A Refinement of the Myerson Value*

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Abstract

The Myerson value is an allocation rule which assigns a vector of payoffs to a collection of conferences. It treats direct and indirect connections of players in conferences equally, and thus assigns the same vector of payoffs to a large class of collections of conferences. This paper proposes and axiomatizes an allocation rule which distinguishes direct connections from indirect ones, and hence provides a refinement of the Myerson value.

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1 Introduction

Myerson (1977, 1980) made a seminal contribution to describe how the outcome of a cooperative game might depend on which groups of players hold cooperative planning conferences. A conference is defined as a set of two or more players and a collection of conferences is called a conference structure.[1] Myerson (1977, 1980) augmented a cooperative game by a conference structure and defined another cooperative game where the conference structure determines which coalitions are feasible. The feasible coalition is the one in which any pair of players are either *directly* or *indirectly* connected (i.e. path connected) by the conferences contained in the coalition. Myerson (1977, 1980) showed that the Shapley value of the induced cooperative game can be characterized by two axioms: fairness and component efficiency. This allocation rule is referred to as the Myerson value in the subsequent literature.[2]

We emphasize that the Myerson value treats direct and indirect connections equally. For example, consider a conference structure $\{\{1,2\},\{2,3\}\}$. Player 1 and player 2 are directly connected in the sense that they have a chance of direct communication in a conference $\{1,2\}$, and so are player 2 and player 3. On the other hand, player 1 and player 3 are not directly connected but indirectly connected in the sense that they have a chance of indirect communication via an intermediary, player 2. Now suppose that a conference $\{1,2,3\}$ is added to the conference structure, by which player 1 and player 3 are directly connected. But in the construction of the Myerson value, payoff allocations are the same in the two cases.

This paper proposes a refinement of the Myerson value which distinguishes direct and indirect connections. Similar to Myerson (1977, 1980), we augment a cooperative game by a conference structure and define another cooperative game where the conference structure determines which coalitions are feasible. But different from Myerson (1977, 1980), the feasible coalition is the one in which any pair of players are *directly* connected by the conferences contained in the coalition. In the main result, we show that the Shapley value of the induced cooperative game can be characterized by three axioms: fairness, complete component efficiency, and no contribution by unconnected players. The latter two new axioms describe the behavior of the allocation rule distinguishing direct and indirect connections. To establish the main result, we take advantage of the idea of potentials for cooperative games originated by Hart and Mas-Colell (1989). We prove that if an allocation rule satisfies the three axioms, then it is represented in terms of the marginal contributions of the potential for the induced cooperative game, which leads us to the main result. Also in the main result, we provide a characterization of the potential for the induced cooperative game, which extends the result of Hart and Mas-Colell (1989).

[1] Myerson (1977) considered special conferences with exactly two players and regarded a conference structure as a network, while Myerson (1980) considered general conferences and nontransferable utility.

[2] The study of allocation rules with partial cooperation possibilities is well-documented since Aumann and Drèze (1974). For other allocation rules, see Meesen (1988), Borm et al. (1992), Hamiache (1999), Bilbao and López (2006), and the review by Slikker and van den Nouweland (2001), among others.

The organization of the paper is as follows. Preliminary definitions and results are summarized in section 2. Conference structures and allocation rules are introduced in section 3. The main result is stated in section 4, which is proved in section 5. In section 6, we compare our result and that of Myerson (1977, 1980) and show that our allocation rule is in fact a refinement of the Myerson value. In the same section, we point out some connection of our result to the network games of Jackson and Wolinsky (1996).

2 Preliminaries

Let $N = \{1, \ldots, n\}$ be a set of players. A subset $S \in 2^N$ is referred to as a coalition. A game v is a function from 2^N to $\mathbb{R}$ with $v(\emptyset) = 0$. The unanimity game on $T \in 2^N$ is denoted by u_T and defined as

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

A collection of coalitions $\mathcal{P} \subseteq 2^N$ is partially ordered with the set inclusion relation. Regard $[u_X(Y)]_{X,Y\in\mathcal{P}}$ as a $|\mathcal{P}| \times |\mathcal{P}|$ matrix and observe that it is non-singular and thus invertible. The *Möbius function* of $\mathcal{P}$ is defined as a function $\mu_\mathcal{P} : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ such that the matrix $[\mu_\mathcal{P}(X, Y)]_{X,Y\in\mathcal{P}}$ is the inverse matrix of $[u_X(Y)]_{X,Y\in\mathcal{P}}$;[3] that is, for $X, Y \in \mathcal{P}$, it holds that

$$\sum_{T\in\mathcal{P}} \mu_\mathcal{P}(X, T)u_T(Y) = \sum_{T\in\mathcal{P}} u_X(T)\mu_\mathcal{P}(T, Y) = \begin{cases} 1 & \text{if } X = Y, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

It is known that the Möbius function $\mu_\mathcal{P}$ is determined inductively by the following rule:[4]

$$\mu_\mathcal{P}(X, Y) = \begin{cases} 1 & \text{if } X = Y, \\ 0 & \text{if } X \not\subseteq Y, \\ -\displaystyle\sum_{T\in\mathcal{P}: X\subseteq T\subset Y} \mu_\mathcal{P}(X, T) & \text{if } X \subset Y. \end{cases} \tag{2}$$

For the special case of $\mathcal{P} = 2^N$, it holds that

$$\mu_\mathcal{P}(X, Y) = \begin{cases} (-1)^{|X|-|Y|} & \text{if } X \subseteq Y, \\ 0 & \text{otherwise.} \end{cases}$$

The following result is referred to as *the principle of Möbius inversion.*

[3] A function $\zeta_\mathcal{P} : \mathcal{P}\times\mathcal{P} \to \mathbb{R}$ such that $\zeta_\mathcal{P}(X, Y) = u_X(Y)$ for all $X, Y \in \mathcal{P}$ is called the *zeta function* of $\mathcal{P}$. The zeta function and the Möbius function are defined on any partially ordered set. See a textbook on combinatorics such as Lint and Wilson (1992).

[4] When $X \subset Y$ and $|Y| - |X| = 1$, this formula requires that $\mu_\mathcal{P}(X, Y) = -\mu_\mathcal{P}(X, X) = -1$, and once $\mu_\mathcal{P}(X, Y)$ is determined for $X \subset Y$ with $|Y| - |X| \le k$, then the formula determines $\mu_\mathcal{P}(X, Y)$ for $X \subset Y$ with $|Y| - |X| = k + 1$, and so on.

Lemma 1 *For any function $v : \mathcal{P} \to \mathbb{R}$, if $f : \mathcal{P} \to \mathbb{R}$ is given by*

$$f(X) = \sum_{T \in \mathcal{P}} v(T)\mu_{\mathcal{P}}(T, X) \textit{ for all } X \in \mathcal{P}, \tag{3}$$

then it holds that

$$v(X) = \sum_{T \in \mathcal{P}} f(T)u_T(X) \textit{ for all } X \in \mathcal{P}. \tag{4}$$

Conversely, for any function $f : \mathcal{P} \to \mathbb{R}$, if $v : \mathcal{P} \to \mathbb{R}$ is given by (4)*, then* (3) *holds.*

The principle of Möbius inversion can be easily checked because (1) and (3) imply that

$$\begin{aligned}\sum_{T \in \mathcal{P}} f(T)u_T(X) &= \sum_{T \in \mathcal{P}} \left(\sum_{T' \in \mathcal{P}} v(T')\mu_{\mathcal{P}}(T', T) \right) u_T(X) \\ &= \sum_{T' \in \mathcal{P}} v(T') \left(\sum_{T \in \mathcal{P}} \mu_{\mathcal{P}}(T', T)u_T(X) \right) = v(X) \text{ for all } X \in \mathcal{P},\end{aligned}$$

and (1) and (4) imply that

$$\begin{aligned}\sum_{T \in \mathcal{P}} v(T)\mu_{\mathcal{P}}(T, X) &= \sum_{T \in \mathcal{P}} \left(\sum_{T' \in \mathcal{P}} f(T')u_{T'}(T) \right) \mu_{\mathcal{P}}(T, X) \\ &= \sum_{T' \in \mathcal{P}} f(T') \left(\sum_{T \in \mathcal{P}} u_{T'}(T)\mu_{\mathcal{P}}(T, X) \right) = f(X) \text{ for all } X \in \mathcal{P}.\end{aligned}$$

The principle of Möbius inversion for the special case of $\mathcal{P} = 2^N$ leads us to the well known fact that any game v is uniquely represented as a linear combination of unanimity games (Shapley, 1953):

$$v = \sum_{T \in 2^N} \beta_T u_T \text{ where } \beta_T = \sum_{T \in 2^N : S \subseteq T} (-1)^{|T|-|S|} v(S).$$

Denote by $\delta_i v(S)$ the marginal contribution of player $i \in S$ to $v(S)$; that is,

$$\delta_i v(S) = v(S) - v(S \backslash \{i\}).$$

The Shapley value of v is the vector of payoffs $\phi(v) \in \mathbb{R}^N$ given by the following formula (Shapley, 1953):

$$\phi_i(v) = \sum_{S \in 2^N : i \in S} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} \delta_i v(S) \text{ for all } i \in N.$$

In particular, the Shapley value of u_T is given by

$$\phi_i(u_T) = \begin{cases} 1/|T| & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

Since the Shapley value is linear in games, we have an alternative formula for the Shapley value of $v = \sum_{T \in 2^N} \beta_T u_T$ as follows:

$$\phi_i(v) = \sum_{T \in 2^N} \beta_T \phi_i(u_T) = \sum_{T \in 2^N : i \in T} \beta_T / |T|. \tag{5}$$

A potential for a game v is a game p such that

$$\sum_{i \in S} \delta_i p(S) = v(S) \tag{6}$$

for all $S \in 2^N$. Hart and Mas-Colell (1989) showed the following result.[5]

Proposition 1 *There exists a unique potential p for $v = \sum_{T \in 2^N} \beta_T u_T$, which is given by*

$$p = \sum_{T \in 2^N} \frac{\beta_T}{|T|} u_T.$$

Moreover, the vector of the marginal contributions $(\delta_i p(N))_{i \in N}$ coincides with the Shapley value of v; that is,

$$\delta_i p(N) = \phi_i(v) \text{ for all } i \in N.$$

3 Conference structures and allocation rules

To describe how players organize their cooperation, we specify which groups of players are willing and able to confer together for the purpose of planning cooperative actions. Myerson (1980) have used the term a *conference* to refer to any set of two or more players who might meet together to discuss their cooperative plans. So, we define a conference as a coalition with two or more players. A conference structure is then any collection of conferences. The collection of all possible conference structures is denoted by

$$\mathbf{CS} = \{\mathcal{H} \subseteq 2^N \,|\, |H| \geq 2 \text{ for all } H \in \mathcal{H}\}.$$

We write $\mathcal{H}_S = \{H \in \mathcal{H} \,|\, H \subseteq S\}$ and $\mathcal{H}_{-i} = \mathcal{H}_{N \setminus \{i\}}$ for $\mathcal{H} \in \mathbf{CS}$, $S \in 2^N$, and $i \in N$.

We consider two types of connections between players, direct and indirect ones.

Definition 1 Players $i, j \in N$ are said to be *directly $\mathcal{H}$-connected* in a coalition S if $i = j$ or there exists a conference $H \in \mathcal{H}_S$ with $\{i, j\} \subseteq H$. Players $i, j \in N$ are said to be *$\mathcal{H}$-connected* in a coalition S if there exist a sequence of players $i_1, ..., i_m$ with $i = i_1$ and $j = i_m$ such that i_k and i_{k+1} are directly $\mathcal{H}$-connected in S for $k = 1, \ldots, m-1$.

[5] Originally, Hart and Mas-Colell (1989) defined a potential as a real-valued function over the space of games. The value assigned by the potential to the restriction of a game v to a coalition S corresponds to $p(S)$ in this paper.

Thus, two players are directly $\mathcal{H}$-connected in S if they can be coordinated by direct communication; and two players are $\mathcal{H}$-connected in S if they can be coordinated either by direct communication or by indirect communication via intermediaries.[6] By definition, two players are (directly) $\mathcal{H}$-connected in S if and only if they are (directly) $\mathcal{H}_S$-connected in S. Also by definition, if two players are (directly) $\mathcal{H}$-connected in S then they are (directly) $\mathcal{H}$-connected in T with $S \subseteq T$.

The above notions of connectedness for players induce the corresponding notions for coalitions.[7]

Definition 2 A coalition $S \in 2^N$ is said to be $\mathcal{H}$-*complete* if any pair of players in S are directly $\mathcal{H}$-connected in S. A coalition $S \in 2^N$ is said to be $\mathcal{H}$-*connected* if any pair of players in S are $\mathcal{H}$-connected in S.

By definition, any singleton is $\mathcal{H}$-complete and $\mathcal{H}$-connected. Note that S is $\mathcal{H}$-complete if and only if it is $\mathcal{H}_S$-complete, and similarly, S is $\mathcal{H}$-connected if and only if it is $\mathcal{H}_S$-connected.

Let $\mathrm{cm}(\mathcal{H}) \in \mathbf{CS}$ denote the collection of all $\mathcal{H}$-complete conferences, and let $\mathrm{cn}(\mathcal{H}) \in \mathbf{CS}$ denote the collection of all $\mathcal{H}$-connected conferences (so, singletons are excluded). Both $\mathrm{cm}(\cdot)$ and $\mathrm{cn}(\cdot)$ are monotonic as operators on $\mathbf{CS}$ in the sense that $\mathrm{cm}(\mathcal{H}) \subseteq \mathrm{cm}(\mathcal{H}')$ and $\mathrm{cn}(\mathcal{H}) \subseteq \mathrm{cn}(\mathcal{H}')$ if $\mathcal{H} \subseteq \mathcal{H}'$. It follows that

$$\mathcal{H} \subseteq \mathrm{cm}(\mathcal{H}) \subseteq \mathrm{cn}(\mathcal{H}), \tag{7}$$

since any pair of players in $S \in \mathcal{H}$ are directly $\mathcal{H}$-connected in S and any pair of players in $S \in \mathrm{cm}(\mathcal{H})$ are (directly) $\mathcal{H}$-connected in S. Furthermore, we can show the properties below.[8]

Lemma 2 *Players $i, j \in N$ are directly $\mathcal{H}$-connected in a coalition S if and only if they are directly $\mathrm{cm}(\mathcal{H})$-connected in S. Thus, it holds that*

$$\mathrm{cm}(\mathcal{H}) = \mathrm{cm}(\mathrm{cm}(\mathcal{H})).$$

Proof. If $i = j$ then the above claim holds trivially. Suppose that $i, j \in N$ with $i \neq j$ are directly $\mathcal{H}$-connected in S. Then, there exists $H \in \mathcal{H}_S$ with $\{i, j\} \subseteq H$. Since $H \in \mathrm{cm}(\mathcal{H})_S$, they are also directly $\mathrm{cm}(\mathcal{H})$-connected in S. Conversely, suppose that $i, j \in N$ with $i \neq j$ are directly $\mathrm{cm}(\mathcal{H})$-connected in S. Then, there exists $H \in \mathrm{cm}(\mathcal{H})_S$ with $\{i, j\} \subseteq H$. Since H is $\mathcal{H}$-complete, there exists $T \in \mathcal{H}_H \subseteq \mathcal{H}_S$ with $\{i, j\} \subseteq T$. This implies that i and j are directly $\mathcal{H}$-connected in S. The equivalence of the direct $\mathcal{H}$-connected relation and the direct $\mathrm{cm}(\mathcal{H})$-connected relation implies the equivalence of $S \in \mathrm{cm}(\mathcal{H})$ and $S \in \mathrm{cm}(\mathrm{cm}(\mathcal{H}))$. ∎

[6] In Myerson (1980), players $i, j \in N$ are said to be $\mathcal{H}$-connected in S if $i = j$ or there exists a sequence of conferences $H_1, \ldots, H_m \in \mathcal{H}_S$ such that $i \in H_1$, $j \in H_m$, and $H_k \cap H_{k+1} \neq \emptyset$ for $k = 1, \ldots, m-1$, which is equivalent to the above definition.

[7] The notion of $\mathcal{H}$-completeness is introduced by Kajii et al. (2005) for events, i.e., subsets of the set of states, and used for a characterization of the Choquet integral. The term "complete" is adopted from an analogy to complete graphs. For $S \in 2^N$, consider an undirected graph with a vertex set S such that $\{i, j\} \subseteq S$ is an edge if there is $H \in \mathcal{H}_S$ satisfying $\{i, j\} \subseteq H$. This is a complete graph if and only if S is $\mathcal{H}$-complete.

[8] Kajii et al. (2005) obtained results similar to Lemma 2.

Lemma 3 *Players $i, j \in N$ are $\mathcal{H}$-connected in a coalition S if and only if they are* $\mathrm{cn}(\mathcal{H})$*-connected in S. Thus, it holds that*

$$\mathrm{cn}(\mathcal{H}) = \mathrm{cm}(\mathrm{cn}(\mathcal{H})) = \mathrm{cn}(\mathrm{cn}(\mathcal{H})).$$

Proof. Suppose that $i, j \in N$ are $\mathcal{H}$-connected in S. Then, there exist a sequence of players $i_1, ..., i_m$ with $i = i_1$ and $j = i_m$ such that i_k and i_{k+1} are directly $\mathcal{H}$-connected in S for $k = 1, \ldots, m-1$. Since $\mathcal{H} \subseteq \mathrm{cn}(\mathcal{H})$, i_k and i_{k+1} are directly $\mathrm{cn}(\mathcal{H})$-connected in S for each k. This implies that i and j are $\mathrm{cn}(\mathcal{H})$-connected in S. Conversely, suppose that $i, j \in N$ are $\mathrm{cn}(\mathcal{H})$-connected in S. Then, there exist a sequence of players $i_1, ..., i_m$ with $i = i_1$ and $j = i_m$ such that i_k and i_{k+1} are directly $\mathrm{cn}(\mathcal{H})$-connected in S for $k = 1, \ldots, m-1$. Thus, there exists $S_k \in \mathrm{cn}(\mathcal{H})_S$ with $\{i_k, i_{k+1}\} \subseteq S_k$, which implies that i_k and i_{k+1} are $\mathcal{H}$-connected in S for each k. Since the $\mathcal{H}$-connected relation is transitive, i and j must be $\mathcal{H}$-connected in S. The equivalence of the $\mathcal{H}$-connected relation and the $\mathrm{cn}(\mathcal{H})$-connected relation implies the equivalence of $S \in \mathrm{cn}(\mathcal{H})$ and $S \in \mathrm{cn}(\mathrm{cn}(\mathcal{H}))$, establishing $\mathrm{cn}(\mathcal{H}) = \mathrm{cn}(\mathrm{cn}(\mathcal{H}))$. Since $\mathrm{cn}(\mathcal{H}) \subseteq \mathrm{cm}(\mathrm{cn}(\mathcal{H})) \subseteq \mathrm{cn}(\mathrm{cn}(\mathcal{H}))$ by (7), $\mathrm{cn}(\mathcal{H}) = \mathrm{cm}(\mathrm{cn}(\mathcal{H})) = \mathrm{cn}(\mathrm{cn}(\mathcal{H}))$ must follow. ∎

Note that the $\mathcal{H}$-connected relation in S is an equivalence relation, although the direct $\mathcal{H}$-connected relation in S might not be. For $S \in 2^N$ and $\mathcal{H} \in \mathbf{CS}$, let $S/\mathcal{H}$ denote the partition of S consisting of the equivalence classes induced by the $\mathcal{H}$-connected relation in S; that is,

$$S/\mathcal{H} = \{\{j \in S \,|\, i \text{ and } j \text{ are } \mathcal{H}\text{-connected in } S\} \,|\, i \in N\}.$$

It follows that $S/\,\mathrm{cn}(\mathcal{H}) = S/\mathcal{H} = S/\mathcal{H}_S$ by the equivalence of the $\mathrm{cn}(\mathcal{H})$-connected, $\mathcal{H}$-connected, and $\mathcal{H}_S$-connected relations in S. We call an element of $S/\mathcal{H}$ a component of S. A component of S is a maximal $\mathcal{H}$-connected coalition in S because any pair of players in a $\mathcal{H}$-connected coalition are $\mathcal{H}$-connected in the component to which they both belong.

An allocation rule assigns a vector of payoffs to each conference structure; that is, an allocation rule is a mapping $f : \mathbf{CS} \to \mathbb{R}^N$ where player i's payoff is $f_i(\mathcal{H})$ for $\mathcal{H} \in \mathbf{CS}$. Myerson (1977, 1980) considered the following axioms for an allocation rule f.

Component efficiency (CE)

$$\sum_{i \in S} f_i(\mathcal{H}) = v(S) \text{ if } S \in N/\mathcal{H}.$$

Fairness (F)

$$f_i(\mathcal{H}) - f_i(\mathcal{H}\backslash\{H\}) = f_j(\mathcal{H}) - f_j(\mathcal{H}\backslash\{H\}) \text{ if } i, j \in H \in \mathcal{H}.$$

Balanced contribution (BC)

$$f_i(\mathcal{H}) - f_i(\mathcal{H}_{-j}) = f_j(\mathcal{H}) - f_j(\mathcal{H}_{-i}) \text{ for all } i, j \in N.$$

Component efficiency (CE) says that if S is a component of N, i.e., a maximal $\mathcal{H}$-connected coalition, then the members of S ought to allocate to themselves the total wealth $v(S)$ available to them. Fairness (F) says that all players in a conference gain equally from their agreement to form the conference. Balanced contribution (BC) says that player j's contribution to i always equals i's contribution to j. The next result (Myerson, 1980) shows that BC implies F.

Lemma 4 *If an allocation rule satisfies BC then it satisfies F.*

To characterize an allocation rule satisfying the axioms above, Myerson (1977, 1980) considered a game $r^{\mathcal{H}}$ determined by the collection of $\mathcal{H}$-connected coalitions, which is defined as follows:

$$r^{\mathcal{H}}(S) = \sum_{T \in S/\mathcal{H}} v(T) \text{ for all } S \in 2^N. \tag{8}$$

The game $r^{\mathcal{H}}$ is called the *restricted game* of v. The following result, originally due to Myerson (1977, 1980) and later elaborated by van den Nouweland et al. (1992), is fundamental.

Proposition 2 *The following three statements about an allocation rule f^M are equivalent.*

(i) *f^M satisfies CE and F.*

(ii) *f^M satisfies CE and BC.*

(iii) *$f^M(\mathcal{H})$ is the Shapley value of the restricted game $r^{\mathcal{H}}$. That is, $f^M_i(\mathcal{H}) = \phi_i(r^{\mathcal{H}})$ for all $i \in N$ and $\mathcal{H} \in$* **CS**.

Since the restricted game $r^{\mathcal{H}}$ is uniquely determined from v and $\mathcal{H}$ by (8), each statement in Proposition 2 identifies a unique allocation rule. Especially, this proposition shows that there exists a unique allocation rule satisfying CE and F. This allocation rule is referred to as the Myerson value.

Note that $r^{\mathcal{H}} = r^{\mathrm{cn}(\mathcal{H})}$ because $S/\mathcal{H} = S/\mathrm{cn}(\mathcal{H})$. This means that the Myerson value assigns the same vector of payoffs to different conference structures as far as the collections of $\mathcal{H}$-connected conferences are the same, even if those of $\mathcal{H}$-complete conferences are distinct. In this sense, the Myerson value treats direct and indirect connections equally. For example, let $N = \{1, 2, 3, 4\}$ and

$$\begin{aligned} \mathcal{H}^1 &= \{\{1,2\},\{2,3\},\{3,4\},\{1,2,3,4\}\}, \\ \mathcal{H}^2 &= \{\{1,2\},\{2,3\},\{3,4\},\{1,2,3\},\{1,2,3,4\}\}, \\ \mathcal{H}^3 &= \{\{1,2\},\{2,3\},\{3,4\},\{1,2,3\},\{2,3,4\},\{1,2,3,4\}\}. \end{aligned} \tag{9}$$

Since $\mathrm{cn}(\mathcal{H}^1) = \mathrm{cn}(\mathcal{H}^2) = \mathrm{cn}(\mathcal{H}^3) = \mathcal{H}^3$, the payoff allocations by the Myerson value are identical for all the above conference structures. On the other hand, we have $\mathcal{H}^1 = \mathrm{cm}(\mathcal{H}^1)$, $\mathcal{H}^2 = \mathrm{cm}(\mathcal{H}^2)$, and $\mathcal{H}^3 = \mathrm{cm}(\mathcal{H}^3)$. In the next section, we propose an allocation rule which distinguishes conference structures with distinct collections of $\mathcal{H}$-complete conferences.

4 The main result

Our motivation is similar to Myerson's but we are interested in an allocation rule based upon direct connections. We formalize this idea in terms of the following new axioms and replace CE with them.

Complete component efficiency (CCE)

$$\sum_{i \in S} f_i(\mathcal{H}) = v(S) \text{ if } S \in N/\mathcal{H} \text{ and } S \text{ is } \mathcal{H}\text{-complete.}$$

No contribution by unconnected players (NCU)

$$f_i(\mathcal{H}) = f_i(\mathcal{H}_{-j}) \text{ if } i, j \in N \text{ are not directly } \mathcal{H}\text{-connected in } N.$$

Complete component efficiency (CCE) is in the same spirit as Myerson's component efficiency. However, since we regard direct connections as basic units for communication, a component $S \in N/\mathcal{H}$ can function and allocate the total wealth $v(S)$ if S is $\mathcal{H}$-complete. To put it differently, if S is not $\mathcal{H}$-complete, there are some pairs in S who cannot directly meet, and thus an agreement for cooperation may not occur. Clearly, CE implies CCE, but not vice versa.

No contribution by unconnected players (NCU) implies that player i's payoff remains the same when all conferences containing j, who are not directly $\mathcal{H}$-connected with i, are removed. In other words, player j's contribution to i equals zero. Note by symmetry that $f_i(\mathcal{H}) - f_i(\mathcal{H}_{-j}) = f_j(\mathcal{H}) - f_j(\mathcal{H}_{-i}) (= 0)$, which is the special case of BC. It can be readily seen that the Myerson value does not satisfy NCU because it treats direct and indirect connections equally.

To characterize an allocation rule satisfying the axioms above, we consider a game determined by the collection of $\mathcal{H}$-complete coalitions. Write $\mathrm{cm}^*(\mathcal{H}) = \mathrm{cm}(\mathcal{H}) \cup \{\{i\} \mid i \in N\}$ for the collection of all $\mathcal{H}$-complete coalitions and let $\mu_{\mathrm{cm}^*(\mathcal{H})}$ be the Möbius function of $\mathrm{cm}^*(\mathcal{H})$. Define the following game $v^{\mathcal{H}}$, which we call the *direct-connection restricted* (*d-restricted*) *game* of v:

$$v^{\mathcal{H}} = \sum_{T \in 2^N} \beta_T^{\mathcal{H}} u_T \text{ where } \beta_T^{\mathcal{H}} = \begin{cases} \displaystyle\sum_{S \in \mathrm{cm}^*(\mathcal{H})} \mu_{\mathrm{cm}^*(\mathcal{H})}(S, T) v(S) & \text{if } T \in \mathrm{cm}^*(\mathcal{H}), \\ 0 & \text{if } T \notin \mathrm{cm}^*(\mathcal{H}). \end{cases} \tag{10}$$

We will see in section 6 that the construction of $v^{\mathcal{H}}$ generalizes that of the restricted game $r^{\mathcal{H}}$. The following lemma provides a simple characterization of $v^{\mathcal{H}}$.

Lemma 5 *Let $w = \sum_{T \in 2^N} \gamma_T u_T$ be a game. Then, $w = v^{\mathcal{H}}$ if and only if $w(S) = v(S)$ for all $S \in \mathrm{cm}^*(\mathcal{H})$ and $\gamma_T = 0$ for all $T \notin \mathrm{cm}^*(\mathcal{H})$.*

Proof. Assume that $w(S) = v(S)$ for all $S \in \mathrm{cm}^*(\mathcal{H})$ and $\gamma_T = 0$ for all $T \notin \mathrm{cm}^*(\mathcal{H})$. Then,

$$w(S) = \sum_{T \in \mathrm{cm}^*(\mathcal{H})} \gamma_T u_T(S) = v(S) \text{ for all } S \in \mathrm{cm}^*(\mathcal{H}). \tag{11}$$

By Lemma 1 with $f(X) = \gamma_X$ and $v(X) = w(X)$ restricted to $\mathrm{cm}^*(\mathcal{H})$, (11) is equivalent to

$$\gamma_T = \sum_{S \in \mathrm{cm}^*(\mathcal{H})} \mu_{\mathrm{cm}^*(\mathcal{H})}(S,T)v(S) \text{ for all } T \in \mathrm{cm}^*(\mathcal{H}). \tag{12}$$

By (10) and (12), $\gamma_T = \beta_T^{\mathcal{H}}$ for all $T \in 2^N$ and thus $w = v^{\mathcal{H}}$. Conversely, assume that $w = v^{\mathcal{H}}$ and thus $\gamma_T = \beta_T^{\mathcal{H}}$ for all $T \in 2^N$. Then, (10) implies that $\gamma_T = 0$ for all $T \notin \mathrm{cm}^*(\mathcal{H})$ and (12), the latter of which is equivalent to (11). Therefore, $w(S) = v(S)$ for all $S \in \mathrm{cm}^*(\mathcal{H})$ and $\gamma_T = 0$ for all $T \notin \mathrm{cm}^*(\mathcal{H})$. ∎

Now we are ready to state our main result, which characterizes an allocation rule satisfying CCE, NCU, and F.

Proposition 3 *The following four statements about an allocation rule f are equivalent.*

(i) *f satisfies CCE, NCU, and F.*

(ii) *f satisfies CCE, NCU, and BC.*

(iii) *$f(\mathcal{H})$ is the vector of the marginal contributions of a game $p^{\mathcal{H}}$ satisfying the following two conditions:*

$$\sum_{i \in S} \delta_i p^{\mathcal{H}}(S) = v(S) \text{ if } S \text{ is } \mathcal{H}\text{-complete.} \tag{13}$$

$$\delta_i p^{\mathcal{H}}(S) = \delta_i p^{\mathcal{H}}(S \backslash \{j\}) \text{ if } i, j \in S \text{ are not directly } \mathcal{H}\text{-connected in } S. \tag{14}$$

That is, $f_i(\mathcal{H}) = \delta_i p^{\mathcal{H}}(N)$ for all $i \in N$ and $\mathcal{H} \in \mathbf{CS}$.

(iv) *$f(\mathcal{H})$ is the Shapley value of the d-restricted game $v^{\mathcal{H}}$. That is, $f_i(\mathcal{H}) = \phi_i(v^{\mathcal{H}})$ for all $i \in N$ and $\mathcal{H} \in \mathbf{CS}$.*

Since the d-restricted game $v^{\mathcal{H}}$ is uniquely determined from v and $\mathcal{H}$ by (10), each statement in Proposition 3 identifies a unique allocation rule. Especially, this proposition shows that there exists a unique allocation rule satisfying CCE, NCU, and F. We call this allocation rule the *direct-connection Myerson* (*d-Myerson*) *value.*

Notice the resemblance between $p^{\mathcal{H}}$ in (iii) and the potential for v. The latter satisfies (6) for all coalitions, whereas the former satisfies it for all $\mathcal{H}$-complete coalitions, which is the condition (13). The other condition (14) requires that the marginal contribution of player i to $p^{\mathcal{H}}(S)$ be determined by players who are directly $\mathcal{H}$-connected in S with i. In both of the conditions, the direct $\mathcal{H}$-connected relation is essential. Note that if $\mathcal{H}$ is the finest conference structure (hence any coalition is $\mathcal{H}$-complete), then (13) is identical to (6), and (14) holds trivially because any pair of players are directly $\mathcal{H}$-connected in any coalition containing them. Thus in this case, $p^{\mathcal{H}}$ coincides with the potential for v by Proposition 1. As will be shown in Lemma 9 in the next section, $p^{\mathcal{H}}$ is the potential for $v^{\mathcal{H}}$, which will explain why the allocation rule is uniquely determined.

5 The proof

This section provides the proof of Proposition 3. It proceeds in the following order: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), (iii) $\Leftrightarrow$ (iv), and (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

5.1 (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)

As the next result shows, F and NCU together imply BC. Thus, if an allocation rule satisfies CCE, NCU, and F, then it satisfies CCE, NCU, and BC, establishing (i) $\Rightarrow$ (ii).

Lemma 6 *If an allocation rule f satisfies F and NCU, then it satisfies BC.*

Proof. If $i, j \in N$ are not directly $\mathcal{H}$-connected in N, then NCU implies that $f_i(\mathcal{H}) - f_i(\mathcal{H}_{-j}) = f_j(\mathcal{H}) - f_j(\mathcal{H}_{-i}) = 0$. If $i, j \in N$ are directly $\mathcal{H}$-connected in N, then write $\{H \in \mathcal{H} \,|\, \{i, j\} \subseteq H\} = \{H_1, \ldots, H_k\}$. By applying F repeatedly, we have

$$f_i(\mathcal{H}) - f_i(\mathcal{H}\backslash\{H_1, \ldots, H_k\}) = f_j(\mathcal{H}) - f_j(\mathcal{H}\backslash\{H_1, \ldots, H_k\}). \tag{15}$$

Note that i and j are not directly $\mathcal{H}\backslash\{H_1, \ldots, H_k\}$-connected in N since $\mathcal{H}\backslash\{H_1, \ldots, H_k\} = \{H \in \mathcal{H} \,|\, \{i, j\} \not\subseteq H\}$. Thus, NCU implies that

$$f_i(\mathcal{H}\backslash\{H_1, \ldots, H_k\}) = f_i((\mathcal{H}\backslash\{H_1, \ldots, H_k\})_{-j}) = f_i(\mathcal{H}_{-j}) \tag{16}$$

where the latter equality holds because $(\mathcal{H}\backslash\{H_1, \ldots, H_k\})_{-j} = \{H \in \mathcal{H} \,|\, \{i, j\} \not\subseteq H \text{ and } j \notin H\} = \{H \in \mathcal{H} \,|\, j \notin H\} = \mathcal{H}_{-j}$. Similarly, it follows that $f_j(\mathcal{H}\backslash\{H_1, \ldots, H_k\}) = f_j(\mathcal{H}_{-i})$. By plugging this and (16) into (15), we have established BC. ∎

As noted by Hart and Mas-Colell (1989), BC is a finite difference analogue of the Frobenious integrability condition, i.e., the symmetry of the cross partial derivatives, which suggests that the solution admits a potential. In fact, BC assures the existence of a "potential" in the following sense.[9]

Lemma 7 *If an allocation rule f satisfies BC, then, for each $\mathcal{H} \in \mathbf{CS}$, there exists a game $p^{\mathcal{H}}$ such that $f_i(\mathcal{H}_S) = \delta_i p^{\mathcal{H}}(S)$ for all $i \in S$ and $S \in 2^N$.*

Proof. Define a game $p^{\mathcal{H}}$ by the following rule: for each $S = \{i_1, \ldots, i_k\} \in 2^N$ with $i_1 < \cdots < i_k$, $p^{\mathcal{H}}(S) = \sum_{l=1}^{k} f_{i_l}(\mathcal{H}_{\{i_1,\ldots,i_l\}})$. Note that, by construction, if $i = \max S$ then $f_i(\mathcal{H}_S) = p^{\mathcal{H}}(S) - p^{\mathcal{H}}(S\backslash\{i\}) = \delta_i p^{\mathcal{H}}(S)$.

We show by induction that $f_i(\mathcal{H}_S) = \delta_i p^{\mathcal{H}}(S)$ for all $i \in S$ and $S \in 2^N$. If $|S| = 1$ and $S = \{i\}$, then $f_i(\mathcal{H}_{\{i\}}) = p^{\mathcal{H}}(\{i\}) - p^{\mathcal{H}}(\emptyset) = \delta_i p^{\mathcal{H}}(\{i\})$. Suppose as an induction hypothesis that

[9] Consider a vector-valued mapping $F : \mathbb{R}^n \to \mathbb{R}^n$. In vector analysis, a function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be a potential of F if $F = (\partial f / \partial x_i)_{i=1}^n$.

$f_i(\mathcal{H}_S) = \delta_i p^{\mathcal{H}}(S)$ for all $i \in S$ and $S \in 2^N$ with $|S| \leq k < n$. Let $S = \{i_1, \ldots, i_{k+1}\} \in 2^N$ with $i_1 < \cdots < i_{k+1}$. For every $i \in S$, by applying BC (with $\mathcal{H}_S$ instead of $\mathcal{H}$), we have

$$\begin{aligned} f_i(\mathcal{H}_S) &= f_{i_{k+1}}(\mathcal{H}_S) - f_{i_{k+1}}((\mathcal{H}_S)_{-i}) + f_i((\mathcal{H}_S)_{-i_{k+1}}) \\ &= f_{i_{k+1}}(\mathcal{H}_S) - f_{i_{k+1}}(\mathcal{H}_{S\setminus\{i\}}) + f_i(\mathcal{H}_{S\setminus\{i_{k+1}\}}). \end{aligned} \tag{17}$$

By the construction of $p^{\mathcal{H}}$,

$$f_{i_{k+1}}(\mathcal{H}_S) = p^{\mathcal{H}}(S) - p^{\mathcal{H}}(S\setminus\{i_{k+1}\}). \tag{18}$$

By the induction hypothesis,

$$f_{i_{k+1}}(\mathcal{H}_{S\setminus\{i\}}) = p^{\mathcal{H}}(S\setminus\{i\}) - p^{\mathcal{H}}(S\setminus\{i, i_{k+1}\}), \tag{19}$$

$$f_i(\mathcal{H}_{S\setminus\{i_{k+1}\}}) = p^{\mathcal{H}}(S\setminus\{i_{k+1}\}) - p^{\mathcal{H}}(S\setminus\{i, i_{k+1}\}). \tag{20}$$

Plugging (18), (19), and (20) into (17), we have

$$f_i(\mathcal{H}_S) = p^{\mathcal{H}}(S) - p^{\mathcal{H}}(S\setminus\{i\}) = \delta_i p^{\mathcal{H}}(S),$$

which completes the proof. ∎

A "potential" in Lemma 7 is shown to satisfy (13) and (14) if an allocation rule satisfies CCE and NCU in addition.

Lemma 8 *Let an allocation rule f satisfy CCE and NCU. Suppose that there exists a game $p^{\mathcal{H}}$ such that $f_i(\mathcal{H}_S) = \delta_i p^{\mathcal{H}}(S)$ for all $i \in S$ and $S \in 2^N$. Then, $p^{\mathcal{H}}$ satisfies* (13) *and* (14).

Proof. Suppose that S is $\mathcal{H}$-complete. Then $S \in N/\mathcal{H}_S$ and S is $\mathcal{H}_S$-complete. By CCE, $\sum_{i\in S} f_i(\mathcal{H}_S) = \sum_{i\in S} \delta_i p^{\mathcal{H}}(S) = v(S)$. Therefore, $p^{\mathcal{H}}$ satisfies (13). To show that $p^{\mathcal{H}}$ satisfies (14), suppose that $i, j \in S$ are not directly $\mathcal{H}$-connected in S. Then, they are not directly $\mathcal{H}_S$-connected in N. Thus, by NCU, $\delta_i p^{\mathcal{H}}(S) = f_i(\mathcal{H}_S) = f_i((\mathcal{H}_S)_{-j}) = f_i(\mathcal{H}_{S\setminus\{j\}}) = \delta_i p^{\mathcal{H}}(S\setminus\{j\})$. Therefore, $p^{\mathcal{H}}$ satisfies (14). ∎

By Lemma 7 and Lemma 8, if an allocation rule f satisfies CCE, NCU, and BC, then there exists a game $p^{\mathcal{H}}$ satisfying (13) and (14) such that $f_i(\mathcal{H}) = f_i(\mathcal{H}_N) = \delta_i p^{\mathcal{H}}(N)$, which establishes (ii) ⇒ (iii).

5.2 (iii) ⇔ (iv)

We shall show below that a game $p^{\mathcal{H}}$ which satisfies the conditions in (iii) must be the potential for the d-restricted game $v^{\mathcal{H}}$. This suffices to establish (iii) ⇔ (iv) since $\delta_i p^{\mathcal{H}}(N) = \phi_i(v^{\mathcal{H}})$ holds by Proposition 1.

Lemma 9 *There exists a unique game $p^{\mathcal{H}}$ satisfying* (13) *and* (14)*. The game $p^{\mathcal{H}}$ coincides with the potential for the d-restricted game $v^{\mathcal{H}}$.*

Proof. We first show that the potential for $v^{\mathcal{H}}$ does satisfy (13) and (14). So let $p^{\mathcal{H}}$ be the potential for $v^{\mathcal{H}} = \sum_{T \in 2^N} \beta_T^{\mathcal{H}} u_T$. Then by Proposition 1, $p^{\mathcal{H}} = \sum_{T \in 2^N} (\beta_T^{\mathcal{H}}/|T|) u_T$. Observe that $\sum_{i \in S} \delta_i p^{\mathcal{H}}(S) = v^{\mathcal{H}}(S) = v(S)$ if S is $\mathcal{H}$-complete, where the first equality holds since $p^{\mathcal{H}}$ is the potential for $v^{\mathcal{H}}$ and the second equality holds by Lemma 5. This is the condition (13). Next, observe that, since $\beta_T^{\mathcal{H}} = 0$ for all $T \notin \mathrm{cm}^*(\mathcal{H})$,

$$\begin{aligned}\delta_i p^{\mathcal{H}}(S) &= p^{\mathcal{H}}(S) - p^{\mathcal{H}}(S \backslash \{i\}) \\ &= \sum_{T \in \mathrm{cm}^*(\mathcal{H})_S} \beta_T^{\mathcal{H}}/|T| - \sum_{T \in \mathrm{cm}^*(\mathcal{H})_{S \backslash \{i\}}} \beta_T^{\mathcal{H}}/|T| \\ &= \sum_{T \in \mathrm{cm}^*(\mathcal{H})_S : i \in T} \beta_T^{\mathcal{H}}/|T|, \end{aligned} \tag{21}$$

and similarly,

$$\delta_i p^{\mathcal{H}}(S \backslash \{j\}) = \sum_{T \in \mathrm{cm}^*(\mathcal{H})_{S \backslash \{j\}} : i \in T} \beta_T^{\mathcal{H}}/|T|. \tag{22}$$

Now suppose that $i, j \in S$ are not directly $\mathcal{H}$-connected in S. Then, there is no $T \in \mathrm{cm}^*(\mathcal{H})_S$ such that $\{i, j\} \subseteq T$ because any pair of players in $T \in \mathrm{cm}^*(\mathcal{H})_S$ are directly $\mathcal{H}$-connected in T and thus in S. This implies that $\{T \in \mathrm{cm}^*(\mathcal{H})_S \mid i \in T\} = \{T \in \mathrm{cm}^*(\mathcal{H})_{S \backslash \{j\}} \mid i \in T\}$ and thus $\delta_i p^{\mathcal{H}}(S) = \delta_i p^{\mathcal{H}}(S \backslash \{j\})$ by (21) and (22). This is the condition (14).

To complete the proof, we show that a game $p^{\mathcal{H}}$ satisfying (13) and (14) is unique, by constructing $p^{\mathcal{H}}$ recursively such that in the k-th step we determine the unique value of $p^{\mathcal{H}}(S)$ with $|S| = k$ from $p^{\mathcal{H}}(S')$ with $|S'| \leq k - 1$. Start with $p^{\mathcal{H}}(\emptyset) = 0$ since $p^{\mathcal{H}}$ is a game. Consider the k-th step with $k \geq 1$ and pick any S with $|S| = k$. Suppose that S is $\mathcal{H}$-complete. Then, (13) is rewritten as

$$p^{\mathcal{H}}(S) = n^{-1} \left(v(S) + \sum_{i \in S} p^{\mathcal{H}}(S \backslash \{i\}) \right).$$

Since $p^{\mathcal{H}}(S \backslash \{i\})$ on the right hand side is uniquely calculated for each $i \in N$ in the previous step, so is $p^{\mathcal{H}}(S)$ on the left hand side. Suppose that S is not $\mathcal{H}$-complete. Then, there exist two distinct players $i, j \in S$ who are not directly $\mathcal{H}$-connected in S. So, by (14),

$$p^{\mathcal{H}}(S) = p^{\mathcal{H}}(S \backslash \{i\}) + p^{\mathcal{H}}(S \backslash \{j\}) - p^{\mathcal{H}}(S \backslash \{i, j\}). \tag{23}$$

Since the terms on the right hand side are uniquely calculated in the earlier steps, so is $p^{\mathcal{H}}(S)$ on the left hand side. Note that $p^{\mathcal{H}}(S)$ in (23) does not depend upon the choice of i and j because (23) holds for any $i, j \in S$ who are not directly $\mathcal{H}$-connected in S. By the above procedure, we can uniquely determine $p^{\mathcal{H}}$ recursively, which establishes the uniqueness. ∎

5.3 (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i)

Recall that BC implies F by Lemma 4, which establishes (ii) $\Rightarrow$ (i). To prove (iii) $\Rightarrow$ (ii), we use the following lemma.

Lemma 10 *Let $p^{\mathcal{H}}$ be a game satisfying* (13) *and* (14) *for each $\mathcal{H} \in \mathbf{CS}$. Then, $\delta_i p^{\mathcal{H}_S}(S) = \delta_i p^{\mathcal{H}}(S)$ for all $i \in S$ and $S \in 2^N$.*

Proof. By Lemma 9 and Proposition 1, $p^{\mathcal{H}} = \sum_{T\in 2^N}(\beta_T^{\mathcal{H}}/|T|)u_T$ where

$$\beta_T^{\mathcal{H}} = \begin{cases} \displaystyle\sum_{S\in \mathrm{cm}^*(\mathcal{H})} \mu_{\mathrm{cm}^*(\mathcal{H})}(S,T)v(S) & \text{if } T \in \mathrm{cm}^*(\mathcal{H}), \\ 0 & \text{if } T \notin \mathrm{cm}^*(\mathcal{H}). \end{cases}$$

Observe that if $T \subseteq R$ then $\mu_{\mathrm{cm}^*(\mathcal{H})}(S,T) = \mu_{\mathrm{cm}^*(\mathcal{H}_R)}(S,T)$. This is because the recursive construction of $\mu_{\mathrm{cm}^*(\mathcal{H})}(S,T)$ in (2) implies $\mu_{\mathrm{cm}^*(\mathcal{H})}(S,T) = \mu_{\mathrm{cm}^*(\mathcal{H})_R}(S,T)$ and the definition of $\mathcal{H}$-completeness implies $\mathrm{cm}^*(\mathcal{H})_R = \mathrm{cm}^*(\mathcal{H}_R)$. Therefore, $\beta_T^{\mathcal{H}} = \beta_T^{\mathcal{H}_R}$ if $T \subseteq R$ and thus $p^{\mathcal{H}}(S) = \sum_{T\subseteq S}\beta_T^{\mathcal{H}}/|T| = \sum_{T\subseteq S}\beta_T^{\mathcal{H}_R}/|T| = p^{\mathcal{H}_R}(S)$ if $S \subseteq R$. This implies that $\delta_i p^{\mathcal{H}_S}(S) = \delta_i p^{\mathcal{H}}(S)$. ∎

We are ready to establish (iii) $\Rightarrow$ (ii).

Lemma 11 *Let f be an allocation rule stated in* (iii)*. Then, f satisfies CCE, NCU, and BC.*

Proof. If $i \neq j$, then they are not directly $\mathcal{H}_{-j}$-connected in N. Thus $f_i(\mathcal{H}_{-j}) = \delta_i p^{\mathcal{H}_{-j}}(N) = \delta_i p^{\mathcal{H}_{-j}}(N\backslash\{j\})$ by (14). By setting $S = N\backslash\{j\}$ in Lemma 10, we have $\delta_i p^{\mathcal{H}_{-j}}(N\backslash\{j\}) = \delta_i p^{\mathcal{H}}(N\backslash\{j\})$. Therefore, $f_i(\mathcal{H}_{-j}) = \delta_i p^{\mathcal{H}}(N\backslash\{j\})$, which implies BC because

$$\begin{aligned} f_i(\mathcal{H}) - f_i(\mathcal{H}_{-j}) &= \delta_i p^{\mathcal{H}}(N) - \delta_i p^{\mathcal{H}}(N\backslash\{j\}) \\ &= p^{\mathcal{H}}(N) - p^{\mathcal{H}}(N\backslash\{i\}) - p^{\mathcal{H}}(N\backslash\{j\}) + p^{\mathcal{H}}(N\backslash\{i,j\}) \\ &= \delta_j p^{\mathcal{H}}(N) - \delta_j p^{\mathcal{H}}(N\backslash\{i\}) \\ &= f_j(\mathcal{H}) - f_j(\mathcal{H}_{-i}). \end{aligned}$$

If $i, j \in N$ are not directly $\mathcal{H}$-connected in N, then $\delta_i p^{\mathcal{H}}(N) = \delta_i p^{\mathcal{H}}(N\backslash\{j\})$ by (14), and the above equation is reduced to $f_i(\mathcal{H}) - f_i(\mathcal{H}_{-j}) = 0$, which is NCU.

It remains to prove that f satisfies CCE. Let $S \in N/\mathcal{H}$ be $\mathcal{H}$-complete. We first show that $\delta_i p^{\mathcal{H}}(N) = \delta_i p^{\mathcal{H}}(S)$ for $i \in S$. Let $N\backslash S = \{j_1, \dots, j_m\}$ and $T_k = N\backslash\{j_1, \dots, j_k\}$ for $k = 1, \dots, m$. Since $i, j_1 \in N$ are not directly $\mathcal{H}$-connected in N, it holds that $\delta_i p^{\mathcal{H}}(N) = \delta_i p^{\mathcal{H}}(N\backslash\{j_1\}) = \delta_i p^{\mathcal{H}}(T_1)$ by (14). Similarly, since $i, j_k \in T_{k-1}$ are not directly $\mathcal{H}$-connected in T_{k-1}, it holds that $\delta_i p^{\mathcal{H}}(T_{k-1}) = \delta_i p^{\mathcal{H}}(T_{k-1}\backslash\{j_k\}) = \delta_i p^{\mathcal{H}}(T_k)$ by (14) for $k = 2, \dots, m$. Therefore, $\delta_i p^{\mathcal{H}}(N) = \delta_i p^{\mathcal{H}}(T_1) = \cdots = \delta_i p^{\mathcal{H}}(T_m) = \delta_i p^{\mathcal{H}}(S)$. Then, we have $\sum_{i\in S} f_i(\mathcal{H}) = \sum_{i\in S}\delta_i p^{\mathcal{H}}(N) = \sum_{i\in S}\delta_i p^{\mathcal{H}}(S) = v(S)$ by (13), which is CCE. ∎

6 Discussions

6.1 A characterization of $v^{\mathcal{H}}$

We summarize a characterization of the d-restricted game $v^{\mathcal{H}}$.

Lemma 12 *Fix a game v and $\mathcal{H} \in \mathbf{CS}$. The following four statements about a game $w = \sum_{T \in 2^N} \gamma_T u_T$ are equivalent.*

(i) $w = v^{\mathcal{H}}$*, i.e., w is the d-restricted game.*

(ii) $w(S) = v(S)$ *if S is $\mathcal{H}$-complete and $\gamma_T = 0$ if T is not $\mathcal{H}$-complete.*

(iii) $\{\gamma_T\}_{T \in 2^N}$ *is determined recursively by the following rule:*

1. $\gamma_{\{i\}} = v(\{i\})$ *for all $i \in N$.*
2. *For $T \in 2^N$ with $|T| \geq 2$,*
   - $\gamma_T = v(T) - \sum_{S \subset T} \gamma_S$ *if T is $\mathcal{H}$-complete,*
   - $\gamma_T = 0$ *if T is not $\mathcal{H}$-complete.*

(iv) *w satisfies the following two conditions:*

$$w(S) = v(S) \text{ if } S \text{ is } \mathcal{H}\text{-complete.} \tag{24}$$

$$\delta_i w(S) = \delta_i w(S \backslash \{j\}) \text{ if } i, j \in S \text{ are not directly } \mathcal{H}\text{-connected in } S. \tag{25}$$

Proof. Lemma 5 established (i) $\Leftrightarrow$ (ii). So we prove (ii) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv).

(ii) $\Leftrightarrow$ (iii): The rule in (iii) is rewritten as the condition that if S is $\mathcal{H}$-complete then $v(S) = \sum_{T \subseteq S} \gamma_T = w(S)$ and if T is not $\mathcal{H}$-complete then $\gamma_T = 0$, which is (ii).

(ii) $\Leftrightarrow$ (iv): Let w be as stated in (ii). Then, the condition (24) is obviously satisfied. If $i, j \in S$ are not directly $\mathcal{H}$-connected in S, then, as shown in the proof of Lemma 9, we have $\{T \in \mathrm{cm}^*(\mathcal{H})_S \mid i \in T\} = \{T \in \mathrm{cm}^*(\mathcal{H})_{S \backslash \{j\}} \mid i \in T\}$. Since $\gamma_T = 0$ for all $T \notin \mathrm{cm}^*(\mathcal{H})$, a calculation similar to (21) and (22) shows that

$$\delta_i w(S) = \sum_{T \in \mathrm{cm}^*(\mathcal{H})_S : i \in T} \gamma_T = \sum_{T \in \mathrm{cm}^*(\mathcal{H})_{S \backslash \{j\}} : i \in T} \gamma_T = \delta_i w(S \backslash \{j\}),$$

which is the condition (25).[10] Thus, (ii) implies (iv).

Suppose that w satisfies the conditions in (iv). To prove that (iv) implies (ii), it suffices to show that w is uniquely determined because $v^{\mathcal{H}}$ is the unique game that satisfies the conditions

[10] Kajii et al. (2005) considered a condition similar to (25) and called it modularity for $\mathcal{H}$-decomposition pairs. They showed that $\gamma_T = 0$ for all $T \notin \mathrm{cm}^*(\mathcal{H})$ if and only if w is modular for $\mathcal{H}$-decomposition pairs. It can be readily shown that (25) and modularity for $\mathcal{H}$-decomposition pairs are equivalent.

in (ii) by Lemma 5, which also satisfies the conditions in (iv) as discussed above. To show the uniqueness, we construct w recursively such that in the k-th step we determine the unique value of $w(S)$ with $|S| = k$ from $w(S')$ with $|S'| \leq k-1$. Start with $w(\emptyset) = 0$. Consider the k-th step with $k \geq 1$ and pick any S with $|S| = k$. If S is $\mathcal{H}$-complete, then $w(S) = v(S)$ by (24). If S is not $\mathcal{H}$-complete, then there exist $i, j \in S$ who are not directly $\mathcal{H}$-connected in S and so, by (25),

$$w(S) = w(S \backslash \{i\}) + w(S \backslash \{j\}) - w(S \backslash \{i, j\}). \tag{26}$$

Since the terms on the right hand side are uniquely calculated in the earlier steps, so is $w(S)$ on the left hand side. Note that $w(S)$ in (26) does not depend upon the choice of i and j because (26) holds for any $i, j \in S$ who are not directly $\mathcal{H}$-connected in S. By the above procedure, we can uniquely determine w recursively, which establishes the uniqueness. ∎

6.2 The Myerson value and the d-Myerson value

As the next result shows, we can derive the restricted game $r^{\mathcal{H}}$ from the d-restricted game $v^{\mathcal{H}}$.

Lemma 13 *For each $\mathcal{H} \in \mathbf{CS}$, it holds that $v^{\mathrm{cn}(\mathcal{H})} = r^{\mathcal{H}}$.*

Proof. We prove that

$$v^{\mathrm{cn}(\mathcal{H})}(S) = \sum_{T \in 2^N} \beta_T^{\mathrm{cn}(\mathcal{H})} u_T(S) = \sum_{T \in S/\mathcal{H}} v(T) \text{ for all } S \in 2^N. \tag{27}$$

Let us write $\mathrm{cn}^*(\mathcal{H}) = \mathrm{cn}(\mathcal{H}) \cup \{\{i\} \,|\, i \in N\}$, which is the collection of all $\mathcal{H}$-connected coalitions. Note that $\beta_T^{\mathrm{cn}(\mathcal{H})} = 0$ for all $T \notin \mathrm{cm}^*(\mathrm{cn}(\mathcal{H}))$ by Lemma 5. Since $\mathrm{cm}(\mathrm{cn}(\mathcal{H})) = \mathrm{cn}(\mathcal{H})$ by Lemma 3, it follows that $\mathrm{cm}^*(\mathrm{cn}(\mathcal{H})) = \mathrm{cm}(\mathrm{cn}(\mathcal{H})) \cup \{\{i\} \,|\, i \in N\} = \mathrm{cn}(\mathcal{H}) \cup \{\{i\} \,|\, i \in N\} = \mathrm{cn}^*(\mathcal{H})$. Thus, for each $S \in 2^N$,

$$v^{\mathrm{cn}(\mathcal{H})}(S) = \sum_{T \subseteq S} \beta_T^{\mathrm{cn}(\mathcal{H})} = \sum_{T \in \mathrm{cm}^*(\mathrm{cn}(\mathcal{H}))_S} \beta_T^{\mathrm{cn}(\mathcal{H})} = \sum_{T \in \mathrm{cn}^*(\mathcal{H})_S} \beta_T^{\mathrm{cn}(\mathcal{H})}. \tag{28}$$

Observe that each $T \in \mathrm{cn}^*(\mathcal{H})_S$ is a $\mathcal{H}$-connected coalition contained in S and thus there exists $R \in S/\mathcal{H}$ such that $T \subseteq R$ because any pair of players in T are $\mathcal{H}$-connected in S and thus they are $\mathcal{H}$-connected in the component of S to which they both belong. Note that such $R \in S/\mathcal{H}$ is unique. Hence we have

$$\sum_{T \in \mathrm{cn}^*(\mathcal{H})_S} \beta_T^{\mathrm{cn}(\mathcal{H})} = \sum_{R \in S/\mathcal{H}} \left(\sum_{T \subseteq R} \beta_T^{\mathrm{cn}(\mathcal{H})} \right) = \sum_{R \in S/\mathcal{H}} v^{\mathrm{cn}(\mathcal{H})}(R). \tag{29}$$

Observe that $R \in S/\mathcal{H}$ is $\mathrm{cn}(\mathcal{H})$-complete because if R is a singleton then it is so by definition and if $|R| \geq 2$ then any $i, j \in R$ are $\mathcal{H}$-connected in R and thus $R \in \mathrm{cn}(\mathcal{H}) = \mathrm{cm}(\mathrm{cn}(\mathcal{H}))$ by Lemma 3, which with Lemma 5 implies that

$$v^{\mathrm{cn}(\mathcal{H})}(R) = v(R). \tag{30}$$

By (28), (29), and (30), we have (27), completing the proof. ▮

This lemma implies that the Shapley value of $r^{\mathcal{H}}$ and that of $v^{\mathrm{cn}(\mathcal{H})}$ coincide. Therefore, by Proposition 2 and Proposition 3, the d-Myerson value can be regarded as a refinement of the Myerson value in the following sense.

Lemma 14 *Let f^M be the Myerson value and f be the d-Myerson value. Then,*

$$f^M(\mathcal{H}) = f(\mathrm{cn}(\mathcal{H})) \text{ for all } \mathcal{H} \in \mathbf{CS}.$$

We shall supply an example illustrating the difference between the two allocation rules. Consider again the example in section 3, i.e., $\mathcal{H}^1, \mathcal{H}^2$, and $\mathcal{H}^3$ specified in (9) with $N = \{1, 2, 3, 4\}$, and define a game

$$v = \alpha u_{\{1,2,3\}} + \beta u_{\{2,3,4\}} + \gamma u_{\{1,2,3,4\}}.$$

Using the construction method (iii) of Lemma 12, we have $v^{\mathcal{H}^1} = (\alpha+\beta+\gamma)u_{\{1,2,3,4\}}$, $v^{\mathcal{H}^2} = \alpha u_{\{1,2,3\}} + (\beta+\gamma)u_{\{1,2,3,4\}}$, and $v^{\mathcal{H}^3} = v$. The payoff vectors given by the d-Myerson value can be found by calculating the Shapley value of these games, and using the formula (5), we have them in the following table.

	player 1	players 2 and 3	player 4
$f(\mathcal{H}^1)$	$(\alpha+\beta+\gamma)/4$	$(\alpha+\beta+\gamma)/4$	$(\alpha+\beta+\gamma)/4$
$f(\mathcal{H}^2)$	$\alpha/3+(\beta+\gamma)/4$	$\alpha/3+(\beta+\gamma)/4$	$(\beta+\gamma)/4$
$f(\mathcal{H}^3)$	$\alpha/3+\gamma/4$	$(\alpha+\beta)/3+\gamma/4$	$\beta/3+\gamma/4$

If $\alpha > 0$ then $f_i(\mathcal{H}^2) > f_i(\mathcal{H}^1)$ for $i \in \{1, 2, 3\}$ and $f_4(\mathcal{H}^2) < f_4(\mathcal{H}^1)$. That is, addition of a conference $\{1, 2, 3\}$ to $\mathcal{H}^1$ decreases the payoff of player 4 and increases those of players in $\{1, 2, 3\}$ if the dividend of $\{1, 2, 3\}$ is positive. On the other hand, if $\beta > 0$ then $f_i(\mathcal{H}^3) > f_i(\mathcal{H}^2)$ for $i \in \{2, 3, 4\}$ and $f_1(\mathcal{H}^3) < f_1(\mathcal{H}^2)$. That is, addition of another conference $\{2, 3, 4\}$ to $\mathcal{H}^2$ decreases the payoff of player 1 and increases those of players in $\{2, 3, 4\}$ if the dividend of $\{2, 3, 4\}$ is positive. Note that the above differences of payoffs do not appear in the Myerson value because $\mathrm{cn}(\mathcal{H}^1) = \mathrm{cn}(\mathcal{H}^2) = \mathrm{cn}(\mathcal{H}^3) = \mathcal{H}^3$ and thus $f^M(\mathcal{H}^1) = f^M(\mathcal{H}^2) = f^M(\mathcal{H}^3) = f(\mathcal{H}^3)$ by Lemma 14.

Let us conclude with a final remark on the comparison of the two allocation rules. In some applications, it may make sense to require $\sum_{i\in S} f_i(\mathcal{H}) = v(S)$ to hold for all $S \in N/\mathcal{H}$. The d-Myerson value, however, do not satisfy CE and thus $\sum_{i\in S} f_i(\mathcal{H}) > v(S)$ is certainly possible for $S \in N/\mathcal{H}$ which is not $\mathcal{H}$-complete. Our suggestion to avoid this difficulty is simple: adopt $f(\mathcal{H} \cup (N/\mathcal{H}))$ as the payoff vector for $\mathcal{H}$ instead of $f(\mathcal{H})$. Since each $S \in N/\mathcal{H}$ is $(\mathcal{H} \cup (N/\mathcal{H}))$-complete and $S \in N/(\mathcal{H} \cup (N/\mathcal{H}))$, it holds that $\sum_{i\in S} f_i(\mathcal{H} \cup (N/\mathcal{H})) = v(S)$ for all $S \in N/\mathcal{H}$ by CCE. We believe that this is not an ad hoc treatment because $\sum_{i\in S} f_i(\mathcal{H}) = v(S)$ implies that players in S can cooperate and thus it is natural to add S to $\mathcal{H}$. Note that, for the Myerson value f^M, it holds that $f^M(\mathcal{H} \cup (N/\mathcal{H})) = f^M(\mathcal{H})$ for all $\mathcal{H} \in \mathbf{CS}$ because $N/\mathcal{H} \subseteq \mathrm{cn}(\mathcal{H})$ and thus $\mathrm{cn}(\mathcal{H} \cup (N/\mathcal{H})) = \mathrm{cn}(\mathcal{H})$. So, it is also of interest to compare $f^M(\mathcal{H})$ and $f(\mathcal{H} \cup (N/\mathcal{H}))$. In the above example, we have $f(\mathcal{H}^k \cup (N/\mathcal{H}^k)) = f(\mathcal{H}^k)$ for each k because $\mathcal{H}^k = \mathcal{H}^k \cup (N/\mathcal{H}^k)$ holds.

6.3 Network games and d-restricted games

Let $\mathbf{G}$ be the collection of conference structures each conference of which contains exactly two players:

$$\mathbf{G} = \{\mathcal{G} \in \mathbf{CS} \mid |L| = 2 \text{ for all } L \in \mathcal{G}\}.$$

Each $\mathcal{G} \in \mathbf{G}$ is regarded as a network because $(N, \mathcal{G})$ is an undirected graph with a vertex set N and an edge set $\mathcal{G}$.

Jackson and Wolinsky (1996) called a function $V : \mathbf{G} \to \mathbb{R}$ a *network game* where $V(\mathcal{G})$ is the total wealth when the network $\mathcal{G} \in \mathbf{G}$ is formed. They considered an allocation rule $f : \mathbf{G} \to \mathbb{R}^N$ given by the Shapley value of a game w satisfying $w(S) = V(\mathcal{G}_S)$ for all $S \in 2^N$. They called this allocation rule the Myerson value for network games and gave a characterization similar to that of Myerson (1977).

The following result shows that a special class of network games are represented in terms of the d-restricted game $v^{\mathcal{G}}$.

Lemma 15 *Let $V : \mathbf{G} \to \mathbb{R}$ be a network game with $V(\emptyset) = 0$. Suppose that, for each $\mathcal{G} \in \mathbf{G}$, it holds that*

$$V(\mathcal{G}_S) - V(\mathcal{G}_{S\setminus\{i\}}) = V(\mathcal{G}_{S\setminus\{j\}}) - V(\mathcal{G}_{S\setminus\{i,j\}}) \text{ if } \{i,j\} \subseteq S \text{ and } \{i,j\} \notin \mathcal{G}. \tag{31}$$

Then, there exists a game v such that

$$v^{\mathcal{G}}(S) = V(\mathcal{G}_S) \text{ for all } S \in 2^N \text{ and } \mathcal{G} \in \mathbf{G}. \tag{32}$$

Proof. Let v be a game such that

$$v(S) = V(\{\{i,j\} \mid \{i,j\} \subseteq S\}) \text{ for all } S \in 2^N. \tag{33}$$

Fix $\mathcal{G} \in \mathbf{G}$. Let w be a game such that $w(S) = V(\mathcal{G}_S)$. If S is $\mathcal{G}$-complete, then $\mathcal{G}_S = \{\{i,j\} \mid \{i,j\} \subseteq S\}$ and thus $w(S) = V(\{\{i,j\} \mid \{i,j\} \subseteq S\}) = v(S)$. If $i, j \in S$ are not directly $\mathcal{G}$-connected in S, then $\{i,j\} \notin \mathcal{G}$ and $\delta_i w(S) = \delta_i w(S\setminus\{j\})$ by (31). Thus w satisfies the conditions (24) and (25). The equivalence of (i) and (iv) in Lemma 12 implies $w = v^{\mathcal{G}}$, which completes the proof. ∎

The condition (31) says that if players i and j contained in S are not linked in the network $\mathcal{G}$ then the marginal contribution of i to $V(\mathcal{G}_S)$ equals that to $V(\mathcal{G}_{S\setminus\{j\}})$. The above lemma implies that if a network game V satisfies (31) for all $\mathcal{G} \in \mathbf{G}$ then the Myerson value for V coincides with the d-Myerson value of v give by (33). For example, consider a network game V defined by

$$V(\mathcal{G}) = \sum_{L \in \mathcal{G}} w_L \text{ for all } \mathcal{G} \in \mathbf{G}$$

where $w_L \in \mathbb{R}$ is a constant. It is easy to check that V satisfies (31) for all $\mathcal{G} \in \mathbf{G}$ and that (32) holds for a game v such that

$$v(S) = \sum_{L \subseteq S} w_L \text{ for all } S \in 2^N.$$

7 Concluding remarks

This paper has proposed and axiomatized the d-Myerson value as a refinement of the Myerson value. In so doing, we have introduced the two new axioms, CCE and NCU, and the d-restricted game; the axiomatization of the d-Myerson value is done by replacing the CE axiom in that of the Myerson value with the CCE and NCU axioms, and the d-Myerson value is shown to coincide with the Shapley value of the d-restricted game in place of the restricted game in the Myerson value. As concluding remarks, we point out other possible applications of the CCE and NCU axioms and the d-restricted game.

The position value (Meesen, 1988; Borm et al., 1992) and the Hamiache value (Hamiache, 1999) are allocation rules defined on the collection of networks $\mathbf{G}$. Later, these allocation rules are extended to those on the collection of conference structures $\mathbf{CS}$ (van den Nouweland et al., 1992; Bilbao and López, 2006). Both of them also treat direct and indirect connections equally because the position value is defined in terms of the restricted game and the Hamiache value is axiomatized in terms of the CE axiom. So, it is natural to consider refinements of these allocation rules respecting differences of direct and indirect connections, as we did for the Myerson value. We speculate that the CCE and NCU axioms and the d-restricted game might be employed instead of the CE axiom in the Hamiache value and the restricted game in the position value.

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