Optimal taxation in the extensive model

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Abstract

We study optimal taxation in the general extensive model: the only decision of the participants in the economy is to choose between working (full time) or staying inactive. People differ in their productivities and in other features which determine their work opportunity costs. The qualitative properties of optimal tax schemes are presented, with an emphasis on the role of heterogeneity in the equity-efficiency tradeoff. When the government has a redistributive stance, there are a number of cases where the low skilled workers face *larger* financial incentives to work than in the laissez-faire (*negative* average tax rates). In particular, this occurs whenever the social weights vary continuously with income and the social weight assigned to the less skilled workers is larger than average.

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1 Introduction

Since Mirrlees (1971), the theory of optimal taxation has largely been developed in the intensive model setup, where labor supply is continuous. The properties of the optimal tax scheme now have been thoroughly investigated in this framework (see, among others, (Seade (1977), Seade (1982), Werning (2000), Hellwig (2005)). Comparatively, much less attention has been devoted to the extensive model, in spite of the fact that it may be empirically important as Heckman (1993) convincingly argues. The early work of Diamond (1980) stresses non convexity issues in a simple example, but it had no following until Saez (2002). In this paper, we examine in depth the extensive model of optimal taxation. We derive qualitative properties of optimal schedules from a full-fledged model of labor force participation and stress the impact of heterogeneity on the efficiency-redistribution tradeoff.

First a word of caution is in order. In the intensive model, attention is focussed on the *marginal* tax rates which there determine labor supply. In the extensive model, workers are concerned with *average* tax rates. The average tax rate is equal to 1 minus the ratio (net income at work - subsistence income out of work)/(labor cost). Here 'net income at work' denotes disposable income when taking a full time job, 'subsistence income out of work' is disposable income when not working, including, say in the USA, food stamps and temporary assistance for needy families, and 'labor cost' is the cost of labor to the employer, a proxy for productivity in a competitive environment. Subsidizing work, compared with laissez-faire, is equivalent to have negative *average tax rates*:[1] this means a disposable income larger than the sum of productivity and of the subsistence income one would collect if unemployed.

Another difference between the intensive and the extensive models comes from the available information. In the extensive model, workers have little opportunity to evade the tax: as soon as after tax income is an increasing function of before tax income, they work at their full productivity. Their only choice is between working or not working. Thus the informational asymmetry which prevents the fiscal authority from getting to the first best lies in the determinants of the participation decision which are private knowledge to the (potential) workers. We set a general structural model of discrete choice, which allows for differences in productivity and multidimensional heterogeneity in the participation factors. We derive a convenient reduced form from the structural model, which summarizes the various determinants of participation into a unidimensional *work opportunity cost*. We investigate the relationships between the structural and reduced forms. It turns

[1] Some of the models of the literature have both intensive and extensive components. Then subsidizing labor may take place at the extensive margin (negative average tax rate), at the intensive margin (negative marginal tax rate), at no margins, or at both. The literature is not explicit on the topic.

out that the distribution of work opportunity costs, conditional on productivity and on income out of work, inherits properties from the structural model which are important for the analysis. In a large part of the paper, we work under the simplifying assumption that the distribution of work opportunity costs in the economy is independent of productivity.

The second best program is not concave: the proportion of agents at work, whose opportunity costs are smaller than their gains from working, enters the feasibility constraint and this typically is *not* a concave function of its arguments. A pointwise maximization of the Lagrangian, at each productivity level, yields the second best optima. It turns out that there is no pooling: on the range of productivities where the sets of workers and unemployed agents are both non-negligible, after tax income is a strictly increasing function of productivity. Define the social weight of a group of individuals as their average marginal utilities of income. Then consider the workers of a given productivity level. When their social weight is larger than the marginal cost of public funds, their labor supply is subsidized, i.e. they face a negative tax rate, and they are better off than at the laissez-faire. Conversely when their social weight is smaller than the marginal cost of public funds, they face a positive tax rate and work less than at the laissez-faire. This property follows from the first order condition associated with the income level and can be explained as follows. Let λ be the marginal cost of public funds, the multiplier of the government budget constraint. Consider a small change in the tax schedule in favor of workers of productivity ω, keeping unchanged the situation of the other agents. This reform has two effects on the population of productivity ω: it increases the incomes of the workers, and it brings into the labor force some previously unemployed persons. When the social weight of the workers is larger than the marginal cost of public funds, the first, purely redistributive, effect increases social welfare. The second, distortive, effect comes from the pivotal agents that enter the labor force. They are essentially indifferent between working and not working, and their impact on social welfare comes from the difference between their productivity ω and the cost of putting them to work. For the first order condition to be satisfied, this difference must be negative: the financial incentive to work (income at work minus subsistence income) has to be larger than productivity.

Redistribution comes from restrictions on the social weights of the participants in the economy. A property of the government objective in the intensive model is that social weights decrease with income: the social planner wants to redistribute towards poor people. Here, in the extensive model with multidimensional heterogeneity, utilitarianism and the concavity in income of the utility functions are not enough to warrant this property: the private characteristics that determine participation may be associated with a redistributive motive positively correlated with income. However, we give simple natural supplementary conditions under which the social weights of the workers decrease with income. Then, income being increasing with productivity, social weights are larger for the lower skilled employees at the optimum. If negative tax rates are present at the optimum, they concern an interval of productivities at the bottom of the distribution. The optimum allocation can take one of the two following shapes: either the average tax rate is always positive, so that labor supply is everywhere downwards distorted; or the persons whose skills are at the low end of the distribution face a negative tax rate and are better off than at the laissez-faire. The former case is qualitatively similar to the optimal tax of the intensive Mirrlees model, while the latter is very different. We now discuss when it occurs.

To this end, a further first order condition of the optimal tax program is of interest. It is associated with an equal marginal change of the incomes of everyone in the economy. It says that the marginal cost of public funds is a weighted average, with positive weights summing up to 1, of the social weights of all the agents in the economy. Consider now the benchmark model where the agents get a utility $U(s)$ when unemployed and $U(R-\delta)$ when they work, with s denoting subsistence income, R income at work and δ the work opportunity cost which varies in the population. Then the marginal employees, who are indifferent between working and not working, have the same utility levels as the unemployed, and the social weights of these two groups of agents are equal to $U'(s)$. In such an economy, if there is any redistribution at all, the weights of the most favored agents are larger than the marginal cost of public funds. Provided weights are a continuous function of income, we therefore are in the second case described above: there are negative tax rates and financial incentives to work are higher than under laissez-faire at the bottom of the income distribution. This property obviously is satisfied in a wider range of situations than the just outlined simple model, but it does not always hold. It can break down when social weights are discontinuous: in particular, in the Rawlsian case, where social weights are zero as soon as income is above the minimum, average tax rates are everywhere nonnegative. Similarly, if the weights put on the unemployed agents is (strictly) larger than that of the low income workers, tax rates could also be everywhere positive.

The paper is organized as follows. Section 2 describes the labor supply model, while Section 3 presents the tax instruments and government objective. The first order condition satisfied by the tax schedule are derived in Section 4. Section 5 presents in turn the properties of general second best tax schedules and the restrictions that they satisfy under a redistributive government. Section 6 presents the full set of equations satisfied by an optimum, and Section 7 discusses the role of heterogeneity. The shape of taxes at the bottom of the income distribution and the likelihood of negative taxes or upwards distortion the financial incentives to work are discussed in the conclusion. Proofs are gathered in the appendix.

2 Heterogeneity and the description of the private economy

We consider an economy with a continuum of participants of measure 1. The agents' only decision is whether to work or not. The agents differ along several dimensions. First, they differ by their productivity levels ω, the before tax income that they generate when they work. Second, there is a possibly multidimensional heterogeneity parameter α that describes other idiosyncratic characteristics, such as the costs of going to work. Allowing for a lot of heterogeneity (i.e. a multidimensional parameter α) accords with empirical studies where marital status, family composition, human capital and culture often appear to be determinants of the labor supply decision.

Let c_E be the income of someone who decides to work, c_U her income when unemployed. Utility is described by

$$\begin{cases} \tilde{u}(c_E; \alpha, \omega) & \text{if she participates,} \\ \tilde{v}(c_U; \alpha, \omega) & \text{if she does not work,} \end{cases} \tag{1}$$

and the decision is the one that yields the highest utility. The utility functions $\tilde{u}$ and $\tilde{v}$ are assumed to be twice continuously differentiable. They are increasing and concave in (nonnegative) consumption.

The *work opportunity cost* is the (possibly negative) sum of money which, given to an agent if she works on top of the subsistence income she has while unemployed, makes her indifferent between working or not. Formally, from the monotonicity of $\tilde{u}$, for each agent (α, ω) and for each level of consumption when not working c_U, there exists a number $\delta = \Delta(\alpha, \omega, c_U)$ such that the agent wants to work ($\tilde{u} > \tilde{v}$) if $c_E > c_U + \delta$, does not want to work if $c_E < c_U + \delta$ and is indifferent if $c_E = c_U + \delta$, i.e. δ is solution to the equation[2]

$$\tilde{u}(c_U + \delta, \alpha, \omega) = \tilde{v}(c_U, \alpha, \omega). \tag{2}$$

The threshold δ is the work opportunity cost of the agent. In the terminology of labor supply, the net wage or the net gain from work is the difference $c_E - c_U$, and an agent works whenever her net wage exceeds her opportunity cost of work. The subsistence income or other income unrelated to work is c_U. Labor supply is subject to an income effect when the work opportunity cost $\Delta(\alpha, \omega, c_U)$ depends on other income c_U. If leisure is a normal good, labor supply decreases with other income, i.e. $\Delta(\alpha, \omega, c_U)$ is nondecreasing in c_U.

An economy is defined by a pair of utility functions $(\tilde{u}, \tilde{v})$ and a distribution of characteristics (α, ω). It is convenient to substitute the work opportunity cost δ

[2] We let δ equal to $-c_U$ for persons who always want to work, and δ equal $+\infty$ for persons that never work. We do not rule out $\delta < 0$: some agents may be ready to pay to go to work.

for the multidimensional parameter α. Accordingly, we define the average utilities of the agents who have the same productivity ω and work opportunity cost δ as

$$\begin{cases} u(c_E;\delta,\omega,c_U) &= \mathbb{E}_\alpha\left[\tilde{u}(c_E;\alpha,\omega) \mid \delta = \Delta(\alpha,\omega,c_U),\omega,c_U\right], \\ v(c_U;\delta,\omega) &= \mathbb{E}_\alpha\left[\tilde{v}(c_U;\alpha,\omega) \mid \delta = \Delta(\alpha,\omega,c_U),\omega,c_U\right]. \end{cases} \tag{3}$$

Note that the function u generally depends on the subsistence level c_U, except when there are no income effects (Δ independent of c_U). In general, when the heterogeneity dimension is at least equal to 2, we conjecture that the structural model does not restrict the shape of the reduced form utilities (see example in the first section of the Appendix). However this is not the case when α is of dimension 1 (for notation simplicity, we have dropped the ω argument in the statement of the Lemma):

Lemma 1. *When there is one dimension of heterogeneity (α is scalar), the reduced-form model satisfies*

$$u(c_E;\delta,c_U) \geq v(c_U;\delta) \Longleftrightarrow c_E \geq c_U + \delta. \tag{4}$$

and

$$u_2(c_E;\delta,c_U)/u_3(c_E;\delta,c_U) \text{ does not depend on } c_E. \tag{5}$$

Conversely, if a reduced-form model (u, v) satisfies conditions (4) and (5), then there exists a structural model $(\tilde{u}, \tilde{v})$ with unidimensional heterogeneity satisfying (3).

While the structural model induces little restrictions on the shape of the utility functions, it has important consequences on the distribution of work opportunity costs. For each value of c_U, the distribution of the underlying parameters induces a distribution on (δ, ω). We denote $F(.|\omega, c_U)$ the cumulative distribution of work opportunity costs of persons of productivity ω when out of work income is equal to c_U: $F(c_E - c_U|\omega, c_U)$ is the proportion of agents of productivity ω that want to work when income at work is equal to c_E and income out of work is c_U. Note that, from the monotonicity of $\tilde{v}$, the function $F(c_E - c_U|\omega, c_U)$, seen as a function of c_U, is nonincreasing in c_U. Also, when leisure is a normal good, an increase in c_U, keeping constant the incentive to work $c_E - c_U$, decreases labor supply. To summarize, when F is differentiable with respect to its arguments, we have

$$\frac{\mathrm{d}}{\mathrm{d}c_U} F(c_E - c_U|\omega, c_U) = -\frac{\partial F(\delta|\omega, c_U)}{\partial \delta} + \frac{\partial F(\delta|\omega, c_U)}{\partial c_U} \leq 0, \tag{6}$$

and, when leisure is a normal good,

$$\frac{\partial F(\delta|\omega, c_U)}{\partial c_U} \leq 0.$$

The structural and the reduced forms of the model are linked through two channels. The shape of the function $\Delta(\alpha, \omega, c_U)$ is one of them. The implicit function theorem, when valid, gives

$$\tilde{u}'_1 \frac{\partial \Delta}{\partial \omega} = \tilde{v}'_\omega - \tilde{u}'_\omega \quad \text{and} \quad \tilde{u}'_1 \frac{\partial \Delta}{\partial \alpha} = \tilde{v}'_\alpha - \tilde{u}'_\alpha.$$

The heterogeneity of the work opportunity cost δ, given the person's productivity ω which is observed by the government for the workers, is an important feature of the model (see Section 7 for a discussion): in an interesting model, the utility functions must have a (generically) non zero $\tilde{v}'_\alpha - \tilde{u}'_\alpha$. The other component that determines the reduced form is the distribution of α conditional on ω. It turns out that the model is much easier to analyze when the distribution of work opportunity costs, as well as the reduced form utilities, do not vary with productivity:

Assumption 1. *The functions $u(c_E; \delta, \omega, c_U)$ and $v(c_U; \delta, \omega)$ and the distribution of opportunity costs $F(.|\omega, c_u)$ do not depend on the productivity level ω.*

The main results of the paper are derived under this assumption. Some additional properties are presented in the general case in the last two sections.

Most of the results carry over to a situation where the distributions have mass points, provided that there is genuine heterogeneity in the work opportunity costs (see section 7). To simplify the presentation, we focus on the case where distributions are continuous.

Assumption 2. *The marginal distribution of productivities ω has support $\Omega = [\underline{\omega}, \overline{\omega}]$, an interval of the positive line. Its cumulative distribution function G has a continuous positive derivative g everywhere on the support.*

The distribution of δ, conditional on ω and c_U, is continuous with support $[\underline{\delta}, \overline{\delta}]$, $\underline{\delta} < \overline{\delta}$, and cumulative distribution function $F(.|\omega, c_U)$. Its probability distribution function $f(.|\omega, c_U)$ is positive and continuous everywhere on its support.

Benchmark case: A simple specification of particular interest is the following. The parameter α is unidimensional and the utility functions are defined through a concave twice differentiable increasing function U, such that

$$\tilde{u}(c_E; \alpha, \omega) = U(c_E - \alpha), \quad \tilde{v}(c_U; \alpha, \omega) = U(c_U).$$

Then direct substitutions yield $\delta(\alpha, \omega, c_U) = \alpha$ and

$$u(c_E; \delta, \omega, c_U) = U(c_E - \delta), \qquad v(c_U; \delta, \omega) = U(c_U).$$

The work opportunity cost of agent (α, ω) is simply equal to α. It is independent of the value of the subsistence income c_U: there are no income effects. Also the value attached by the government to the welfare of the unemployed agents only depend on their income, and does not vary with their characteristics (α, ω).

3 Government behavior and tax schedules

Under laissez-faire, subsistence income out of work is zero, and anyone with a work opportunity cost δ smaller than her productivity ω, which then is equal to income, works. To redistribute income, the government announces a nondecreasing[3] income schedule $R(.)$, which associates to any before tax income y, $y > 0$, a nonnegative disposable income $R(y)$ and gives to the non workers a subsistence income s. To the income schedule, one can associate the tax rate $\tau(y)$ faced by the worker

$$\tau(y) = \frac{y - (R(y) - s)}{y} = 1 - \frac{R(y) - s}{y}.$$

At laissez-faire, $(R(y) = y, s = 0)$, the tax rate $\tau(y)$ is equal to zero. A person of productivity ω decides to work when her financial incentive ω is larger than her work opportunity cost $\Delta(\alpha, \omega, 0)$, where the third argument of Δ, the subsistence income, is zero in the absence of any government safety net. In a second best allocation, the person works when

$$R(\omega) - s > \Delta(\alpha, \omega, s).$$

Now the tax rate faced by a person of productivity ω is *positive* if and only if $\omega > R(\omega) - s$, i.e. when the financial incentive to work is smaller at the second best allocation than at laissez-faire. Then, if leisure is a normal good, that is Δ is nonincreasing in its third argument, labor supply is unambiguously distorted downwards compared to laissez-faire: any unemployed person at laissez-faire, with $\omega < \Delta(\alpha, \omega, 0)$, is still unemployed at second best, since $R(\omega) - s < \omega < \Delta(\alpha, \omega, 0) \leq \Delta(\alpha, \omega, s)$.

When the tax rate is *negative*, the financial incentive to work is larger at the second best than at laissez-faire, $\omega < R(\omega) - s$. In this circumstance, every agent of productivity ω, with or without a job, prefers the second best to laissez-faire $(\max[\tilde{u}(R(\omega); \alpha, \omega), \tilde{v}(s; \alpha, \omega)] \geq \max[\tilde{u}(\omega; \alpha, \omega), \tilde{v}(0; \alpha, \omega)])$. But, contrary to what one might expect by symmetry with the positive rate case, labor supply is not necessarily distorted upwards: if income effects are strong enough, so that utility at home, as well as the work opportunity cost, increase a lot with subsistence income, labor supply may be smaller at the second best allocation, even though financial incentives to work are larger than at laissez-faire. Of course, in

[3] We have assumed that the workers always produce at their productivity level ω. A slightly different presentation would have the workers indifferent to generate any ex-ante income in the interval$[0, \omega]$, and the government announce a possibly sometimes decreasing tax function $\tilde{R}(.)$. Then a worker would choose to produce y solution to $\max_{y \leq \omega} \tilde{R}(y)$, as in the extensive model her work opportunity cost is independent of y. The schedule $R(.)$, defined by $R(y) = \max_{z \leq y} \tilde{R}(z)$, dominates $\tilde{R}$: workers facing R have no interest to produce $y < \omega$. In the language of mechanism design theory, when an agent can mimic any agent of lower productivity, incentive compatibility is equivalent to the monotonicity of the income schedule.

the absence of income effects, when work opportunity costs are independent of the level of subsistence income, labor supply distortion is of the opposite sign of the tax rate, compared with laissez-faire.[4]

The utility functions are normalized so that the objective of the utilitarian government is the sum of the utilities of the participants in the economy, which takes the form

$$\begin{aligned}\mathbb{E}_{\omega,\alpha}\max\left[\tilde{u}(R(\omega);\alpha,\omega),\tilde{v}(s;\alpha,\omega)\right] &= \mathbb{E}_{\omega,\alpha}\left\{\tilde{u}\ \mathbb{1}_{R(\omega)-s\geq\delta}+\tilde{v}\ \mathbb{1}_{R(\omega)-s<\delta}\right\}\\ &= \mathbb{E}_{\omega,\delta}\left\{u\ \mathbb{1}_{R(\omega)-s\geq\delta}+v\ \mathbb{1}_{R(\omega)-s<\delta}\right\},\end{aligned}$$

where the second line follows from the law of iterated expectations, remarking that the participation decision only depends on δ and ω. The government objective thus is the expectation with respect to $G(\omega)$ of

$$\int_{\underline{\delta}}^{R(\omega)-s} u(R(\omega);\delta,\omega,s)\,\mathrm{d}F(\delta|\omega,s)+\int_{R(\omega)-s}^{\overline{\delta}} v(s;\delta,\omega)\,\mathrm{d}F(\delta|\omega,s). \tag{7}$$

The feasibility constraint is

$$\int_{\Omega}[\omega-R(\omega)+s]F(R(\omega)-s|\omega,s)\,\mathrm{d}G(\omega)=s. \tag{8}$$

Without loss of generality, we can impose the additional constraint: $R(\omega)-s\geq\underline{\delta}$ for all $\omega\in\Omega$.[5]

Let λ be the multiplier associated with the feasibility constraint (8). The Lagrangian of the problem is $\mathbb{E}_{\omega}L(R(\omega),s;\omega)$, with

$$\begin{aligned}L(R,s;\omega) &= \int_{\underline{\delta}}^{R-s} u(R;\delta,\omega,s)\,\mathrm{d}F(\delta|\omega,s)+\int_{R-s}^{\overline{\delta}} v(s;\delta,\omega)\,\mathrm{d}F(\delta|\omega,s)\\ &\quad +\lambda[\omega-R+s]F(R-s|\omega,s)-\lambda s.\end{aligned}$$

The optimization takes place over (R,s,λ), where R is a nonnegative nondecreasing function and s and λ are nonnegative. The function L is not concave in its

[4]Formally, the labor supply of agents with productivity ω is (weakly) distorted upwards if and only if

$$\omega\geq\Delta(\alpha,\omega,0)\implies R(\omega)\geq s+\Delta(\alpha,\omega,s).$$

When the function R is increasing, which is typically the case, the previous condition is equivalent to

$$R(\Delta(\alpha,\omega,0))\geq s+\Delta(\alpha,\omega,s),$$

which shows that the tax rate is unambiguously linked with the distortion of labor supply only when there are no income effects, Δ independent of its third argument.

[5]The objective does not depend on the precise value taken by $R(\omega)$ in the region where the set of workers is negligible, i.e. whenever $R(\omega)-s$ is smaller than or equal to the minimal work opportunity cost $\underline{\delta}$. That is, if $R(w)-s<\underline{\delta}$ is optimal, then $R(w)-s=\underline{\delta}$ is optimal as well.

arguments, since the cdf of the work opportunity cost F is typically not concave. Any utilitarian optimum however must satisfy first order necessary conditions, which turn out to be especially fruitful here. In the next two sections, we concentrate on the study of $R(.)$, for given values of s and λ, before commenting briefly on the full program in Section 6.

4 First order conditions for $R(\omega)$

The derivative of L with respect to R plays a central role in the analysis

$$\frac{\partial L}{\partial R}(R, s; \omega) = \lambda[\omega - R + s] f(R - s|\omega, s) - F(R - s|\omega, s) \left[\lambda - p_E(R, s|\omega)\right], \quad (9)$$

where $p_E(R, s|\omega)$ is the average social weight of the working agents of productivity ω:

$$\begin{aligned} p_E(R, s|\omega) &= \mathbb{E}_\alpha \left[\tilde{u}_1'(R; \alpha, \omega) \mid \delta \leq R - s, \omega, s \right] \\ &= \frac{1}{F(R - s|\omega, s)} \int_{\underline{\delta}}^{R-s} u_1'(R; \delta, \omega, s) \, \mathrm{d}F(\delta|\omega, s). \end{aligned} \quad (10)$$

The expression of $\partial L / \partial R$ has a direct economic interpretation. The first term $\lambda[\omega - R + s] f(R - s|\omega, s)$ is the gain in government income obtained from the new $f(R - s|\omega, s)$ workers who participate following an increase in R: they produce ω, they are paid $R(\omega)$ but do not receive the subsistence income s anymore. The second term $F(R - s|\omega, s)[\lambda - p_E(R, s)]$ is the loss on the existing workers: the marginal cost is λ per worker, while the social value of this distribution of income is equal to the average social weight of the employees of productivity ω.

At any point ω where R is strictly increasing (no pooling), it satisfies the first order condition for a pointwise maximum

$$\frac{\partial L}{\partial R}(R, s; \omega) = 0. \quad (11)$$

The average tax rate supported by the workers of productivity ω is $\tau(\omega) = [\omega - R(\omega) + s]/\omega$, so that the first order condition can be rewritten as

$$\omega - R + s = \omega \tau(\omega) = \frac{F(R - s|\omega, s)}{f(R - s|\omega, s)} \left[1 - \frac{p_E(R, s|\omega)}{\lambda}\right]. \quad (12)$$

At a point ω where the tax schedule $R(\omega)$ is strictly increasing and $R - s$ is in the interval $(\underline{\delta}, \overline{\delta})$, i.e. some, but not all, agents of productivity ω want to work, we have

- either $p_E(R(\omega), s|\omega) < \lambda$, the financial incentive to work $R(\omega) - s$ is smaller than before tax income ω: it is distorted downwards compared to laissez-faire;

- or $p_E(R(\omega), s|\omega) = \lambda$, the financial incentive to work $R(\omega) - s$ equals before tax income ω: financial incentives to work are the same as in laissez-faire and the workers are better off than at the laissez-faire;

- or $p_E(R(\omega), s|\omega) > \lambda$, the financial incentive to work $R(\omega) - s$ is larger than before tax income ω: it is distorted upwards compared to laissez-faire and all the persons of productivity ω workers are better off.

Social weights larger than λ, corresponding to a group of employees whose average social weight is larger than the marginal cost of public funds, receive a financial incentive to work $R(\omega) - s$ larger than their productivity ω, distorted upwards compared with laissez-faire.[6]

Remark: When opportunity costs vary with productivities (Assumption 1 is not satisfied), the optimum may involve pooling, with regions where R stays constant because of the monotonicity condition. In a pooling interval $[\omega_1, \omega_2]$, whenever $R - s$ does not hit the lower bound $\max_{\omega \in [\omega_1, \omega_2]} \underline{\delta}(\omega)$, the first order conditions become

$$\int_{\omega_1}^{\omega_2} \frac{\partial L}{\partial R}(R, s; \omega) \, \mathrm{d}G(\omega) = 0 \tag{13}$$

and

$$\int_{\omega}^{\omega_2} \frac{\partial L}{\partial R}(R, s; \omega) \, \mathrm{d}G(\omega) \leq 0 \tag{14}$$

for all $\omega_1 \leq \omega \leq \omega_2$. Considering a neighborhood of the extremities of the pooling interval, the last condition implies

$$\frac{\partial L}{\partial R}(R, s; \omega_1) \geq 0 \quad \text{and} \quad \frac{\partial L}{\partial R}(R, s; \omega_2) \leq 0. \tag{15}$$

Note that the first order condition, under regularity conditions, can be differentiated into

$$\frac{\partial^2 L}{\partial R^2} R' + \frac{\partial^2 L}{\partial R \partial \omega} = 0.$$

At a local maximum, $\partial^2 L / \partial R^2$ is negative, so that the monotony of R supposes $\partial^2 L / \partial R \partial \omega \geq 0$. Conversely, pooling only prevails if there is a region where $\partial^2 L / \partial R \partial \omega$ is negative (see equations 15).

[6] In the intensive model also, the sign of the *marginal* tax rate follows from the comparison of the cost of public funds with the average value of social weights in a certain group of workers. But the relevant groups differs in the intensive and extensive models. In the former case, what matters is the population of workers whose productivity is greater than or equal to a given level; in the latter, only workers with a given productivity are considered. This difference follows from the informational structures: the workers' productivities are observed in the extensive model, whereas this information has to be extracted in the intensive case.

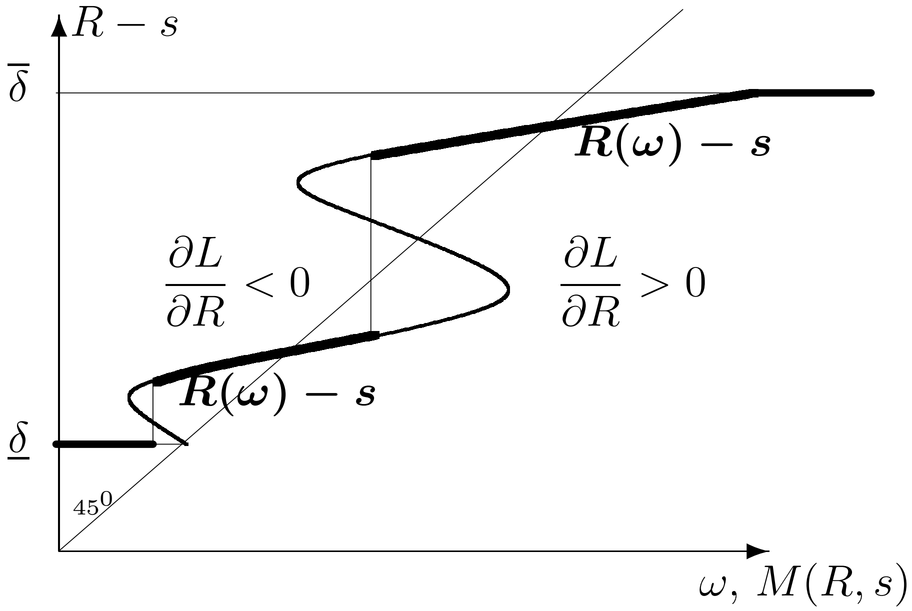

Figure 1: Financial incentives to work

5 The shape of optimal tax schemes

5.1 Unrestricted second best tax schedules

Under assumptions 1 and 2, one can describe more precisely the qualitative properties of the optimal tax schedule. The geometrical construction of a schedule is shown on Figure 1. For $R - s$ in the support of the distribution of work opportunity cost, let:

$$M(R,s) = R - s + \frac{F(R-s)}{f(R-s)} \left[1 - \frac{p_E(R,s)}{\lambda}\right]. \tag{16}$$

The first order condition $\partial L/\partial R = 0$ is equivalent to the equality $\omega = M(R,s)$. With $R - s$ on the vertical axis, we draw the function M on the x-axis, which is also the ω-axis. By construction, for ω smaller than the minimum of M, on the left of the graph of M, the derivative of the Lagrangian with respect to R is negative, while on the right of the graph it is positive. To look for the pointwise maximum of the Lagrangian, draw the vertical line through ω. When it does not intersect the graph of M (i.e. ω is outside the range of M), then the optimum is to have everybody unemployed ($R - s \leq \underline{\delta}$) when the line is on the left of the graph, to have everyone employed ($R - s \geq \overline{\delta}$) when it is on the right of the graph. Consider now an intersection point: if the slope of M is negative at the point, this is a local minimum ($\partial L/\partial R$ is negative for smaller R's, positive for larger R's); the local maxima are the intersection points where the slope of M is positive. The value of $R(\omega)$ is the one that yields the higher L, comparing all the local maxima and the two corners (full employment, or no employment). This procedure is justified by the following proposition.

Proposition 1. *Consider an economy that satisfies Assumptions 1 and 2.*

Then, given the subsistence income s and the marginal cost of public funds λ, pointwise maximization of the Lagrangian for each ω yields the optimal tax scheme. There is no pooling: for any ω such that $\underline{\delta} < R(\omega) - s < \overline{\delta}$, $\omega' > \omega$ implies $R(\omega') > R(\omega)$.

Furthermore, the optimal tax scheme is continuous whenever M is non-decreasing. If the support of productivities is included in the range of M, the tax scheme is discontinuous whenever M is decreasing on part of its domain.

The no-pooling property holds in the interval of productivities for which there are non negligible sets of both employees and unemployed, say $[\omega_\ell, \omega_h]$. Then R is increasing and no employee has interest in faking a smaller productivity than her own: she then would receive a smaller after tax income. People with lower productivities than ω_ℓ are unemployed and receive the subsistence income s: they are treated as all the unemployed whose productivities are not observed by the government. Persons with a higher productivity than ω_h are all working. Typically, the optimal function R is constant for ω larger than ω_h, and the high skilled workers under consideration are indifferent between working at their full productivities ω or at any productivity in the interval $[\omega_h, \omega]$. We suppose that they do not shirk.

It may be useful to spell out what happens at a discontinuity point of the tax schedule, for instance when one jumps from $R_- < \omega + s$ to $R_+ > \omega + s$ at some productivity ω. At R_-, there is a (relatively) small number of employees, $F(R_- - s|s)$ and their average social weights $p_E(R_-, s)$ is smaller than λ, since we supposed that the point lies below the 45 degree line on the graph. At R_+, there are many more employees: all the persons with a work opportunity cost in the interval $[R_- - s\ ,\ R_+ - s]$ who are unemployed when their productivities are slightly smaller than ω take a job when their productivity is larger than ω. Since the M curve then is above the 45 degree line, the average social weight of the employees $p_E(R_+, s)$ is larger then λ. For a discontinuity to occur, when the distributions are smooth (Assumption 2), the average social weight of the persons of work opportunity costs in $[R_- - s\ ,\ R_+ - s]$ has to be larger than that of persons with work opportunity costs lower than $(R_- - s)$.

The previous remark indicates that, as in Stiglitz (1982), the social weights p_E play a crucial role in shaping the income tax schedule. We therefore now look into the relationships of social weights with the government objective.

5.2 Redistribution

5.2.1 The social weights of the employees

Typically in the intensive model the more productive agents are richer and better off at the optimum so that with identical preferences social weights decrease with income. This behavior of the social weights, p_E decreases with R, is less straightforward in the extensive model.[7]

Indeed brute differentiation yields

$$\begin{aligned}\frac{\partial p_E}{\partial R} &= \frac{1}{F(R-s|\omega,s)}\int_{\underline{\delta}}^{R-s} u''_{11}(R;\delta,\omega,s)\,\mathrm{d}F(\delta|\omega,s) \\ &\quad -\frac{f(R-s|\omega,s)}{F(R-s|\omega,s)}p_E(R,s|\omega) \\ &\quad +\frac{1}{F(R-s|\omega,s)}u'_1(R;R-s,\omega,s)f(R-s|\omega,s).\end{aligned}$$

The first two terms are negative, following the intuition, but the last term is positive: an increase in R brings newcomers into employment, whose weights may be high. Nevertheless one can derive the desired property under two assumptions. The first one states that the new entrants in the labor force do not have too high a marginal utility of income:

Assumption 3. *The marginal utility of income of the new entrants into the labor force is nonincreasing in R for all s and ω: $u'_1(R;R-s,\omega,s)$ is nonincreasing in R.*

This assumption does not seem overly restrictive. For instance, in the benchmark case described at the end of Section 2, the marginal social utility of newcomers equals $U'(s)$ and does not depend on R. More generally, the assumption is satisfied whenever the marginal utility of income $u'_1(R;\delta,\omega,s)$ is decreasing in the opportunity cost of working δ. The second assumption, the log concavity of the cumulative distribution of work opportunity costs, is a stronger requirement.

Assumption 4. *The distribution of work opportunity costs is log-concave.*

We show in the Appendix:

Proposition 2. *Under Assumptions 3 and 4, the average social weight of the employed of productivity ω, $p_E(R,s|\omega)$, is a nonincreasing function of R.*

[7]We provide here structural foundations to Saez (2002)'s statement, page 1049:

> *If the government values redistribution, then the lower the earnings level of the individual, the higher the social marginal value of an extra dollar for that individual. As a result, the weights g_i are decreasing in i.*

5.2.2 Utilitarian tax schedules

We proceed to describing the shapes of the optimal tax schedules under the two assumptions 3 and 4, which ensure that the social weights of the employees decrease with their incomes.[8]

Proposition 3. *Under Assumptions 1 to 4, if different from laissez-faire, the optimal income tax schedule satisfies one of the two following properties in addition to those of Proposition 1:*

1. *either the incentive to work is always smaller than productivity, and labor supply is everywhere distorted downwards, for $\omega > \underline{\delta} + s$;*

2. *or there is a value $R_m > \underline{\delta} + s$ of income and an associated productivity $\omega_m = R_m + s$ such that labor supply is distorted downwards for $\omega > \omega_m$, while the financial incentive to work is undistorted or distorted upwards for $\omega_m > \omega > \underline{\delta} + s$.*

In both cases, in the region where labor supply is distorted downwards, after tax income is a continuous function of before tax income, with a slope smaller than 1.

Figure 2, where $\underline{\omega} \leq \underline{\delta}$, illustrates the foregoing proposition. Case 1 corresponds to the dotted curve. The graph of M lies below the 45 degree line; all the workers are taxed; their financial incentive to work is distorted downwards compared to laissez-faire, where they would be better off than in the second best. This situation is typical of a highly redistributive government and holds in particular under the Rawlsian objective, where p_E is zero (see e.g. Choné and Laroque (2005)). Case 2 is associated with the bold solid curve, which here is drawn in a situation where the M curve is monotonic so that incentives to work $R(\omega) - s$ are a continuous increasing function of productivity.[9] There is a low skilled region, $\underline{\delta} \leq \omega \leq R_m - s$, where labor supply is distorted upwards.

Proposition 3 rules out discontinuities of the tax schedule in the region where labor supply is distorted downwards, but it does not preclude them below R_m for low incomes, when financial incentives to work are distorted upwards. The following example, which satisfies all the Assumptions of the paper, illustrates this fact with a uniform distribution of opportunity costs:

[8] Other properties of the optimal tax schedules, in relationship with the Rawlsian criterion may be worth recalling. Theorem 6 of Choné and Laroque (2005) applies here: all the utilitarian optimal incentive schemes are located above the Rawlsian curve. Theorem 3 of Laroque (2005) also applies: any incentive scheme above the Laffer curve which does not overtax and such that $R(\omega) - s \leq \omega$ corresponds to a second best optimal allocation. Note that in a benchmark model, from the above results, none of these allocations satisfy a utilitarian criterion. All the utilitarian optimal allocations are such that $R(\omega) - s > \omega$ for some ω's, a property discussed in Remark 2.3 of Laroque (2005).

[9] It is similar to Figure IIa in Saez (2002), who discusses from a more applied perspective the occurrence of negative marginal tax rates.

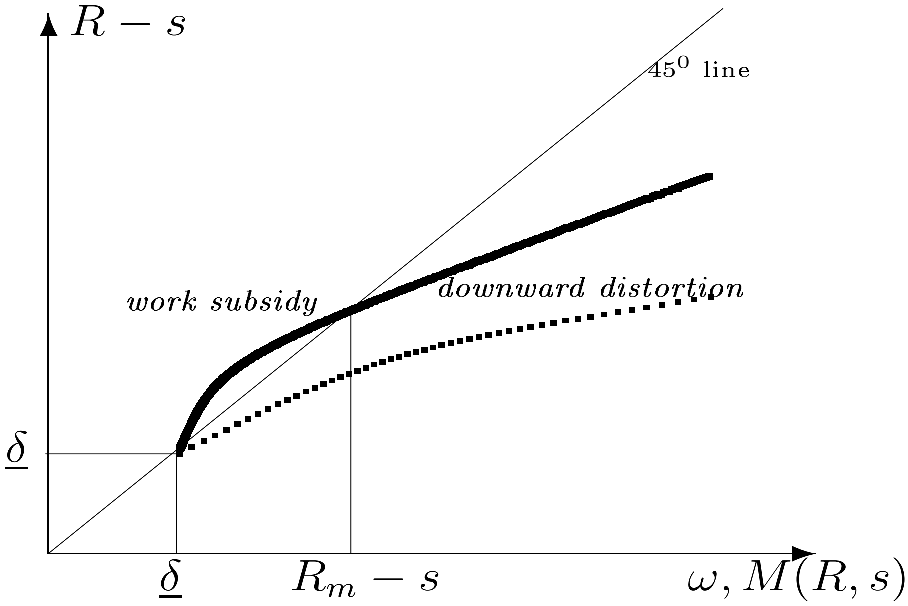

Figure 2: Two 'well behaved' optimal tax schemes

Example 1. *Consider a benchmark economy satisfying Assumptions 1 and 2. Suppose that the opportunity cost δ is uniformly distributed on $[\underline{\delta}, \overline{\delta}]$ and that $s + \underline{\delta} < R_m < s + \overline{\delta}$.*

Then $R(\omega)$ is increasing and concave whenever some agents of productivity ω work, i.e. on the set $\{\omega | R(\omega) - s > \underline{\delta}\}$. Two cases arise depending on the social weight of the unemployed,[10] p_U:

1. *If $p_U \leq 2\lambda$, none of the agents of productivity smaller than $\underline{\delta}$ work at the optimum, $R(\underline{\delta}) - s = \underline{\delta}$, and the slope of the income schedule at $\underline{\delta}$ satisfies: $R'_+(\underline{\delta}) = 1/[2\lambda - p_U] > 0$.*

2. *If $p_U > 2\lambda$, there exists ω_0, $\underline{\omega} \leq \omega_0$, such that $R(\omega) - s > \underline{\delta}$ for all $\omega \geq \omega_0$ and $R(\omega) - s \leq \underline{\delta}$ for productivities smaller than ω_0. When $\underline{\omega} < \omega_0$, there is an upward discontinuity in the incentives to work at ω_0.*

The situation where the social weights of the unemployed agents are high ($p_U > 2\lambda$) is shown on Figure 3, the bold line representing the optimal tax scheme. None of the agents with very low productivities, $\omega < \omega_0$, work. But for all ω larger than or equal to ω_0, a fraction of the agents does. In fact the

[10]We note the (unconditional) social weight of the whole population of unemployed agents

$$p_U = \mathbb{E}_{\alpha,w} \left[\tilde{v}_1'(s; \alpha, \omega) \mathbb{1}_{R(\omega)-s<\delta} \right].$$

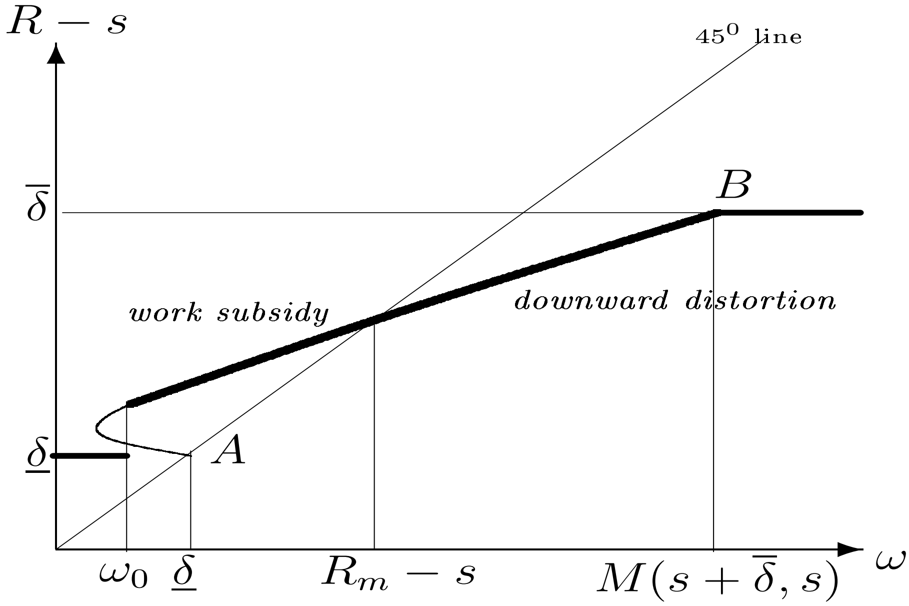

Figure 3: Strong upward distortion (uniformly distributed opportunity costs)

upward distortion to labor supply here is particularly strong: some agents with productivity smaller than the minimal cost of going to work participate in the labor force. Note that after tax income is a concave function of income, implying a progressive tax system. There is an upward discontinuity in the tax schedule at ω_0.

6 The full program

6.1 The marginal cost of public funds

The preceding section has studied the shape of the optimal tax schemes given the levels of subsistence income s and of the marginal cost of public funds λ. We need two equations to determine these two quantities. One is the feasibility condition (8). The other is a first order necessary condition, associated with a small translation of all incomes, i.e. an equal marginal change in both s and $R(\omega)$ for all ω. It is proved in the Appendix, without restrictions on the dependence of work opportunity costs on productivities (Assumption 1 is not used), allowing for pooling. The social weights of the employees, defined in (10), and of the unemployed

$$p_U(R, s|\omega) = \mathbb{E}_\alpha \; [\tilde{v}_1'(s; \alpha, \omega) \mid \delta > R - s, \; \omega, s \,] \,, \tag{17}$$

are the ingredients of the Lemma.

Lemma 2. *Under Assumption 2, for all ω, there exists $\rho(\omega) < 1$ such that, at an optimum*

$$\int_\Omega \{(1-\rho)\ F\ p_E(R(\omega), s|\omega) + (1-F)\ p_U(R(\omega), s|\omega)\} \mathrm{d}G(\omega) = \int_\Omega \lambda(1-\rho F)\mathrm{d}G(\omega), \tag{18}$$

where F stands for $F(R(\omega) - s|\omega, s)$.

Absent income effects, ρ is equal to 0 for all ω. When leisure is a normal good, ρ is a non positive function of ω.

In the case where work opportunity costs do not depend on the level of the subsistence income (no income effects), the lemma indicates that the familiar equality of the marginal cost of public funds to the average of the agents marginal utilities of income holds here: ρ is equal to zero, and (18) simplifies into

$$\int_{\underline{\delta}}^{R-s} u_1'(R; \delta, \omega, s)\, \mathrm{d}F(\delta|\omega, s) + \int_{R-s}^{\overline{\delta}} v_1'(s; \delta, \omega)\, \mathrm{d}F(\delta|\omega, s) = \lambda.$$

In the presence of income effects, the coefficients of p_E and of p_U are weighted. The weights are nonnegative and sum up to the coefficient of λ: $F(1-\rho) + (1-F) = 1 - \rho F$. When leisure is a normal good, ρ is negative and the employees are given a higher importance than in the absence of income effects.

6.2 When are negative tax rates optimal?

When does a redistributive government, in an economy that satisfies all the assumptions of the paper, implement negative taxes? The theory helps to answer the question. In general, utility may depend in a complicated way on a person's characteristics (productivity, work opportunity cost) and on her work status and income. However, the fact that the marginal employees are indifferent between working and not working, with work opportunity costs smaller than that of the unemployed, holds in any labor supply model. Suppose that the utilities of the unemployed are a nondecreasing function of their work opportunity costs, as in the benchmark model. Then a redistributive government would impute them a lower social weight than that of the marginal employees. From (18), it follows that the social weight of the lower income employees is larger than the marginal cost of public funds if there is any redistribution at all.[11] Then, if social weights vary continuously with income, there is a range of productivities which include the lower skills, where the social weights of the workers is larger than the marginal cost of public funds. It follows that there are *upward labor supply distortions for*

[11] Otherwise all the social weights on the left hand side of (18) are at most equal to λ, so that for the equality to hold, all weights must be equal to λ, which corresponds to the laissez-faire equilibrium.

low productivity workers at the optimum, and more 'working poors' than at the laissez-faire equilibrium.

On the other hand, situations where the social weight attached to the unemployed agents is larger than that attached to the employees abound: for instance this may be the case in the presence of 'involuntary' unemployment, or when a large opportunity cost to work is associated with a handicap (the marginal social weight $v'(s; \delta, \omega)$ is increasing with δ). It is then easy to think of economies where at the optimum the average social weight of the unemployed is larger than the marginal cost of public funds and the social weight of the lowest paid workers is smaller. In these economies, after tax income is everywhere smaller than productivity.

Similarly, the analysis has proceeded under the assumption that the social welfare function is smooth, so that the distribution of the agents' weights has no mass point. The case of a Rawlsian planner who puts all the weight on the least favored agent in the economy corresponds here to a situation where p_E is equal to zero everywhere. Then the tax rate is always positive. This is in line with the results of Choné and Laroque (2005). On the other hand, other social choice criteria, such as the one advocated in Fleurbaey and Maniquet (2006), may put the weight on the deserving 'working poors'. As a consequence, in an intensive model with multiple dimensions of heterogeneity, Fleurbaey and Maniquet (2006) show that optimal taxation implies subsidizing as much as possible the poorest workers.

It would be of interest to know whether and when the subsidy result still holds in the *mixed* situation where both the extensive and intensive margins operate. Boone and Bovenberg (2004) analyze such a model where the utility is quasi linear. There is a fixed cost of searching for a job work which is constant across the population, and the random outcome of search creates heterogeneity. They find cases where work is subsidized (Section 4.3), but do not characterize them in terms of economic fundamentals. More work is needed in this area.

7 An example without heterogeneity

Homburg (2002) considers the limit case where everyone has the same work opportunity cost, say δ_0, a situation where Assumption 2 does not hold. In the benchmark model (see end of Section 2), the objective takes the form

$$L(R, s; \omega) = \begin{cases} U(s) - \lambda s & \text{if } R - s < \delta_0, \\ U(R - \delta_0) + \lambda[\omega - R] & \text{if } R - s > \delta_0, \end{cases}$$

The Lagrangian is equal to any of the two above quantities when $R - s = \delta_0$: the agent is indifferent between working or not, and the planner can choose the preferred outcome. The problem is to maximize the integral of the Lagrangian subject to the feasibility constraint over R, R nondecreasing.

Absent pooling, the first order condition (11) with respect to $R(\omega)$ for an unconstrained worker of productivity ω would be:

$$U'(R(\omega) - \delta_0) = \lambda.$$

Since the solution $R(\omega)$ does not depend on productivity, there is full pooling. Furthermore the feasibility constraint (8) here becomes

$$\int_\Omega [\omega - R + s] \mathbb{1}_{\text{workers}} \, \mathrm{d}G(\omega) = s.$$

An optimum is characterized by two quantities (R, s) linked by the feasibility constraint. Furthermore it has to specify the set of workers when $R - s$ is equal to δ_0. Consequently, the first order conditions lead to three possibilities:

1. If $R - \delta_0 < s$, nobody works. By feasibility s is equal to zero. The value of social welfare, $U(0)$, is at a global minimum when the economy is productive enough.

2. If $R - \delta_0 > s$, everybody works. The constant income R is equal to the average productivity in the economy $\int_\Omega \omega \, \mathrm{d}G(\omega)$. The social welfare is equal to
$$U\left(\int_\Omega \omega \, \mathrm{d}G(\omega) - \delta_0\right),$$
which is only defined when $\int_\Omega \omega \, \mathrm{d}G(\omega) \geq \delta_0$.

3. Finally, when there is indifference between working or not, $R - \delta_0 = s$, and looking at the expression of $L(R, s; \omega)$, the planner decides to put to work the agents with productivity ω at least as large as $R - s$. The feasibility condition gives the value of s
$$\int_\Omega \max[\omega - \delta_0, 0] \, \mathrm{d}G(\omega) = s \geq \int_\Omega \omega \, \mathrm{d}G(\omega) - \delta_0.$$

This last case is the optimum whenever there is a non negligible set of agents with $\omega > \delta_0$. Then there is no upward distortion of labor supply. The utilitarian optimum does not leave any surplus to the workers and everyone is treated equally. Heterogeneity in the form of some dispersion of work opportunity costs gives more scope for redistribution, associated with the unknown value of δ.

8 Conclusion

It appears that in the extensive model, negative *average* tax rates and upward distortions of the financial incentives to work of the less skilled workers are the rule rather than the exception. This property occurs in particular in the standard benchmark model, provided social weights vary continuously with the utility levels, when the unemployed workers have chosen not to work. This is in sharp contrast with the results that follow from the intensive model a la Mirrlees.

The results obtained here may hopefully be useful in the literature that relates empirical (extensive) labor supply models, taxation and government preferences, and looks for government preferences that may rationalize the observed tax system.

A natural extension of the framework would be to allow for both an extensive and intensive margins in the labor supply. This should be the subject of further work.

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A Appendix

A.1 Relationship between the structural and the reduced forms

Proof of Lemma 1: Consider the structural model (1) and assume that α is scalar and that, for any given c_U, the map $\alpha \to \delta(\alpha; c_U)$ is one-to-one. We note $\delta^{-1}(.; s)$ the inverse function: $\delta \to \delta^{-1}(\delta; c_U)$. We have

$$u(c_E; \delta, c_U) = \tilde{u}(c_E; \delta^{-1}(\delta; c_U)) \text{ and } v(c_U; \delta) = \tilde{v}(c_U; \delta^{-1}(\delta; c_U)). \tag{19}$$

Condition (2) is equivalent to condition (4). It follows from (19) that

$$u_2(c_E; \delta, c_U) = \frac{\tilde{u}_2(c_E; \delta^{-1}(\delta; c_U))}{\delta_1(\delta^{-1}(\delta, c_U); c_U)} \text{ and } u_3(c_E; \delta, c_U) = \frac{\tilde{u}_2(c_E; \delta^{-1}(\delta; c_U))}{\delta_2(\delta^{-1}(\delta, c_U); c_U)}.$$

which implies (5). Now, note that

$$u(c_E; \delta(\alpha, c_U), c_U) = \tilde{u}(c_E, \alpha).$$

It follows that, for any given (c_E, α), the function $s \to \delta(\alpha; c_U)$ satisfies the following ordinary differential equation:

$$u_2(c_E; \delta(\alpha; c_U), c_U)\delta_s(\alpha; c_U) + u_3(c_E; \delta(\alpha; c_U), c_U) = 0. \tag{20}$$

The income effect, that is the dependence of $\delta(\alpha; c_U)$ on c_U, is given by (20).

Conversely, consider two functions $u(c_E; \delta, c_U)$ and $v(c_U; \delta)$ satisfying conditions (4) and (5). Thanks to (5), the ordinary differential equation (20) *does not* depend on c_E. For a given initial condition at some point x_0, the solution of this equation does not depend on c_E. Using α to parameterize the initial condition, say $\delta(\alpha; x_0) = \alpha$, we get a family of functions $c_U \to \delta(\alpha; c_U)$ that characterize the behavior of the agent and the income effect. We define

$$\begin{cases} \tilde{u}(c_E; \alpha) = u(c_E; \delta(\alpha; c_U), c_U) \\ \tilde{v}(c_U; \alpha) = v(c_U; \delta(\alpha; c_U)). \end{cases}$$

Thanks to (20), the definition of $\tilde{u}$ is consistent, that is $\tilde{u}$ does not depend on c_U.
■

When the dimension of α is greater than 1, the condition (5) is not necessary any more, as the following example shows. We fix $M > 1$ and suppose that (α_1, α_2) is uniformly distributed on the square $[0, 1]^2$. Let $\tilde{u}$ and $\tilde{v}$ be defined by

$$\begin{cases} \tilde{u}(c_E; \alpha) = -\dfrac{1}{M + c_E - \alpha_1} \\ \tilde{v}(c_U; \alpha) = -\dfrac{1}{M + (1 + \alpha_2)c_U}. \end{cases} \tag{21}$$

The functions $\tilde{u}$ and $\tilde{v}$ are strictly increasing and strictly concave in income for all $\alpha = (\alpha_1, \alpha_2)$. It is easy to check that (2) holds with $\delta(\alpha_1, \alpha_2; c_U) = \alpha_1 + c_U\alpha_2$ and that α_1, conditionally on δ, is uniformly distributed.[12] If we note $[\underline{\alpha}_1(\delta, c_U), \overline{\alpha}_1(\delta, c_U)]$ the support of the conditional distribution, we have

$$u(c_E; \delta, c_U) = \frac{-1}{\overline{\alpha}_1(\delta, c_U) - \underline{\alpha}_1(\delta, c_U)} \int_{\underline{\alpha}_1(\delta, c_U)}^{\overline{\alpha}_1(\delta, c_U)} \frac{1}{M + c_R - \alpha_1} R(\omega) - s\alpha_1.$$

In the region where $c_U < \delta < 1$, we have $\underline{\alpha}_1(\delta, c_U) = \delta - c_U$, $\overline{\alpha}_1(\delta, c_U) = \delta$ and

$$u(c_E; \delta, c_U) = \frac{1}{c_U} \ln \frac{M + c_E - \delta}{M + c_E - \delta + c_U}$$

which does not satisfy (5).

A.2 Proof of Proposition 1

Pointwise maximization leads to the global maximum in the absence of pooling, so that the only thing to prove is the no pooling property. It is a straightforward consequence of single crossing. Under Assumption 1,

$$\frac{\partial^2 L}{\partial R \partial \omega} = \lambda f(R - s) > 0.$$

Then

$$L(R(\omega'), \omega') - L(R(\omega'), \omega) - L(R(\omega), \omega') + L(R(\omega), \omega) = \int_{R(\omega)}^{R(\omega')} \int_{\omega}^{\omega'} \frac{\partial^2 L}{\partial R \partial \omega} \,\mathrm{d}R \,\mathrm{d}\omega.$$

The left hand side is nonnegative from pointwise maximization. The right hand side is of the same sign as $R(\omega') - R(\omega)$. ■

A.3 Proof of Proposition 2

We have to show that

$$p_E(R, s|\omega) = \frac{1}{F(R - s|\omega, s)} \int_{\underline{\delta}(\omega)}^{R-s} \frac{\partial u}{\partial R}(R; \delta) \,\mathrm{d}F(\delta|\omega, s) = E_\delta \left[\frac{\partial u}{\partial R} | \delta \leq R - s, \omega, s\right] \tag{22}$$

decreases with R. To simplify notations, we drop the variables s and ω which are kept constant in the proof. Now an integration by parts yields

$$p_E(R) = \frac{\partial u}{\partial R}(R; R - s) - \frac{1}{F(R - s)} \int_{\underline{\delta}(\omega)}^{R-s} \frac{\partial^2 u}{\partial R \partial \delta}(R; \delta) F(\delta) \,\mathrm{d}\delta,$$

[12] Leisure is a normal good as δ increases with c_U.

or

$$p_E(R) = \frac{\partial u}{\partial R}(R; R-s) - E_\delta \left[\frac{\partial^2 u}{\partial R \partial \delta}(R;\delta) \frac{\tilde{F}(\delta|\omega)}{f(\delta|\omega)} |\delta \leq R - s \right].$$

Now differentiating (22) gives

$$\frac{\partial p_E}{\partial R}(R) = E_\delta \left[\frac{\partial^2 u}{\partial R^2}(R;\delta) |\delta \leq R-s \right] - \frac{f(R-s)}{F(R-s)} p_E(R) + \frac{f(R-s)}{F(R-s)} \frac{\partial u}{\partial R}(R; R-s).$$

Substituting the expression obtained for $p_E(R)$:

$$\frac{\partial p_E}{\partial R}(R) = E_\delta \left[\frac{\partial^2 u}{\partial R^2}(R;\delta) + \frac{\partial^2 u}{\partial R \partial \delta}(R;\delta) \frac{f(R-s)}{F(R-s)} \frac{F(\delta)}{f(\delta)} |\delta \leq R-s \right].$$

By the log-concavity of F, the factor

$$\frac{f(R-s)}{F(R-s)} \frac{F(\delta)}{f(\delta)}$$

is smaller than 1. By concavity of the utility function, the first term is negative. By the assumption on the marginal utility of income of the newcomer into the labor force

$$\frac{\partial^2 u}{\partial R^2}(R;\delta) + \frac{\partial^2 u}{\partial R \partial \delta}(R;\delta)$$

is non positive. The result follows. ∎

A.4 Proof of Proposition 3

We first prove the last statement of the Proposition. Since F is log-concave, whenever $p_E(R) < \lambda$, the last term in the expression of M in (16) is the product of two positive nondecreasing functions. The function M therefore is increasing, with a slope larger than 1.

We now show the first part of the Proposition. If $p_E(R)$ is smaller than λ for all R such that $R - s$ is in the support of δ, case 1 holds. Otherwise $\lambda \geq p_E(R)$ for all R implies that $\omega - R(\omega) + s \leq 0$, which in turn by feasibility (8) is only possible if all the inequalities are equalities and s is equal to zero. ∎

A.5 Example 1

Since there are no income effects ($\partial F / \partial s = 0$), equation (18) writes

$$\lambda = \int \int U'[s + \max(0, R(\omega) - s - \delta)] \, \mathrm{d}F(\delta) \, \mathrm{d}G(\omega).$$

Then

$$p_E(R) = \frac{1}{F(R-s)} \int_{\underline{\delta}}^{R-s} U'[R-\delta] \frac{\mathrm{d}\delta}{\overline{\delta}-\underline{\delta}} = \frac{U(R-\underline{\delta})-U(s)}{R-s-\underline{\delta}}.$$

Integrating and substituting yields

$$M(R,s) = 2(R-s) - \underline{\delta} - \frac{U(R-\underline{\delta})-U(s)}{\lambda}.$$

The function $M(R,s)$ is strictly convex in R and $M'_R(s+\underline{\delta},s) = 2 - p_U(s)/\lambda$.

1) Case $p_U(s) \leq 2\lambda$. $M(R,s)$ is strictly increasing in R and Proposition 3 applies. The convexity of $M(.,s)$ implies the concavity of $R(\omega)$.

2) Case $p_U(s) > 2\lambda$. As in the proof of Proposition 3, we consider the pointwise maximum of $L(R,s;\omega)$ for $R-s \geq \underline{\delta}$. Since it is increasing in ω, it satisfies the monotonicity condition and is the optimum.

Recall that $L(s+\underline{\delta},s;\omega)=0$. Now,

$$\frac{\partial L}{\partial R}(R,s;\omega) = \lambda(\omega - M(R,s))f(R-s) = \frac{\lambda}{\overline{\delta}-\underline{\delta}}\,(\omega - M(R,s))$$

for $\underline{\delta} \leq R-s \leq \overline{\delta}$ is a concave function of R which becomes negative for large enough R. We consider three cases:

a. For $\omega > \underline{\delta}$, $\partial L/\partial R(s+\underline{\delta};\omega)$ is positive. There is a single zero $R(\omega)$ of the derivative, solution to $\omega = M(R,s)$, which maximizes $L(R,s;\omega)$.

b. For $\omega = \underline{\delta}$, $\partial L/\partial R(s+\underline{\delta},s;\omega)$ is equal to zero. $\partial^2 L/\partial R^2(\underline{\delta},s;\omega) = (p_U(s) - 1 - \lambda)/(\overline{\delta}-\underline{\delta})$ is positive, so that there is another root $R(\underline{\delta})$, larger than $s+\underline{\delta}$ ($R = s+\underline{\delta}$ is a local minimum of L). Recall that $L(s+\underline{\delta},s;\omega)$ is equal to zero for all ω: the maximum is positive.

c. Finally consider $\omega < \underline{\delta}$. The function $\partial L/\partial R(\cdot,s;\omega)$ is linear increasing in ω: when ω decreases from $\underline{\delta}$, its smallest root increases, its largest root (a local maximum of L), say $\Delta(\omega)$, decreases, until eventually they both disappear, say at ω_1, $\omega_1 < \underline{\delta}$. Note that $L(\Delta(\omega),s;\omega)$ is an increasing function of ω. Since $L(\underline{\delta},s;\underline{\delta}) = 0$, $L(\Delta(\omega_1),s;\omega_1)$ is negative. Let ω_2, $\omega_2 > \omega_1$, be such that $L(\Delta(\omega_2),s;\omega_2)$ is equal to zero. Define $\omega_0 = \max(\underline{\omega},\omega_2)$, $R(\omega) - s = \Delta(\omega)$ for $\omega_0 \leq \omega \leq \underline{\delta}$, and $R(\omega) - s = \underline{\delta}$ for ω smaller than ω_0.

It is easy to check that the $R(\omega)$ function thus defined indeed is the solution of the problem. ■

A.6 Proof of Lemma 2

Writing the effect of a translation of the overall income (increasing $R(\omega)$ for all ω and s by the same small amount) shows that the expectation in w of

$$\begin{aligned} &F(R(\omega)-s|\omega,s)p_E(R(\omega),s|\omega)+[1-F(R(\omega)-s|\omega,s)]p_U(R(\omega),s|\omega)\\ &\quad+\lambda\left\{[\omega-R+s]\frac{\partial F}{\partial s}(R(\omega)-s|\omega,s)-1\right\} \end{aligned} \tag{23}$$

is zero.

Consider first points ω where $R(.)$ is strictly increasing and the first order condition (11) holds. For these points, we set $\rho=\frac{1}{f}\frac{\partial F}{\partial s}$. Thanks to (6), we get: $\rho\leq 1$. It follows from the first order condition (11) that

$$\lambda[\omega-R+s]\frac{\partial F}{\partial s}(R(\omega)-s|\omega,s)=\lambda[\omega-R+s]\rho f=\rho F(\lambda-p_E).$$

Now consider a pooling interval $[\omega_1,\omega_2]$ where R is constant and equal to $\bar{R}$. Suppose first that $\bar{R}-s\leq\omega_1$. Then using (6) and the first pooling condition (13), we get

$$\lambda\int_{\omega_1}^{\omega_2}(\omega-\bar{R}+s)\frac{\partial F}{\partial s}\,\mathrm{d}G(\omega)\leq\lambda\int_{\omega_1}^{\omega_2}(\omega-\bar{R}+s)f\mathrm{d}G(\omega)=\int_{\omega_1}^{\omega_2}(\lambda-p_E)F\mathrm{d}G(\omega).$$

For $\omega\in[\omega_1,\omega_2]$, we set

$$\rho=\frac{\lambda\int_{\omega_1}^{\omega_2}(\omega-\bar{R}+s)\frac{\partial F}{\partial s}\mathrm{d}G(\omega)}{\int_{\omega_1}^{\omega_2}(\lambda-p_E)F\mathrm{d}G(\omega)}.$$

Since $\int_{\omega_1}^{\omega_2}(\lambda-p_E)F\mathrm{d}G(\omega)>0$, we get $\rho\leq 1$.

Suppose now that $\bar{R}-s\geq\omega_2$. The same computation yields

$$\int_{\omega_1}^{\omega_2}(\omega-\bar{R}+s)\frac{\partial F}{\partial s}\mathrm{d}G(\omega)\geq\lambda\int_{\omega_1}^{\omega_2}(\omega-\bar{R}+s)f\mathrm{d}G(\omega)=\int_{\omega_1}^{\omega_2}(\lambda-p_E)F\mathrm{d}G(\omega).$$

We define ρ as above, and again $\rho\leq 1$. (Notice that, in this case, $\int_{\omega_1}^{\omega_2}(\lambda-p_E)F\mathrm{d}G(\omega)<0$.)

Finally suppose that $\omega_1<\bar{R}-s<\omega_2$. Then, using (6) and the second pooling condition (14), we get

$$\int_{\bar{R}-s}^{\omega_2}\lambda(\omega-\bar{R}+s)\frac{\partial F}{\partial s}\mathrm{d}G\leq\int_{\bar{R}-s}^{\omega_2}\lambda(\omega-\bar{R}+s)f\mathrm{d}G\leq\int_{\bar{R}-s}^{\omega_2}(\lambda-p_E)F\mathrm{d}G.$$

For $\omega\in[\bar{R}-s,\omega_2]$, we define ρ as

$$\rho=\frac{\int_{\bar{R}-s}^{\omega_2}(\omega-\bar{R}+s)\frac{\partial F}{\partial s}\mathrm{d}G(\omega)}{\int_{\bar{R}-s}^{\omega_2}(\lambda-p_E)F\mathrm{d}G(\omega)}$$

and get $\rho \leq 1$. Similarly we have

$$\lambda \int_{\omega_1}^{\bar{R}-s} (\omega - \bar{R} + s) \frac{\partial F}{\partial s} \mathrm{d}G \geq \lambda \int_{\omega_1}^{\bar{R}-s} (\omega - \bar{R} + s) f \mathrm{d}G \geq \int_{\omega_1}^{\bar{R}-s} (\lambda - p_E) F \mathrm{d}G$$

For $\omega \in [\omega_1, \bar{R} - s]$, we define ρ as

$$\rho = \frac{\int_{\omega_1}^{\bar{R}-s} (\omega - \bar{R} + s) \frac{\partial F}{\partial s} \mathrm{d}G(\omega)}{\int_{\omega_1}^{\bar{R}-s} (\lambda - p_E) F \mathrm{d}G(\omega)}$$

and get, again, $\rho \leq 1$. (Notice that $\int_{\omega_1}^{\bar{R}-s} (\lambda - p_E) F \mathrm{d}G(\omega) < 0$.)

Replacing in (23) the term $\lambda[\omega - R + s]\partial F / \partial s$ (or its integral on pooling intervals) by the expressions derived above yields the desired result.

Finally recall that absent income effects: $\partial F / \partial s = \rho = 0$, while when leisure is a normal good, $\partial F / \partial s$ is everywhere less than or equal to 0.

■