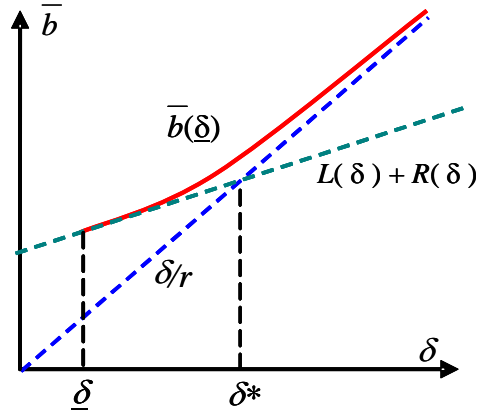
Thus, without moral hazard the principal chooses a stopping time  $\tau$  that solves

$$\bar{b}(\delta_0) = \max_\tau E \left[ \int_0^\tau e^{-rt} \delta_t dt + e^{-r\tau} (L(\delta_\tau) + R(\delta_\tau)) \right]. \quad (9)$$

This is a standard real-option problem that can be solved by the methods of Dixit and Pindyck (1994). Because liquidation is irreversible, it is optimal to trigger liquidation when the expected profitability  $\delta_t$  reaches a critical level of  $\underline{\delta}$  that is below the level  $\delta^*$  such that

$$R(\delta^*) + L(\delta^*) = \delta^*/r$$

See Figure 2.



**Figure 2:** First-best Liquidation Threshold and Value Function

We have the following explicit solution for the first-best liquidation threshold and value of the firm:

**Proposition 1.** *Under the first-best contract, the firm is liquidated if  $\delta \leq \underline{\delta}$  where*

$$\underline{\delta} = \max\left(0, \delta^* - \frac{v\sigma}{\sqrt{2r}}\right).$$

The principal's payoff is  $\bar{b}(\delta_0) - W_0$ , where

$$\bar{b}(\delta) = \frac{\delta}{r} + \left(L(\underline{\delta}) + R(\underline{\delta}) - \frac{\underline{\delta}}{r}\right) \exp\left(-\frac{\sqrt{2r}}{v\sigma}(\delta - \underline{\delta})\right)$$

if  $\delta \geq \underline{\delta}$  and  $\bar{b}(\delta) = L(\delta) + R(\delta)$  otherwise.

*Proof:* Note that  $\bar{b}(\delta)$  is the solution on  $[\underline{\delta}, \infty)$  to the ordinary differential equation

$$r\bar{b}(\delta) = \delta + \frac{1}{2}v^2\sigma^2\bar{b}''(\delta) \tag{10}$$

with boundary conditions

(a)  $\bar{b}(\underline{\delta}) = L(\underline{\delta}) + R(\underline{\delta})$ ,

(b)  $\bar{b}'(\underline{\delta}) = \frac{\lambda + \kappa}{r}$  with  $\underline{\delta} > 0$  (smooth-pasting) or  $\bar{b}'(\underline{\delta}) > \frac{\lambda + \kappa}{r}$  with  $\underline{\delta} = 0$ ,

(c) and  $\bar{b}(\delta) - \delta/r \rightarrow 0$  as  $\delta \rightarrow \infty$ .

Let us show that  $\bar{b}(\delta)$  gives the maximal profit attainable by the principal. For an arbitrary contract  $(C, \tau)$ , consider the process

$$G_t = e^{-rt} \bar{b}(\delta_t) + \int_0^t e^{-rs} \delta_s ds.$$

Let us show that  $G_t$  is a submartingale. Using Ito's lemma, the drift of  $G_t$  is

$$-re^{-rt} \bar{b}(\delta_t) + e^{-rt} \frac{1}{2} v^2 \sigma^2 \bar{b}''(\delta_t) + e^{-rt} \delta_t,$$

which is equal to 0 when  $\delta_t > \underline{\delta}$  and  $-re^{-rt} (L(\delta_t) + R(\delta_t)) + e^{-rt} \delta_t < 0$  when  $\delta_t < \underline{\delta}$ .

Therefore, the principal's expected profit at time 0 is

$$\begin{aligned} E \left[ e^{-r\tau} L(\delta_\tau) + \int_0^\tau e^{-rt} (\delta_t dt - dC_t) \right] &= E \left[ e^{-r\tau} (L(\delta_\tau) + R(\delta_\tau)) + \int_0^\tau e^{-rt} \delta_t dt - W_0 \right] \\ &\leq E[G_\tau] - W_0 \leq G_0 - W_0 \\ &= \bar{b}(\delta_0) - W_0. \end{aligned}$$

The inequalities above become equalities if and only if  $\delta_\tau \leq \underline{\delta}$  and  $\delta_t > \underline{\delta}$  before time  $\tau$ . ■

Our characterization of the first-best contract can be interpreted in terms of the firm's capacity to sustain operating losses. At any moment of time, the firm must be able to withstand a productivity shock of up to  $d\delta = -(\delta - \underline{\delta})$ . From (2), this corresponds to a cash flow shock equal to

$$dX_t - a_t \delta_t dt = \sigma dZ_t = \frac{d\delta_t}{v} = -\left( \frac{\delta - \underline{\delta}}{v} \right). \quad (11)$$

We can view Equation (11) as specify the minimal level of "financial slack" the firm will require in order to avoid inefficient liquidation. This result will play an important role in our implementation of the optimal contract, which we consider next.

### 3. An Implementation

Having characterized the first-best outcome, we now consider the problem of finding the optimal dynamic contract in our setting with both moral hazard and asymmetric information. The task of finding the optimal contract in a setting like ours is complex, because there is a huge space of fully contingent history-dependent contracts to consider. A contract  $(C, \tau)$  must specify how the agent's consumption and the liquidation time depend on the entire history of cash flows. In classic settings with uncertainty only about the agent's effort but not the firm's productivity, there are standard recursive methods to deal with such complexity. These methods rely on dynamic programming using the agent's future expected payoff (a.k.a. continuation value) as a state variable.<sup>6</sup>

But with additional uncertainty and the potential for asymmetric information about the firm's productivity, these standard methods do not apply directly to our model. Thus we will take a different approach. We begin instead by conjecturing a simple and intuitive implementation for the contract. In our setting with moral hazard, if the agent had deep pockets the first-best liquidation policy could be attained by letting the agent own the firm. If the agent's wealth is limited, however, negative cash flow shocks can lead to inefficient liquidation. In order to minimize this inefficiency, it is natural to expect that in an optimal contract, the firm will build up cash reserves until it has an optimal level of financial slack. In this section we consider an implementation based on this intuition, and then show this implementation is incentive compatible. Though our implementation is rather simple, we will then verify the optimality of this contract in the space of all possible contracts in the following section.

---

<sup>6</sup> For example, see Spear and Srivastava (1987), Abreu, Pearce and Stacchetti (1990) (in discrete time) and DeMarzo and Sannikov (2006) and Sannikov (2007a) (in continuous time) for the development of these methods, and Piskorski and Tchisty (2006) and Philippon and Sannikov (2007) for their applications.

### 3.1. Cash Reserves and Payout Policy

Consider an all-equity financed firm that uses cash reserves to provide financial slack.<sup>7</sup> Denote the level of its cash reserves by  $M_t \geq 0$ . Because the firm earns interest at rate  $r$  on these balances, its earnings at date  $t$  are given by

$$dE_t \equiv r M_t dt + dX_t = (r M_t + \delta_t) dt + \sigma dZ_t, \quad (12)$$

where we have assumed the agent's effort  $a_t = 1$ .

If the firm uses these earnings to pay dividends  $dD_t$ , then its cash reserves will grow by

$$dM_t \equiv dE_t - dD_t = (r M_t + \delta_t) dt - dD_t + \sigma dZ_t. \quad (13)$$

Consider a contract in which the firm is forced to liquidate if it depletes its reserves and  $M_t = 0$ .<sup>8</sup> In order to avoid inefficient liquidation, we know from (11) that the firm must have reserves  $M_t$  of at least

$$M^1(\delta_t) \equiv (\delta_t - \underline{\delta})/\nu. \quad (14)$$

Therefore, it is natural to suppose that if  $M_t < M^1(\delta_t)$ , the firm will retain 100% of its earnings in order to increase its reserves and reduce the risk of inefficient liquidation. In that case, dividends are equal to zero:

$$dD_t = 0 \text{ if } M_t < M^1(\delta_t). \quad (15)$$

Suppose the firm achieves the efficient level of reserves, so that  $M_t = M^1(\delta_t)$ . In order to maintain its reserves at the efficient level, using (14) and (2) we must have

$$dM_t = dM^1(\delta_t) = \frac{d\delta_t}{\nu} = \sigma dZ_t. \quad (16)$$

---

<sup>7</sup> We note that our proposed implementation is not unique, nor is it clear that it is optimal. The firm could also maintain financial slack through alternative means, such as a credit line or loan commitment. The analysis would be similar; for convenience we focus on the simplest implementation in terms of cash reserves. See Biais et al (2006) for a similar implementation based on cash reserves in a moral hazard setting without learning.

<sup>8</sup> Over any finite time period, the firm will experience operating losses with probability one; therefore, absent cash or credit, the firm must shut down. However, it is not yet clear whether it is optimal to deny the firm funds and force liquidation if  $M_t = 0$ . We will address optimality in the following section.

That is, to maintain the efficient level of reserves, the firm should adjust its cash balances by the “surprise” component of its earnings. Then from (13), dividends are equal to the firm’s expected earnings:

$$dD_t = E[dE_t] = (rM_t + \delta_t)dt \quad \text{if } M_t = M^1(\delta_t). \quad (17)$$

The following result demonstrates that with this payout policy, the liquidation policy is first-best:

**Proposition 2.** *If  $M_t = M^1(\delta_t)$  and if the firm follows the payout policy (17) after time  $t$ , then  $M_\tau = 0$  if and only if  $\delta_\tau = \underline{\delta}$ .*

*Proof:* Given the payout policy (17), the firm’s cash balance evolves according to (16). Therefore,  $M_\tau = 0$  implies

$$\delta_\tau = \delta_t + \int_t^\tau d\delta_s = \delta_t + v \int_t^\tau dM_s = \delta_t + v(-M_t) = \delta_t - vM^1(\delta_t) = \underline{\delta}.$$

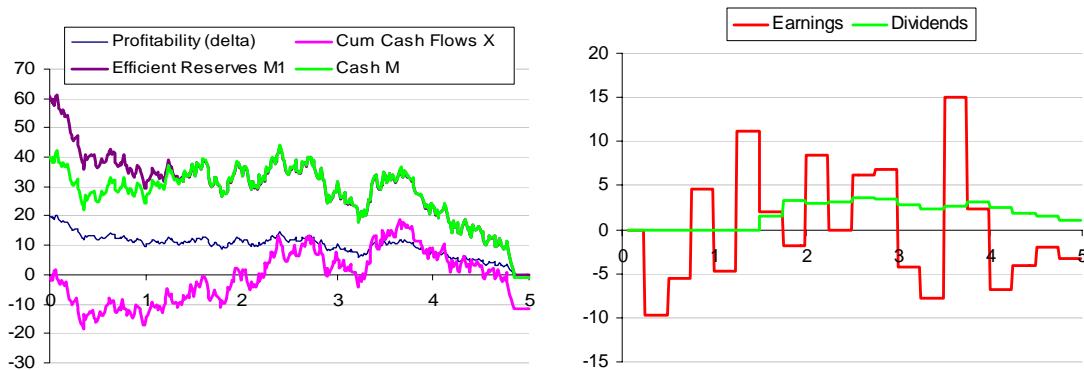
■

Finally, because there is no benefit from maintaining reserves in excess of the amount needed to avoid inefficient liquidation, we assume the firm pays out any excess cash immediately. Thus, we can summarize the firm’s payout policy as follows:

$$dD_t = \begin{cases} 0 & \text{if } M_t < M^1(\delta_t) \\ (rM_t + \delta_t)dt & \text{if } M_t = M^1(\delta_t). \\ M_t - M^1(\delta_t) & \text{if } M_t > M^1(\delta_t) \end{cases} \quad (18)$$

Under the payout policy described by (18), the firm accumulates cash as quickly as possible until it either runs out of cash and is inefficiently liquidated, or its reserves reach the efficient level. Once the efficient level of reserves is attained, the firm begins paying dividends at a rate equal to its expected earnings. It will continue to operate in this fashion unless  $\delta_t$  falls to  $\underline{\delta}$ , in which case  $M_t = 0$  and the firm is liquidated as in the first-best.

Figure 3 presents contract dynamics on an example. Until time 1.5 the firm has cash balances below the efficient level, and it stands the risk of being liquidated inefficiently. However, in this example inefficient liquidation does not happen. At time 1.5 the firm's cash level reaches the efficient target, and the firm initiates dividends. Dividends continue until the firm's profitability falls sufficiently and it is liquidated at date 5. The right panel of Figure 3 illustrates that the total quarterly dividends are significantly smoother than earnings. Note also the qualitative similarity between the results of our model and the pattern of dividends illustrated for GM in Figure 1.



**Figure 3: Contract dynamics when  $r = 5\%$ ,  $\sigma = 15$ ,  $\nu = 33\%$ . The liquidation threshold is  $\underline{\delta} = 0$ .**

### 3.2. Compensation and Incentive Compatibility

The requirement of cash reserves combined with the payout policy described above determines the liquidation time,  $\tau$ , of the contract. To complete this implementation, we need specify the agent's compensation,  $C$ , and then assess whether the contract provides appropriate incentives.

Note that if the agent had unlimited wealth, we could provide the agent with appropriate incentives for effort by paying him a fraction  $\lambda$  of the firm's cash flows. This solution is not possible, however, since the firm's cash flows may be negative and the agent has limited liability. Given the implementation above, a natural alternative to consider is to



pay the agent the fraction  $\lambda$  of the firm's *dividends* (rather than its cash flows), which are always non-negative.

This compensation can be interpreted as providing the agent a fraction  $\lambda$  of the firm's equity, with the proviso that the agent not receive any proceeds from a liquidation should it occur. This outcome could be implemented, for example, by giving outside investors preferred stock with complete priority in the event of liquidation. (Alternatively, the agent may receive a zero interest loan to purchase the shares, which becomes due in the event of liquidation.) We refer to the agent's compensation as equity that is *rescindable in the event of liquidation*.

Now we are ready to consider the agent's incentives. To verify incentive compatibility, we must determine the agent's payoff given different effort choices and payout policies. Consider the case in which the firm is already paying dividends; that is,  $M_t = M^1(\delta_t)$ . Suppose the agent follows the proposed implementation. Then from (17), the agent's expected payoff  $W_t$  is given by

$$\begin{aligned} E \left[ \int_t^\tau e^{-r(s-t)} \lambda dD_s + e^{-r(\tau-t)} R(\underline{\delta}) \right] &= E \left[ \int_t^\tau e^{-r(s-t)} \lambda (rM_s + \delta_s) ds + e^{-r(\tau-t)} R(\underline{\delta}) \right] \\ &= E \left[ \int_t^\tau e^{-r(s-t)} r (\lambda M_s + R(\delta_s)) ds + e^{-r(\tau-t)} (\lambda M_\tau + R(\delta_\tau)) \right] \\ &= \lambda M_t + R(\delta_t) \end{aligned}$$

where the last equation follows from the fact that  $M_s$  and  $\delta_s$  are martingales for  $s > t$ .<sup>9</sup>

Now consider a deviation in which the agent "cashes out" by immediately paying out all cash as a dividend,  $dD_t = M_t$ , and then defaulting. Under this strategy, the agent again receives a payoff of

$$W_t = \lambda M_t + R(\delta_t). \tag{19}$$

Thus there is no incentive for the agent to deviate in this way. We can similarly show that this implementation is robust to other types of deviations for the agent. For example,

---

<sup>9</sup> Note that if  $X_s$  is a martingale, then  $X_t = X_t \left( r \int_t^\tau e^{-r(s-t)} ds + e^{-r(\tau-t)} \right) = E \left[ r \int_t^\tau e^{-r(s-t)} X_s ds + e^{-r(\tau-t)} X_\tau \right]$ .

because (19) implies that the agent's payoff increases by  $\lambda$  for each dollar of additional cash held by the firm, there is no incentive for the agent to shirk and engage in outside activities. The following result establishes the incentive compatibility, with regard to both the agent's effort choice and payout policy, of our proposed implementation:

**Proposition 3.** *Suppose the agent holds a fraction  $\lambda$  of the firm's equity, rescindable in the event of liquidation, and that liquidation occurs if the firm's cash balance falls to zero. Then for any effort strategy and payout policy, the agent's expected payoff is given by (19). Thus, it is optimal for the agent to choose effort  $a_t = 1$  and to adopt the payout policy in (18).*

*Proof:* Consider an arbitrary payout policy  $D$  and effort strategy  $a$ . Define

$$V_t = \int_0^t e^{-rs} (\lambda dD_s + \lambda(1-a_s)\delta_s) ds + e^{-rt}W_t$$

Then using the fact that  $\delta$  is a martingale and that

$$E[dM_t] = rM_t dt + E[dX_t] - dD_t = rM_t dt + a_t \delta_t dt - dD_t,$$

the drift of  $V_t$  is

$$\begin{aligned} E[dV_t] &= e^{-rt} (\lambda dD_t + \lambda(1-a_t)\delta_t ds - rW_t + E[dW_t]) = \\ &= e^{-rt} (\lambda dD_t + \lambda(1-a_t)\delta_t ds - r(\lambda M_t + \lambda \delta_t / r) + \lambda E[dM_t]) \\ &= 0 \end{aligned}$$

and so  $V_t$  is a martingale. Thus, because  $M_\tau = 0$  so that  $W_\tau = R(\delta_\tau)$ , the agent's expected payoff from this arbitrary strategy is

$$\begin{aligned} E \left[ \int_0^\tau e^{-rt} (\lambda dD_t + \lambda(1-a_t)\delta_t dt) + e^{-r\tau} R(\delta_\tau) \right] &= E \left[ V_\tau - e^{-r\tau} W_\tau + e^{-r\tau} R(\delta_\tau) \right] \\ &= E[V_\tau] = V_0 = W_0 \end{aligned}$$

and therefore the implementation is incentive compatible. ■

## 4. Justification of the Optimal Contract.

Because standard methods do not apply directly to our model, in this section we develop a new approach to justify the optimality of our conjectured contract. While the specific solution is unique to our problem, we propose a three-step method to solve similar problems:

- (1) Isolate the necessary incentive constraints, which are most important in limiting the attainable expected profit.
- (2) Show that the conjectured contract solves the principal's optimization problem subject to just the necessary incentive constraints.
- (3) Verify that the conjectured contract is fully incentive-compatible.

We conjectured a contract in the previous section, and verified its full incentive-compatibility in Proposition 3. We need to execute steps 1 and 2 of the verification argument.

Before we proceed, we note that the agent must take action  $a_t = 1$  at all times in the optimal contract. The reason is that because  $\lambda < 1$ , it is cheaper to pay the agent directly rather than let him take actions for private benefit.

**Lemma 1 (High Effort).** In the optimal contract  $a_t = 1$  until time  $\tau$ .

*Proof.* Consider any contract in which sometimes  $a_t < 1$ , and let us show that there exists a better contract. Let us change it, by giving the agent an option to ask the principal for extra payments  $dC_t'$ . If the agent exercises this option at least once, then the agent's wages and termination time  $\tau$  are determined as if the true path of output were

$$\hat{X}_t = X_t - \int_0^t \frac{dC_s'}{\lambda}.$$

Then the agent is indifferent between lowering  $a_t$  below 1 and simply asking the principal for extra money. If he asks for money now whenever he was lowering effort previously, then the agent's strategy is incentive-compatible, and the principal's profit is strictly higher. Therefore, the original contract cannot be optimal. ■

From now on, we restrict attentions to contract with recommended effort  $a_t = 1$ .

#### 4.1. Necessary incentive constraints.

The necessary incentive-compatibility constraints are formulated using appropriately chosen state variables.<sup>10</sup> For our problem, we must include as state variables at least the principal's current belief about the agent's skill  $\hat{\delta}_t$ , which evolves according to

$$d\hat{\delta}_t = v(dX_t - \hat{\delta}_t dt), \quad \hat{\delta}_0 = \delta_0,$$

and the agent's continuation value when the agent follows the recommended strategy ( $a_s$ ) after time  $t$ , and the principal has a correct belief about the agent's skill

$$W_t = E_t \left[ \int_t^\tau e^{-r(s-t)} dC_s + e^{-r(\tau-t)} R(\delta_\tau) \mid \delta_t = \hat{\delta}_t \right] \text{ given strategy } \{a_t = 1\}.$$

The variables  $\hat{\delta}_t$  and  $W_t$  are well-defined for any contract  $(C, \tau)$ , after any history of cash flows  $\{X_s, s \in [0, t]\}$ . However, they do not fully summarize the agent's incentives, which depend on the agent's *deviation payoffs*, the payoff that the agent would obtain if  $\hat{\delta}_t \neq \delta_t$  due to the agent's past deviations. Therefore, we can formulate only necessary conditions for incentive compatibility using the variables  $\hat{\delta}_t$  and  $W_t$ .

Lemma 2A, which is standard in continuous-time contracting, provides a stochastic representation for the dependence of  $W_t$  on the cash flows  $\{X_t\}$  in a given contract  $(C, \tau)$ . The connection between  $W_t$  and  $X_t$  matters for the agent's incentives.

**Lemma 2A (Representation).** *There exists a process  $\{\beta_t, t \geq 0\}$  in  $L^*$  such that*

$$dW_t = rW_t dt - dC_t + \beta_t(dX_t - \hat{\delta}_t dt). \quad (20)$$

*Proof.* See Appendix.

---

<sup>10</sup> For example, in continuous time Sannikov (2007b) solves an agency problem with adverse selection using the continuation values of the two types of agents as state variables, Williams (2007) solves an example with hidden savings using the agent's continuation value and his marginal utility as state variables.

The process  $\beta_t$  determines the agent's exposure to the firm's cash flows shocks and therefore the strength of the agent's incentives under the contract. It is therefore natural to expect that  $\beta_t$  must be sufficiently large for the contract to be incentive compatible. For example, consider a deviation  $\hat{a}_t = 0$ . The agent can gain in two ways from this deviation. First, the agent earns the payoff  $\lambda \delta_t dt$  outside the firm. Second, from (3), the lower output of the firm reduces principal's estimate of the firm's productivity. If this deviation is the agent's first, then

$$d\hat{\delta}_t = v(dX_t - \delta_t dt) = d\delta_t - v\delta_t dt.$$

Given these lowered expectations, the agent can continue to shirk and reduce effort by  $v$  from that point onward and still generate cash flows consistent with the principal's expectations, for an additional expected perpetual gain of  $\lambda v \delta_t dt$ . Because the deviation reduces the agent's contractual payoff by  $\beta_t \delta_t dt$ , this deviation is profitable if

$$\beta_t \delta_t < \lambda \delta_t + \lambda v \delta_t / r = \lambda \delta_t (1 + v/r)$$

Lemma 2B below formalizes this intuition and establishes a necessary condition on  $\beta_t$  for a contract  $(C, \tau)$  to be incentive compatible.

**Lemma 2B (Incentive Compatibility).** *Consider a contract  $(C, \tau)$ , for which the agent's continuation value evolves according to (20). A necessary condition for  $\{a_t = 1\}$  to be incentive-compatible with respect to  $(C, \tau)$  is that  $\beta_t \geq \lambda(1 + v/r)$  while  $\hat{\delta}_t > 0$ .*

*Proof.* See Appendix A.

## 4.2. Verification of Optimality

In this section we verify that our conjectured implementation is indeed an optimal contract. Recall from Section 3 that the agent's payoff in this contract is defined by

$$W_t = \lambda M_t + R(\delta_t),$$

where

$$dM_t = (r M_t + \delta_t) dt + \sigma dZ_t \text{ and } dD_t = 0 \text{ while } M_t < M^1(\delta_t) = (\delta_t - \underline{\delta})/v \text{ and}$$

$$dM_t = \sigma dZ_t, M_t = M^1(\delta_t) \text{ and } dD_t = (rM_t + \delta_t)dt \text{ thereafter.}$$

It follows that

$$\begin{aligned} dW_t &= rW_t dt + \lambda(1 + \nu/r)\sigma dZ_t \text{ and } dC_t = 0 \text{ until } W_t \text{ reaches } W^l(\delta), \text{ and} \\ W_t &= W^l(\delta_t) \text{ and } dC_t = rW_t dt \text{ thereafter,} \end{aligned} \tag{21}$$

where

$$W^l(\delta) = R(\underline{\delta}) + \lambda\left(\frac{1}{\nu} + \frac{1}{r}\right)(\delta - \underline{\delta}).$$

This evolution happens until  $W_t$  reaches  $R(\delta)$ , triggering liquidation.

Let us show that this contract attains the highest expected profit among all contracts that deliver value  $W_0$  to an agent of skill level  $\delta_0$  and satisfy the necessary incentive-compatibility condition of Lemma 2B. The set of such contracts includes all fully incentive-compatible contracts. Since the conjectured contract is incentive-compatible, as shown in Proposition 3, it follows that it is also optimal.

Let us present a roadmap of our verification argument. First, we define a function  $b(W_0, \delta_0)$ , which gives the expected profit that a contract of Section 3 attains for any pair  $(W_0, \delta_0)$  with  $W_0 \geq R(\underline{\delta})$  and  $\delta_0 \geq \underline{\delta}$ . Proposition 4 verifies that this definition is indeed the expected payoff of outside equity holders in our implementation. After that, Proposition 5 shows that the principal's profit in any alternative contract that satisfies the necessary incentive-compatibility condition of Lemma 2B is at most  $b(W_0, \delta_0)$  for any pair  $(W_0, \delta_0)$  with  $W_0 \geq R(\underline{\delta})$  and  $\delta_0 \geq \underline{\delta}$ . It follows that the conjectured contract of Section 3 is optimal.

For  $W \geq R(\delta)$  and  $\delta \geq \underline{\delta}$ , define a function  $b(W, \delta)$  as follows.

- (i) For  $W > W^l(\delta)$ , let  $b(W, \delta) = \bar{b}(\delta) - W$ .
- (ii) For  $W = R(\delta)$ , let  $b(W, \delta) = L(\delta)$ .

Otherwise, for  $\delta > \underline{\delta}$  and  $W \in (R(\delta), W^l(\delta))$ , let  $b(W, \delta)$  solve the equation

$$\begin{aligned} rb(W, \delta) &= \delta + rWb_w(W, \delta) + \\ &\frac{1}{2}\lambda^2\left(1 + \frac{\nu}{r}\right)^2\sigma^2b_{ww}(W, \delta) + \frac{1}{2}\nu^2\sigma^2b_{\delta\delta}(W, \delta) + \lambda\left(1 + \frac{\nu}{r}\right)\nu\sigma^2b_{w\delta}(W, \delta) \end{aligned} \tag{22}$$

with boundary conditions given by (i) and (ii).

For an arbitrary contract  $(C, \tau)$  with an incentive-compatible effort recommendation  $a$ , in which the agent's continuation value evolves according to (21), define the process

$$G_t = \int_0^t e^{-rs} (\delta_s ds - dC_s) + e^{-rt} b(W_t, \delta_t).$$

Note that on the equilibrium path we always have  $\delta_t = \hat{\delta}_t$ .

Lemma 3 helps us prove both Propositions 4 and 5.

**Lemma 3.** *When  $\delta_t \geq \underline{\delta}$  and  $C_t$  is continuous at  $t$ , then*

$$dG_t = e^{-rt} (v b_{\delta}(W_t, \delta_t) + \beta_t b_W(W_t, \delta_t)) \sigma dZ_t - e^{-rt} (b_W(W_t, \delta_t) + 1) dC_t + e^{-rt} \sigma^2 \left( \frac{1}{2} (\beta_t - \lambda(1 + \frac{v}{r}))^2 b_{WW}(W_t, \delta_t) + (\beta_t - \lambda(1 + \frac{v}{r})) (\lambda(1 + \frac{v}{r}) b_{W\delta}(W_t, \delta_t) + v b_{W\delta}(W_t, \delta_t)) \right) dt$$

*Proof.* See Appendix A.

**Proposition 4.** *The conjectured optimal contract of subsection 3.1 attains profit  $b(W_0, \delta_0)$ .*

*Proof.* Under that contract, the process  $G_t$  is a martingale. Indeed, for all  $t > 0$ , the continuous process  $C_t$  increases only when  $W_t = W^l(\delta_t)$  (where  $b_W(W^l(\delta_t), \delta_t) = -1$ ) and  $\beta_t = \lambda(1 + \frac{v}{r})$ , so  $G_t$  is a martingale by Lemma 3. At time 0, the agent consumes positively only in order for  $W_0$  to drop to  $W^l(\delta_0)$ , and  $b_W(W, \delta_0) = -1$  for  $W \geq W^l(\delta_0)$ , so  $G_t$  is a martingale there as well. Therefore, the principal attains the profit of

$$E \left[ e^{-r\tau} b(W_{\tau}, \delta_{\tau}) + \int_0^{\tau} e^{-rt} (\delta_t dt - dC_t) \right] = E[G_{\tau}] = G_0 = b(W_0, \delta_0).$$

QED.

**Proposition 5.** *In any alternative incentive-compatible contract  $(C, \tau)$  the principal's profit is bounded from above by  $b(W_0, \delta_0)$ .*

*Proof.* Let us argue that  $G_t$  is a supermartingale for any alternative incentive-compatible contract  $(C, \tau)$  while  $\delta_t \geq \underline{\delta}$ .

First, whenever  $C_t$  has an upward jump of  $\Delta C_t$ ,  $G_t$  has a jump of  $e^{-rt} (b(W_t + \Delta C_t, \delta_t) - b(W_t, \delta_t) - \Delta C_t) \leq 0$ , since  $b_W(W, \delta) \geq -1$  for all pairs  $(W, \delta)$  (see Appendix B, which shows that  $b$  is concave in  $W$ ).

Second, whenever  $C_t$  is continuous, then  $\beta_t \geq \lambda(1 + \nu/r)$  by Lemma 2B. By Lemma 3, the drift of  $G_t$  is

$$-e^{-rt} (b_W(W_t, \delta_t) + 1) dC_t + e^{-rt} \sigma^2 \left( \frac{1}{2} (\beta_t - \lambda(1 + \frac{\nu}{r}))^2 b_{WW}(W_t, \delta_t) + (\beta_t - \lambda(1 + \frac{\nu}{r})) (\lambda(1 + \frac{\nu}{r}) b_{WW}(W_t, \delta_t) + \nu b_{W\delta}(W_t, \delta_t)) \right) dt < 0$$

since  $b_W(W, \delta) \geq -1$  and, as shown in Appendix B,

$$b_{WW}(W, \delta) \leq 0 \quad \text{and} \quad \lambda(1 + \frac{\nu}{r}) b_{WW}(W, \delta) + \nu b_{W\delta}(W, \delta) \leq 0 \quad (23)$$

for all pairs  $(W, \delta)$ .

Now, let  $\underline{\tau}$  be the earlier of the liquidation time or the time when  $\delta_t$  reaches the level  $\underline{\delta}$ . Then Proposition 1 implies that the principal's profit at time  $\underline{\tau}$  is bounded from above by  $b(W_{\underline{\tau}}, \delta_{\underline{\tau}})$ . It follows that the principal's total expected profit is bounded from above by

$$E \left[ e^{-r\underline{\tau}} b(W_{\underline{\tau}}, \delta_{\underline{\tau}}) + \int_0^{\underline{\tau}} e^{-rt} (\delta_t dt - dC_t) \right] = E[G_{\underline{\tau}}] \leq G_0 = b(W_0, \delta_0).$$

QED

We conclude that Section 3 presents the optimal incentive-compatible contract for any pair  $(W_0, \delta_0)$  such that  $W_0 \geq R(\underline{\delta})$  and  $\delta_0 \geq \underline{\delta}$ . If  $W_0 \geq W^l(\delta_0)$ , then this contract attains the first-best profit, and liquidation always occurs at the efficient level of profitability of  $\delta_t = \underline{\delta}$ . If  $W_0 < W^l(\delta_0)$ , then liquidation happens inefficiently with positive probability.

## 5. Appendix A.

*Proof of Lemma 2A.* Note that



$$V_t = \int_0^t e^{-rs} dC_s + e^{-rt} W_t$$

is a martingale when the agent follows the recommended strategy  $(a_s)$ . By the Martingale Representation Theorem there exists a process  $\{\beta_s, t \geq 0\}$  in  $L^*$  such that

$$V_t = V_0 + \int_0^t e^{-rs} \beta_s (dX_s - \hat{\delta}_s ds),$$

since  $dX_s - \hat{\delta}_s ds = \sigma dZ_s$  under the strategy  $(a_s = 1)$ . Differentiating with respect to  $t$ , we find that

$$\begin{aligned} dV_t &= e^{-rt} dC_t + e^{-rt} dW_t - r e^{-rt} W_t dt = e^{-rt} \beta_t (dX_t - \hat{\delta}_t dt) \Rightarrow \\ dW_t &= r W_t dt - dC_t + \beta_t (dX_t - \hat{\delta}_t dt). \end{aligned}$$

This expression shows how  $W_t$  determined by  $X_t$  (since  $(\hat{\delta}_t)$  itself is determined by  $X_t$ ), and therefore it is valid even if the agent followed an alternative strategy in the past. In this case  $W_t$  is interpreted as the continuation value that the agent would have gotten after a history of cash flows  $\{X_s, s \leq t\}$  if his estimate of the firm's profitability coincided with the principal's, and he planned to follow strategy  $(a=1)$  after time  $t$ . QED.

*Proof of Lemma 2B.* Suppose that  $\beta_t < \lambda(1 + v/r)$  while  $\hat{\delta}_t > 0$  on a set of positive measure. Let us show that the agent has a strategy  $(\hat{a})$  that attains an expected payoff greater than  $W_0$ . Let  $\hat{a}_t = a_t = 1$  when  $\beta_t \geq \lambda(1 + v/r)$  and  $\hat{a}_t = 0$  when  $\beta_t < \lambda(1 + v/r)$  before the time  $\hat{t}$  when the agent is fired or  $\hat{\delta}_t$  reaches 0, whichever happens sooner. After  $\hat{\delta}_t$  reaches 0 but before the agent is fired, let him put effort 0. Define the process

$$V_t = e^{-rt} \left( W_t + (\delta_t - \hat{\delta}_t) \frac{\lambda}{r} \right) + \int_0^t e^{-rs} (dC_s + \lambda(1 - \hat{a}_s) \delta_s ds).$$

If the agent follows the strategy described above, then before time  $\hat{t}$ ,

$$d\delta_t - d\hat{\delta}_t = v(dX_t - \hat{a}_t \delta_t dt - (dX_t - \hat{\delta}_t dt)) = v(\hat{\delta}_t - \hat{a}_t \delta_t) \Rightarrow \delta_t \geq \hat{\delta}_t,$$

$$dW_t = rW_t dt - dC_t + \beta_t(\hat{a}_t \delta_t dt + \sigma dZ_t - \hat{\delta}_t dt),$$

and

$$\begin{aligned} \frac{dV_t}{e^{-rt}} &= -r \left( W_t + (\delta_t - \hat{\delta}_t) \frac{\lambda}{r} \right) dt + \\ &rW_t dt - dC_t + \beta_t(\hat{a}_t \delta_t dt + \sigma dZ_t - \hat{\delta}_t dt) + \lambda v(\hat{\delta}_t - \hat{a}_t \delta_t) + \\ &dC_t + \lambda(1 - \hat{a}_t) \delta_t dt = (\beta_t - \lambda - v \frac{\lambda}{r})(\hat{a}_t \delta_t - \hat{\delta}_t) dt + \sigma dZ_t. \end{aligned}$$

The drift of  $V_t$  is nonnegative, and it is positive when  $\beta_t < \lambda(1 + v/r)$  (so that  $\hat{a}_t = 0$ ).

At time  $\hat{\tau}$  the agent gets the payoff of  $W_{\hat{\tau}} + (\delta_{\hat{\tau}} - \hat{\delta}_{\hat{\tau}}) \frac{\lambda}{r}$ : if he gets fired first this number

is  $R(\delta_{\hat{\tau}}) = \underbrace{W_{\hat{\tau}}}_{R(\hat{\delta}_{\hat{\tau}})} + (\delta_{\hat{\tau}} - \hat{\delta}_{\hat{\tau}}) \frac{\lambda}{r}$ , and if  $\hat{\delta}_{\hat{\tau}}$  reaches 0 first, then by putting effort 0 thereafter,

he gets  $W_{\hat{\tau}} + \delta_{\hat{\tau}} \frac{\lambda}{r}$ . Therefore, the agent's total payoff from the strategy  $(\hat{a})$  is

$$E \left[ e^{-r\hat{\tau}} \left( W_{\hat{\tau}} + (\delta_{\hat{\tau}} - \hat{\delta}_{\hat{\tau}}) \frac{\lambda}{r} \right) + \int_0^{\hat{\tau}} e^{-rs} (dC_s + \lambda(1 - \hat{a}_s) \delta_s ds) \right] = E[V_{\hat{\tau}}] > V_0 = W_0.$$

We conclude that  $\beta_t \geq \lambda(1 + v/r)$  when  $\hat{\delta}_t > 0$  is a necessary condition for the incentive compatibility of the agent's strategy. ■

*Proof of Lemma 3.* Note that for  $\delta \geq \underline{\delta}$ , the function  $b$  satisfies partial differential equation (22) even if  $W > W^l(\delta)$ . Indeed, since  $b(W, \delta) = \bar{b}(\delta) - W$  and  $b_W = -1$  in that region, the equation reduces to

$$r(\bar{b}(\delta) - W) = \delta - rW + \frac{1}{2} v^2 \sigma^2 \bar{b}''(\delta).$$

This equation holds by the definition of  $\bar{b}$ .

When  $C_t$  is continuous at  $t$ , then using Ito's lemma,

$$\begin{aligned}
db(W_t, \delta_t) &= (rW_t dt - dC_t) b_w(W_t, \delta_t) + \sigma^2 \left( \frac{1}{2} \beta_t^2 b_{ww}(W_t, \delta_t) + \frac{1}{2} \nu^2 b_{\delta\delta}(W_t, \delta_t) + \beta_t \nu b_{w\delta}(W_t, \delta_t) \right) dt \\
&\quad + (\nu b_\delta(W_t, \delta_t) + \beta_t b_w(W_t, \delta_t)) \sigma dZ_t = rb(W_t, \delta_t) dt - \delta_t dt - b_w(W_t, \delta_t) dC_t + \\
&\quad \sigma^2 \left( \frac{1}{2} (\beta_t - \lambda(1 + \frac{\nu}{r}))^2 b_{ww}(W_t, \delta_t) dt + (\beta_t - \lambda(1 + \frac{\nu}{r})) (\lambda(1 + \frac{\nu}{r}) b_{ww}(W_t, \delta_t) + \nu b_{w\delta}(W_t, \delta_t)) \right) dt,
\end{aligned}$$

where the second equality follows from (22). From the definition of  $G_t$ , it follows that Lemma 3 correctly specifies how  $G_t$  evolves. QED

## 6. Appendix B.

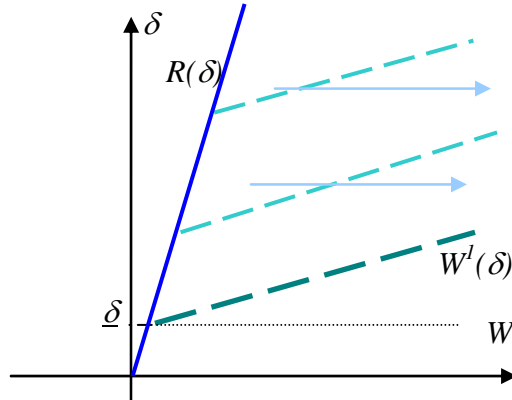
We must show that for all pairs  $(\delta, W)$ , the function  $b(\delta, W)$  satisfies

$$b_{ww}(W, \delta) \leq 0 \text{ and } \lambda(1 + \frac{\nu}{r}) b_{ww}(W, \delta) + \nu b_{w\delta}(W, \delta) \leq 0.$$

It is useful to understand the dynamics of the pair  $(\delta_t, W_t)$  under a conjectured optimal contract first. From (2) and (21), the pair  $(\delta_t, W_t)$  follows

$$\begin{aligned}
d\delta_t &= \nu \sigma dZ_t \text{ and } dW_t = rW_t dt + \lambda(1 + \nu/r) \sigma dZ_t \text{ until } W_t \text{ reaches } W^l(\delta_t), \\
&\text{and } W_t = W^l(\delta_t) \text{ thereafter.} \tag{24}
\end{aligned}$$

When  $W_t$  reaches the level  $R(\delta_t)$ , termination results. The lines parallel to  $W^l(\delta)$  are the paths of the joint volatilities of  $(W_t, \delta_t)$ . Due to the positive drift of  $W_t$ , the pair  $(W_t, \delta_t)$  moves across these lines in the direction of increasing  $W_t$ . See the figure below for reference.



The phase diagram of  $(W_t, \delta_t)$  provides two important directions: the direction of joint volatilities, in which  $dW/d\delta = \lambda(1 + \frac{V}{r})/\nu$ , and the direction of drifts, in which  $W$  increases but  $\delta$  stays the same. We need to prove that  $b_W(\delta, W)$  weakly decreases in both of these directions.

To study how  $b_W(W, \delta)$  depends on  $(W, \delta)$ , it is useful to know that  $b_W(W_b, \delta_t)$  is a martingale (Lemma 4) and that  $b_W(R(\delta), \delta)$  increases in  $\delta$  (Lemma 5).

**Lemma 4.** *When the evolution of  $(W_b, \delta_t)$  is given by (24), then  $b_W(W_b, \delta_t)$  is a martingale.*

*Proof.* Differentiating the partial differential equation for  $b(W, \delta)$  with respect to  $W$ , we obtain

$$0 = rWb_{WW}(W, \delta) + \frac{1}{2}\lambda^2\left(1 + \frac{V}{r}\right)^2\sigma^2b_{WWW}(W, \delta) + \frac{1}{2}\nu^2\sigma^2b_{W\delta\delta}(W, \delta) + \lambda\left(1 + \frac{V}{r}\right)\nu\sigma^2b_{WW\delta}(W, \delta).$$

The right hand side of this equation is the drift of  $b_W(W, \delta)$  when  $W_t < W^l(\delta_t)$  by Ito's lemma. When  $W_t = W^l(\delta_t)$ , then  $b_W(W_b, \delta_t) = -1$  at all times. Therefore,  $b_W(W_b, \delta_t)$  is always a martingale. QED

**Lemma 5.**  *$b_W(R(\delta), \delta)$  weakly increases in  $\delta$ .*

*Proof.* Note that

$$b(W_0, \delta_0) = \bar{b}(\delta_0) - W_0 - E\left[e^{-r\tau}(\bar{b}(\delta_\tau) - L(\delta_\tau) - R(\delta_\tau)) \mid \delta_0, W_0\right]. \quad (25)$$

That is, the principal's profit equals first-best minus the loss of payoff due to early inefficient liquidation. Let us show that for all  $\varepsilon > 0$ ,  $b(R(\delta_0) + \varepsilon, \delta_0) - b(R(\delta_0), \delta_0)$  increases in  $\delta_0$ . Consider the processes  $(W_t^i, \delta_t^i)$  ( $i = 1, 2$ ) that follow (24) starting from values  $\delta_0^1$  and  $\delta_0^2 = \delta_0^1 + \Delta$  and  $W_0^i = R(\delta_0^i) + \varepsilon$ . Then for any path of  $Z$ , the process for  $i = 1$  ends up in liquidation at a sooner time  $\tau_1$  and at a higher value of  $\delta_{\tau_1}^1$ . Indeed, from the law of

motion (24), it is easy to see that while the difference  $\delta_t^2 - \delta_t^1$  stays constant at all times,  $W_t^2 - W_t^1$  becomes larger than  $\lambda/r\Delta$  after time 0, where  $\lambda/r$  is the slope of  $R(\delta)$ . Since

$$\bar{b}(\delta) - L(\delta) - R(\delta)$$

increases in  $\Delta$ , it follows that

$$E\left[e^{-r\tau_1}(\bar{b}(\delta_{\tau_1}^1) - L(\delta_{\tau_1}^1) - R(\delta_{\tau_1}^1)) \mid \delta_0^1, W_0^1\right] \geq E\left[e^{-r\tau_2}(\bar{b}(\delta_{\tau_2}^2) - L(\delta_{\tau_2}^2) - R(\delta_{\tau_2}^2)) \mid \delta_0^2, W_0^2\right].$$

As a result,

$$\begin{aligned} & b(R(\delta_0^1) + \varepsilon, \delta_0^1) - b(R(\delta_0^1), \delta_0^1) = \\ & -E\left[e^{-r\tau_1}(\bar{b}(\delta_{\tau_1}^1) - L(\delta_{\tau_1}^1) - R(\delta_{\tau_1}^1)) \mid \delta_0^1, W_0^1\right] + \bar{b}(\delta_0^1) - L(\delta_0^1) - R(\delta_0^1) \leq \\ & -E\left[e^{-r\tau_2}(\bar{b}(\delta_{\tau_2}^2) - L(\delta_{\tau_2}^2) - R(\delta_{\tau_2}^2)) \mid \delta_0^2, W_0^2\right] + \bar{b}(\delta_0^2) - L(\delta_0^2) - R(\delta_0^2) = \\ & b(R(\delta_0^2) + \varepsilon, \delta_0^2) - b(R(\delta_0^2), \delta_0^2), \end{aligned}$$

where we used (25) to derive the first and the last inequality. QED

We can use Lemmas 4 and 5 to reach conclusions about how  $b_W(W, \delta)$  changes as  $W$  increases or as  $\delta$  and  $W$  increase in the direction  $dW/d\delta = \lambda(1 + \frac{\nu}{r})/\nu$ .

**Lemma 6.**  $b_W(W, \delta)$  weakly decreases in  $W$ .

*Proof.* Let us show that for any  $\delta_0 \geq \underline{\delta}$ , for any two values  $W_0^1 < W_0^2$ ,  $b_W(W_t^1, \delta_0) \geq b_W(W_t^2, \delta_0)$ .

Consider the processes  $(W_t^i, \delta_t^i)$  ( $i = 1, 2$ ) that follow (24) starting from values  $(W_0^i, \delta_0)$  and  $(W_0^2, \delta_0)$  for  $\delta_0^1 < \delta_0^2$ . Then for any path of  $Z$ , we have  $W_t^2 - W_t^1 \geq 0$  until time  $\tau_1$  when  $W_t^1$  reaches the level of  $R(\delta_t)$ . The time when  $W_t^2$  reaches the level of  $R(\delta_t)$  is  $\tau_2 \geq \tau_1$ . Since  $W_{\tau_1}^2 \geq W_{\tau_1}^1 = R(\delta_{\tau_1})$ , it follows that  $\delta_{\tau_2} \leq \delta_{\tau_1}$  and  $W_{\tau_2}^2 \leq W_{\tau_1}^1$ . Using Lemmas 4 and 5,

$$b_W(W_0^1, \delta_0) = E\left[b_W(R(\delta_{\tau_1}), \delta_{\tau_1})\right] \geq E\left[b_W(R(\delta_{\tau_2}), \delta_{\tau_2})\right] = b_W(W_0^2, \delta_0).$$

QED

**Lemma 7.**  $b_W(W, \delta)$  weakly decreases in the direction, in which  $W$  and  $\delta$  increase according to  $dW/d\delta = \lambda(1 + \frac{\nu}{r})/\nu$ .

*Proof.* Consider starting values  $(W_0^i, \delta_0^i)$  that satisfy

$$\delta_0^2 - \delta_0^1 = \Delta > 0 \text{ and } W_0^2 - W_0^1 = \Delta\lambda(1 + \frac{\nu}{r})/\nu.$$

Starting from those values, let the processes  $(W_t^i, \delta_t^i)$  ( $i = 1, 2$ ) follow (\*). Then for any path of  $Z$ , at all times  $\delta_t^2 - \delta_t^1 = \Delta$  and  $W_t^2 - W_t^1 \geq \Delta\lambda(1 + \frac{\nu}{r})/\nu$  (with equality after time 0 only if  $W_t^2 = W^l(\delta_t^2)$  and  $W_t^1 = W^l(\delta_t^1)$ ). Therefore, the time  $\tau_1$  when  $W_t^1$  reaches the level of  $R(\delta_t^1)$  occurs at least as soon as the time  $\tau_2$  when  $W_t^2$  reaches the level of  $R(\delta_t^2)$ . Also, since  $W_{\tau_1}^2 \geq W_{\tau_1}^1 + \Delta\lambda(1 + \frac{\nu}{r})/\nu > R(\delta_{\tau_1}^1) + \Delta\lambda/r$ , it follows that  $\delta_{\tau_2} \leq \delta_{\tau_1}$  and  $W_{\tau_2}^2 \leq W_{\tau_1}^1$ . Using Lemmas 4 and 5,

$$b_w(W_0^1, \delta_0^1) = E[b_w(R(\delta_{\tau_1}^1), \delta_{\tau_1}^1)] \geq E[b_w(R(\delta_{\tau_2}^2), \delta_{\tau_2}^2)] = b_w(W_0^2, \delta_0^2).$$

QED

## 7. References.

- Abreu, D., Pearce, D., and Stacchetti, E. (1990) "Toward a Theory of Discounted Repeated Games with Imperfect Monitoring." *Econometrica* Vol. 58, pp. 1041-1063
- Albuquerque, R. and Hopenhayn, H. A. (2004) "Optimal Lending Contracts and Firm Dynamics," *Review of Economic Studies*, Vol. 72(2), No. 247: 285-315.
- Allen, F. and R. Michaely, 2003, Payout Policy, in G. Constantinides, M. Harris, and R. Stulz, eds., *Handbooks of Economics*, North-Holland.
- Atkeson, Andrew (1991) "International Lending with Moral Hazard and the Risk of Repudiation," *Econometrica* 59, 1069-1089.
- Atkeson, Andrew, and Harold Cole (2005) "A Dynamic Theory of Optimal Capital Structure and Executive Compensation," NBER working paper No. 11083.
- Benartzi, Shlomo, Roni Michaely, and Richard Thaler, 1997, Do changes in dividends signal the future or the past? *Journal of Finance* 52: 1007-1043
- Benartzi, Shlomo, Gustavo Grullon, Roni Michaely, and Richard Thaler, 2005, Dividend changes do not signal changes in future profitability, *Journal of Business* 78: 1659-1682

- Biais, B., T. Mariotti, G. Plantin, and J.-C. Rochet (2007): “Dynamic Security Design: Convergence to Continuous Time and Asset Pricing Implications,” *Review of Economic Studies*, vol. 74(2), pages 345-390
- Biais, B., T. Mariotti, J.-C. Rochet and S. Villeneuve (2007) “Environmental Risk Insurance under Dynamic Moral Hazard,” working paper, Université de Toulouse
- Brav, A., Graham, J., Harvey, C. and R. Michaely, 2005, Payout Policy in the 21st Century, *Journal of Financial Economics*
- Clementi, G. L. and Hopenhayn, H. A. (2006) “A Theory of Financing Contracts and Firm Dynamics,” *Quarterly Journal of Economics*, 121, 229-265.
- DeMarzo, Peter, and Michael Fishman (2003) “Optimal long-term financial contracting,” forthcoming in *Review of Financial Studies*.
- DeMarzo, Peter, and Michael Fishman, (2007) “Agency and optimal investment dynamics,” *Review of Financial Studies*, 20:151-188.
- DeMarzo, P. and Y. Sannikov (2006) “Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model,” *Journal of Finance* 61: 2681-2724.
- Dixit, A. K., and R.S. Pindyck (1994), *Investment under Uncertainty*. Princeton University Press.
- Fama, Eugene and Harvey Babiak, 1968, Dividend policy: an empirical analysis, *Journal of the American Statistical Association* 63: 1132-1161
- Grullon, Gustavo and Roni Michaely, 2002, Dividends, share repurchases and the substitution hypothesis, *Journal of Finance*, 57 , 1649-84.
- Grullon, Gustavo, Roni Michaely and Bhaskaran Swaminathan, 2002, Are dividend changes a sign of firm maturity? *The Journal of Business*, 75, 387-424.
- Healy, Paul M. and Krishna G. Palepu, 1988, Earnings information conveyed by dividend initiations and omissions, *Journal of Financial Economics* 21: 149-176
- Lintner, John., 1956, Distribution of Incomes of Corporations Among Dividends, Retained Earnings, and Taxes, *American Economic Review*, 46(2), 97-113.
- Miller, Merton H. and Franco Modigliani (1961), “Dividend Policy, Growth, and the Valuation of Shares,” *The Journal of Business* 34(4), 411-433.
- Philippon, T. and Y. Sannikov (2007) “Financial Development and the Timing of IPOs,” working paper, New York University.

Piskorski, T., and A. Tchisty (2006): "Optimal Mortgage Design," working paper, New York University.

Praveen Kumar, Bong-Soo Lee: Discrete dividend policy with permanent earnings  
Financial Management, Autumn, 2001

Roberts, Michael R. and Michaely, Roni, "Corporate Dividend Policies: Lessons from Private Firms" (February 22, 2007). Available at SSRN:  
<http://ssrn.com/abstract=927802>

Sannikov, Yuliy (2007a) "A Continuous-Time Version of the Principal-Agent Problem," forthcoming in the *Review of Economic Studies*.

Sannikov, Yuliy (2007b) "Agency Problems, Screening and Increasing Credit Lines," working paper, University of California at Berkeley.

Spear, S. and Srivastava, S. (1987) "On Repeated Moral Hazard with Discounting," *Review of Economic Studies*, Vol. 54, 599-617

Williams, Noah (2007) "On Dynamic Principal-Agent Problems in Continuous Time," working paper, Princeton University.