An Efficient Dynamic Auction for Heterogeneous Commodities

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Abstract

This paper proposes a new dynamic design for auctioning multiple heterogeneous commodities. An auctioneer wishes to allocate K types of commodities among n bidders. The auctioneer announces a vector of current prices, bidders report back quantities demanded at these prices, and the auctioneer adjusts the prices. Units are credited to bidders at the current prices as their opponents' demands decline, and the process continues until every commodity market clears.

Bidders, rather than being assumed to behave as price-takers, are permitted to strategically exercise their market power. Nevertheless, with pure private values, the proposed auction yields Walrasian equilibrium prices. An efficient outcome results, as from a Vickrey-Clarke-Groves mechanism, but bidders are only required to report their demands along a one-dimensional path of prices, rather than their utilities over the entire (K-dimensional) consumption set.

Theoretically, the auction provides a new foundation for Walrasian equilibrium in an exchange economy, offering a modern perspective on the fictitious Walrasian auctioneer. Practically, it provides an efficient method for simultaneously auctioning two or more related types of items, including securities (e.g., three-month and six-month Treasury bills), communications spectrum (e.g., paired and unpaired spectrum) and energy (e.g., contracts of different durations).

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In earlier work (Ausubel, 1997, 2000), I proposed an efficient ascending auction design for multiple homogeneous items. In environments where bidders have pure private values and diminishing marginal values, this dynamic auction yields outcomes coinciding with that of the (sealed-bid) Vickrey (1961) auction, but offers advantages of simplicity, transparency and privacy preservation. Moreover, in some environments where bidders have interdependent values for the items, this dynamic auction continues to yield efficient outcomes and thus outperforms even the Vickrey auction.

However, situations abound in diverse industries in which heterogeneous (but related) commodities are auctioned. On a typical Monday, the U.S. Treasury sells in excess of $10 billion in three-month bills and $10 billion in six-month bills.[1] Current practice is to auction the three-month and six-month bills separately in two independent sealed-bid auctions. In the European UMTS/IMT-2000 spectrum auctions, governments sold both paired and unpaired 3G spectrum, located at similar frequencies but apparently exhibiting markedly different values. Some governments auctioned these together in fixed bundles, while other governments auctioned the paired spectrum followed by the (less valuable) unpaired spectrum. In the quarterly Electricité de France (EDF) generation capacity auctions that began in 2001, EDF sells base-load electricity contracts and peak-load electricity contracts, of five different durations each, simultaneously in one dynamic auction procedure.

The current paper proposes an efficient dynamic auction method for heterogeneous items. The starting point for the new design is a venerable trading procedure, often associated in general equilibrium theory with the fictitious *Walrasian auctioneer*, and sometimes implemented in modern times as an *ascending clock auction*. An auctioneer wishes to allocate K types of heterogeneous commodities among n bidders. The auctioneer announces a price vector, p, and bidders respond by reporting the quantity vectors that they wish to transact at these prices. The auctioneer then calculates the excess demand, and increases or decreases each component of the price vector according as the excess demand is positive or negative (*Walrasian tâtonnement*). This iterative process continues until a price vector is reached at which excess demand is zero, and trades occur only at the final price vector.

In both the fictitious Walrasian auctioneer construct and in real-world ascending clock auctions, bidders' payments are linear in the quantities awarded: if bidder i wins quantity vector q_i in an auction with final price vector p, bidder i pays $p \cdot q_i$. Unfortunately, a strategic agent who faces linear prices in the auction then possesses an incentive to underreport her true demand at the announced prices, and this incentive increases in her market share. This is most straightforwardly seen in the case of homogeneous goods (i.e., $K = 1$), where there is a growing body of both theoretical arguments and empirical evidence.

[1] For example, on 10 June 2002, the U.S. Treasury auctioned $17 billion in three-month bills and $15 billion in six-month bills (Press Release, Department of the Treasury, Bureau of the Public Debt, web-posted at http://www.publicdebt.treas.gov).

(See detailed discussions for sealed-bid auctions in Ausubel and Cramton, 1996, and for ascending auctions in Ausubel, 1997.) Consequently, when agents have market power, the Walrasian auction procedure typically does not result in Walrasian outcomes. The current paper circumvents this problem by extending and generalizing an approach introduced in Ausubel (1997): units are credited to bidders at the current prices whenever the opposing bidders' demands decline. Specified properly, this nonlinear pricing rule restores the incentive for strategic bidders to bid as price-takers, yielding efficient outcomes even when bidders have market power.[2]

One of the objectives of the current paper is, thus, to provide both a solution to an outstanding theoretical auction question and a new practical auction design. In a recent article, Bikhchandani and Mamer (1997, pp. 405–406) ask:

> "Do there exist simple market mechanisms (i.e., mechanisms that assign a price to each object) which efficiently allocate multiple indivisible objects when market clearing prices exist? … Whether there are simple incentive compatible market mechanisms which converge to a competitive equilibrium (whenever one exists) under the more general condition that buyers may want to consume more than one object is an open question."

Meanwhile, Gul and Stacchetti (2000, p. 69) conclude the Introduction of their recent article by stating:

> "More importantly, we show that no dynamic auction can reveal sufficient information to implement the Vickrey mechanism if all Gross Substitutes preferences are allowed. Thus, the unit demand case of Demange *et al.* [1986] and the multiple homogeneous goods case of Ausubel [1997] are the most general environments for which generalizations of the English auction can be used to implement efficient, strategy-proof allocations."[3]

In this paper, I will put forth an affirmative answer to Bikhchandani and Mamer's question, while disagreeing with the spirit (but not the letter) of Gul and Stacchetti's conclusion. A simple market mechanism is provided: the auctioneer announces price vectors, bidders are asked to respond with their naïve demands, and there is no benefit to bidders from strategizing further. Moreover, the dynamic mechanism economizes on information in the sense that bidders need only to report their demands at a one-dimensional set of price vectors, and it maintains privacy in the sense that bidders avoid the need to report demands at prices above the market-clearing prices. While the answer provided here may

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[2] The extension of an efficient dynamic auction to $K \geq 2$ commodities poses at least two significant obstacles. First, unlike in the homogeneous goods case, a bidder may now wish to increase her demand for a given commodity along the path toward equilibrium, as prices of substitute commodities increase. Thus, units that once appeared to be "clinched" by another bidder may later be "unclinched", and the auction rules need to reflect this scenario. Second, K simultaneous auctions are effectively conducted, and it is unclear how the progress of one auction should affect the clinching of units in another. Surprisingly, this paper establishes that it suffices to independently calculate the crediting of different commodities; the only formal interaction among the K auctions needs to occur through the simultaneous bidding and the price adjustment rule. With these obstacles resolved, an efficient dynamic auction design for heterogeneous commodities emerges.

[3] They also conclude their article: "Finally, we showed that in general, no efficient, dynamic auction can extract enough information to implement any strategy-proof mechanism." (Gul and Stacchetti, 2000, p. 83).

superficially appear to be in conflict with Gul and Stacchetti's conclusion, there is no formal conflict with their theorems.[4] Indeed, the elegant analysis of Gul and Stacchetti is utilized as a needed input into the current analysis. Nevertheless, my interpretation of the current results is opposite to Gul and Stacchetti's conclusion. The economic environment of multiple *heterogeneous* commodities is apparently amenable to a generalization of the English auction that can be used to implement efficient, strategy-proof allocations.[5]

The current paper will also seek to offer a modern perspective on the Walrasian auctioneer. Indeed, economists have long been hostile toward this modeling device. Arrow (1959, p. 43) notes at once the motivation for the fictitious auctioneer and the logical problem he creates: "It is not explained whose decision it is to change prices in accordance with [Walrasian tâtonnement]. Each individual participant in the economy is supposed to take prices as given and determine his choices as to purchases and sales accordingly; there is no one left over whose job it is to make a decision on price." Arrow and Hahn (1971, p. 322) elaborate that the auctioneer and perfect competition together produce "the paradoxical problem that a perfect competitor changes prices that he is supposed to take as given."

The current research suggests one way out of the paradox. Instead of an implicit fictitious auctioneer, consider an explicit auction mechanism that uses Walrasian tâtonnement for price adjustment and uses the payment rule proposed in this paper. Then, economic agents — even though conscious that they can and do change prices — find it in their interest to take prices as given at every moment and to report their true demands relative to the current prices. As a result, the auction reaches the same Walrasian allocation of goods *as if* agents were price takers and the fictitious auctioneer were present — however, the agents' payment are generally lower than in the Walrasian model.

Thus, the current paper relates to a variety of strands of the literature. First, it connects most directly with several recent papers seeking to extend or explain my analysis in Ausubel (1997). Perry and Reny (2001) adapt my ascending-bid design to more general environments of homogeneous goods with interdependent values. Bikhchandani and Ostroy (2001a,b) and Bikhchandani *et al.* (2001) formulate the auction problem as a linear programming problem and reinterpret my homogeneous goods design as a primal-dual algorithm. Ausubel (1996) makes a first attempt to extend the design from homogeneous to

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[4] The precise statement of Theorem 6 of Gul and Stacchetti (2000) only excludes implementing *literally* the Vickrey-Clarke-Groves (VCG) outcome by an auction in which price traces a *single* ascending trajectory. By contrast, for the case of substitutes, Theorems 2 and 2′ of the current paper yield the outcome of a modified VCG mechanism using a single ascending trajectory, while Theorems 4 and 4′ of the current paper yield literally the VCG outcome using potentially n distinct ascending trajectories generated in parallel. Hence, the conflict herein is only with Gul and Stacchetti's interpretation, and not literally with their theorem.

[5] The modified VCG mechanism is also a Clarke-Groves mechanism, and so it is also strategy-proof. However, payoffs differ from the standard VCG payoffs by an additive term that depends on other bidders' reports. If the additive term is positive, the modified VCG mechanism may violate a voluntary participation constraint. If the additive term is negative, the modified VCG mechanism will yield lower expected revenues than the VCG mechanism. Thus, there are advantages to obtaining the standard VCG payoffs, but doing so incurs the cost of using the parallel auction procedure with n distinct ascending trajectories.

heterogeneous goods. Second, this paper relates to the literature on efficient auction design. This includes the classic work of Vickrey (1961), Clarke (1971) and Groves (1973), who provide static dominant-strategy mechanisms for private values settings, as well as recent papers examining the possibility or impossibility of efficient mechanisms with interdependent values, including Maskin (1992), Ausubel (1999), Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001) and Perry and Reny (2002). Third, this paper also relates to the literature exploring Walrasian equilibrium in auction environments with discrete goods. This includes the early work of Kelso and Crawford (1982), as well as recent work by Bikhchandani and Mamer (1997), Gul and Stacchetti (1999, 2000) and Milgrom (2000). Fourth, this paper relates to the literature exploring dynamic package bidding — rather than the clock auction — as an approach to auctioning heterogeneous items, papers such as Banks, Ledyard and Porter (1989), Ausubel and Milgrom (2002), and Parkes and Ungar (2002).

Finally, the current paper connects with the venerable literature on tâtonnement stability and price adjustment processes, which seeks to understand the forces operating in an economy that may drive it toward an equilibrium. The most famous early attempt to treat convergence to equilibrium was made by Walras (1874). Classical results include articles by Arrow, Block and Hurwicz (1959), who demonstrate the global stability of Walrasian tâtonnement under the assumption of gross substitutes, and Scarf (1960), who provides (non-substitutes) counterexamples for which Walrasian tâtonnement fails to converge from any starting point other than the equilibria. Hahn (1982) provides a nice survey of the classical literature on tâtonnement stability. Indeed, one way to view the current paper is that it introduces a methodology enabling the economist to convert competitive results on tâtonnement stability into game-theoretic results involving strategic agents. It is hoped that the methodology may ultimately enable us to import significant portions of the existing literature on stability of price adjustment processes into a strategic framework.

The paper examines two economic environments, each containing bidders with quasilinear utilities and pure private values. In the first environment, the commodities are perfectly divisible. Price is adjusted as a differential equation using the classic specification of Walrasian tâtonnement and bidders submit bids in continuous time. Surprisingly, it is unnecessary to assume that bidders display substitutes preferences; strictly concave utility functions are sufficient for obtaining the following results in a continuous environment. If a bidder's opponents bid sincerely, then the bidder's payoff is path independent and equals a constant translation of social surplus at the final allocation (Lemma 2), implying that the bidder maximizes her payoff by bidding sincerely (Theorem 1). Hence, sincere bidding by every bidder is an equilibrium, yielding a Walrasian price vector and an efficient allocation (Theorem 2). With appropriate choice of the initial price, it yields exactly the Vickrey-Clarke-Groves (VCG) payoff to a given bidder (Theorem 3). Finally, a procedure for n parallel auctions is provided which yields exactly the VCG payoff to all n bidders (Theorem 4).

In the second environment, the commodities are discrete. Then it becomes necessary to assume that agents have substitutes preferences; otherwise, Walrasian prices might fail to exist (see Kelso and Crawford, 1982, Bikhchandani and Mamer, 1997, Gul and Stacchetti, 1999, and Milgrom, 2000). Adjusting prices using a variation on Gul and Stacchetti's (2000) adaptation of Walrasian tâtonnement to discrete commodities, Theorems 1′–4′ (analogous to Theorems 1–4) obtain. The sincere bidding equilibrium survives iterated elimination of weakly dominated strategies and, with incomplete information and a "full support" assumption, it is the unique outcome of iterated weak dominance (Theorem 5′). Moreover, since prices only ascend along the adjustment path, the auction design exhibits the same advantage of privacy preservation as in the homogeneous goods case.

The paper is organized as follows. Section 1 illustrates the new dynamic auction. Section 2 specifies the model. Sections 3 and 4 develop the auction in generality. Sections 5 and 6 prove theorems for the continuous environment. Section 7 specifies and proves theorems for the discrete environment. Section 8 offers two approximations to the full n parallel auctions procedure. Section 9 concludes.

1. An Illustration of the Efficient Dynamic Auction

We illustrate the new dynamic auction procedure for heterogeneous commodities using an example in which $K = 2$. There are two types of commodities, denoted A and B. Real-world examples fitting this description may include the sale of three-month and six-month Treasury bills, the sale of paired and unpaired telecommunications spectrum, or the sale of base-load and peak-load electricity. Suppose that there are $n = 3$ bidders and let the supply vector equal (10,8). The auctioneer initially announces a price vector of $p_1 = (3,4)$, and subsequently adjusts the price vector to $p_2 = (4,5)$, $p_3 = (5,7)$, $p_4 = (6,7)$, and finally $p_5 = (7,8)$. The bidders' quantities demanded at these price vectors are shown in Table 1:

Price Vector	Bidder 1	Bidder 2	Bidder 3
$p_1 = (3,4)$	(5,4)	(5,4)	(5,4)
$p_2 = (4,5)$	(4,4)	(5,4)	(4,3)
$p_3 = (5,7)$	(4,3)	(4,4)	(4,1)
$p_4 = (6,7)$	(4,3)	(4,4)	(3,2)
$p_5 = (7,8)$	(4,2)	(3,4)	(3,2)

Table 1: Price and Quantity Vectors for Illustrative Example with $K = 2$

The *crediting* of units to bidders occurs as follows. First, consider Bidder 1. When the price vector advances from $p_1 = (3,4)$ to $p_2 = (4,5)$, the sum of the quantity vectors demanded by Bidder 1's opponents decreases from (10,8) to (9,7). Thus, 1 unit of commodity A and 1 unit of commodity B can be thought of as becoming available to Bidder 1 at the current price of $p_2 = (4,5)$. The auction algorithm takes this literally, by *crediting* 1 unit of commodity A at a price of 4, and 1 unit of commodity B at a price of 5, to Bidder 1. Next, consider Bidder 2. When the price vector advances from p_1 to p_2, the sum of the quantity vectors demanded by Bidder 2's opponents decreases from (10,8) to (8,7). Thus, 2 units of commodity A and 1 unit of commodity B can be thought of as becoming available to Bidder 2 at the current price. The auction algorithm takes this literally, by *crediting* 2 units of commodity A at a price of 4, and 1 unit of commodity B at a price of 5, to Bidder 2. Finally, consider Bidder 3. When the price vector advances from p_1 to p_2, the sum of the quantity vectors demanded by Bidder 3's opponents decreases from (10,8) to (9,8). Thus, 1 unit of commodity A and 0 units of commodity B can be thought of as becoming available to Bidder 3 at the current price. Again, the auction algorithm takes this literally, by *crediting* 1 unit of commodity A at a price of 4 and 0 units of commodity B at a price of 5, to Bidder 3.

The process continues as the price vector advances. One interesting moment occurs when the price advances from $p_3 = (5,7)$ to $p_4 = (6,7)$. Observe that Bidder 3's demand vector changes from (4,1) to (3,2), while the other bidders' demand vectors remain constant. In particular, Bidder 3's demand for commodity B *increases*, meaning that 1 *fewer* unit of commodity B remains available for Bidders 1 and 2. Consequently, the auction algorithm needs to take this literally, by *debiting* 1 unit of commodity B at the current price of 7 from each of Bidders 1 and 2.

The entire progression of units credited and debited is summarized in Table 2:

Price Vector	Bidder 1	Bidder 2	Bidder 3
$P_1 = (3,4)$	Initialization	Initialization	Initialization
$P_2 = (4,5)$	1 unit of A credited at 4 1 unit of B credited at 5 Cumulative payment = 9	2 units of A credited at 4 1 unit of B credited at 5 Cumulative payment = 13	1 unit of A credited at 4 0 units of B credited at 5 Cumulative payment = 4
$P_3 = (5,7)$	1 unit of A credited at 5 2 units of B credited at 7 Cumulative payment = 28	0 units of A credited at 5 3 units of B credited at 7 Cumulative payment = 34	1 unit of A credited at 5 1 unit of B credited at 7 Cumulative payment = 16
$P_4 = (6,7)$	1 unit of A credited at 6 1 unit of B *debited* at 7 Cumulative payment = 27	1 unit of A credited at 6 1 unit of B *debited* at 7 Cumulative payment = 33	0 units of A credited at 6 0 units of B credited at 7 Cumulative payment = 16
$P_5 = (7,8)$	1 unit of A credited at 7 0 units of B credited at 8 Cumulative payment = 34	0 units of A credited at 7 1 unit of B credited at 8 Cumulative payment = 41	1 unit of A credited at 7 1 unit of B credited at 8 Cumulative payment = 31

Table 2: Credits and Debits for Illustrative Example with $K = 2$

At $p_5 = (7,8)$, supply and demand are now in balance for both commodities. Thus, p_5 becomes the final price. Bidders 1, 2 and 3 receive their quantity vectors of (4,2), (3,4) and (3,2), respectively, demanded at the final price. Observe that, for each bidder, the quantity vector demanded at the final price equals the sum of all units credited or debited along the way. However, since many of the credits and debits occurred at earlier prices, bidders' payments do *not* generally equal their final demands evaluated at the final prices. Rather, the bidders' payments are related to those from the VCG mechanism, justifying the sincere bidding assumed in this section by making it incentive compatible.

2. THE MODEL

A seller wishes to allocate units of each of K heterogeneous commodities among a set of n bidders, $N \equiv \{1, \ldots, n\}$. The seller's available supply of commodities is denoted by $S = (S^1, \ldots, S^K) \in \mathbb{R}^K_{++}$. Bidder i's consumption set, X_i, is assumed to be a compact, convex[6] subset of $\mathbb{R}^K_+$, and bidder i's consumption bundle is denoted by $x_i = (x_i^1, \ldots, x_i^K) \in X_i$. The following assumptions are made for the divisible commodities model:[7]

(A1) *Pure private values*: Bidder i's value, $U_i(x_i)$, for consumption vector x_i does not change when bidder i learns other bidders' information.

(A2) *Quasilinearity*: Bidder i's utility from receiving the consumption vector x_i in return for the payment y_i is given by $U_i(x_i) - y_i$.

(A3) *Monotonicity*: The function $U_i : X_i \to \mathbb{R}$ is increasing.

(A4) *Concavity*: The function $U_i : X_i \to \mathbb{R}$ is concave.

(A5) *Continuity*: The function $U_i : X_i \to \mathbb{R}$ is continuous.

The price vector will be denoted by $p = (p^1, \ldots, p^K) \in \mathbb{R}^K$. Bidder i's *indirect net utility function*, $V_i(p)$, and true *demand correspondence*, $Q_i(p)$, are defined respectively by:

$$V_i(p) = \max_{x_i \in X_i} \{U_i(x_i) - p \cdot x_i\}, \tag{1}$$

$$\text{and} \qquad Q_i(p) = \arg\max_{x_i \in X_i} \{U_i(x_i) - p \cdot x_i\}. \tag{2}$$

Observe that $V_i(p)$ is well defined and $Q_i(p)$ is nonempty. If the correspondence $Q_i(p)$ is single valued (as will be the case when we strengthen (A4) to assume strictly concave utility in Sections 5 and 6,

[6] More precisely, convexity will be assumed when considering the divisible commodities model emphasized in Sections 5 and 6; obviously, convexity in $\mathbb{R}^K$ will *not* be assumed for the discrete commodities model studied in Section 7, since then $X_i \subset \mathbb{Z}^K$.

[7] In addition, to avoid arcane difficulties, it is assumed that the consumption sets of the bidders and the available supply S of commodities are such that there exists a feasible allocation of S among all the bidders and, for each i, there exists a feasible allocation of S among the bidders $j \neq i$.

below), then we may also refer to the solution of Eq. (2) as the *demand function* $q_i(p)$.

Since $U_i(\bullet)$ is continuous and concave, its conjugate function, $-V_i(\bullet):\mathbb{R}^K \to \mathbb{R}$, is continuous, closed and concave (Rockafellar, 1970, Thm. 12.2 and p. 308). We have:

$$\partial V_i(p) = -Q_i(p), \text{ for all } p \in \mathbb{R}^K, \tag{3}$$

i.e., x is a subgradient of V_i at p if and only if $-x$ is an element of bidder i's true demand correspondence at p (Rockafellar, 1970, Thm. 23.5). Note that Eq. (3) is merely a general version of $\nabla V_i(p) = -q_i(p)$, Roy's identity as restricted to quasilinear utility. To see this, consider the case where $U_i(\bullet)$ is twice continuously differentiable and strictly concave. Then demand is a continuously differentiable function, $q_i(\bullet)$. Furthermore, since $V_i(p) = U_i(q_i(p)) - p \cdot q_i(p)$, the Envelope Theorem implies $\nabla V_i(p) = -q_i(p)$.

With every time $t \in [0,\infty)$, we associate a price vector $p(t)$, and each bidder i selects a bid $x_i(t)$. We say that bidder i bids *sincerely* if her bid always belongs to her true demand correspondence:

> *Sincere Bidding*.[8] Bidder i is said to bid sincerely relative to utility function $U_i(\bullet)$ if, at every time $t \in [0,\infty)$, her bid $x_i(t) \in Q_i(p(t)) = \arg\max_{x_i \in X_i} \{U_i(x_i) - p(t) \cdot x_i\}$.

Next, we define two notions of efficient outcomes for this auction environment, the first taken from general equilibrium theory and the second taken from game theory:

DEFINITION 1. *A* ***Walrasian equilibrium*** *is a price vector* p^* *and a consumption bundle* $\{x_i^*\}_{i=1}^n$ *for every bidder such that* $x_i^* \in Q_i(p^*)$, *for i* = 1, …, *n, and* $\sum_{i=1}^n x_i^* = S$.

DEFINITION 2. *The* ***Vickrey-Clarke-Groves (VCG) mechanism*** *is the following procedure: each bidder i reports a valuation function,* $U_i : X_i \to \mathbb{R}$, *to the auctioneer. The auctioneer assigns a consumption bundle,* x_i^*, *to each bidder i and charges a payment of* $y_i^* = U_i(x_i^*) - W^* + W_{-i}^*$, *where:*

$$\{x_i^*\}_{i=1}^n \in \arg\max\left\{\sum_{i=1}^n U_i(x_i) : x_i \in X_i \text{ and } \sum_{i=1}^n x_i = S\right\},$$
$$W^* = \max\left\{\sum_{i=1}^n U_i(x_i) : x_i \in X_i \text{ and } \sum_{i=1}^n x_i = S\right\}, \text{ and}$$
$$W_{-i}^* = \max\left\{\sum_{j \neq i} U_j(x_j) : x_j \in X_j \text{ and } \sum_{j \neq i} x_j = S\right\}.$$

The ***VCG payoffs*** *(if reports are truthful) are the payoffs* $W^* - W_{-i}^*$.

Assumptions (A1)–(A5) guarantee the existence of Walrasian equilibrium. If bidders' reports are constrained to satisfy Assumptions (A1)–(A5), then the VCG mechanism is well defined. It is well known that truthful reporting is a dominant strategy equilibrium of the VCG mechanism. By the First

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[8] If bidder i's utility function is *strictly* concave (as will be assumed in Sections 5 and 6), then the demand correspondence $Q_i(p)$ is single valued, and so sincere bidding simply means that $x_i(t) = Q_i(p(t))$ for all $t \in [0,T]$.

Fundamental Welfare Theorem, any Walrasian equilibrium allocation $\{x_i^*\}_{i=1}^n$ is welfare maximizing, so (apart from nonuniqueness issues[9]) the Walrasian and VCG allocations coincide. However, the payments in the VCG mechanism are generally less than the linear calculation $p^* \cdot x_i^*$ of the Walrasian equilibrium.

Finally, we define a modification of the VCG mechanism that will be useful in characterizing the outcomes of the auction proposed in this paper. It is somewhat related to the notion of a Vickrey auction with a reserve price discussed in Ausubel and Cramton (1999):

DEFINITION 3. *The* ***modified VCG mechanism with price of p*(0)** *is the following procedure: each bidder i reports a valuation function,* $U_i : X_i \to \mathbb{R}$*, to the auctioneer. The auctioneer assigns a consumption bundle,* x_i^**, to each bidder i and charges a payment of* $y_i^* = U_i(x_i^*) - W^{**} + W_{-i}^{**}$*, where:*

$$\{x_i^*\}_{i=1}^n \in \arg\max \left\{ \sum\nolimits_{i=1}^n \left(U_i(x_i) - p(0) \cdot x_i \right) \,:\, x_i \in X_i \text{ and } \sum\nolimits_{i=1}^n x_i = S \right\},$$

$$W^{**} = \max \left\{ \sum\nolimits_{i=1}^n \left(U_i(x_i) - p(0) \cdot x_i \right) \,:\, x_i \in X_i \text{ and } \sum\nolimits_{i=1}^n x_i = S \right\}, \text{ and}$$

$$W_{-i}^{**} = \max \left\{ \sum\nolimits_{j \neq i} \left(U_j(x_j) - p(0) \cdot x_j \right) \,:\, x_j \in X_j \right\}.$$

The modified VCG mechanism with price of $p(0)$ has the following interpretation. The calculation of W^{**} is similar to the calculation of W^* in Definition 1: social surplus is calculated with all bidders present and the supply constraint of $\sum_{i=1}^n x_i = S$ is maintained, but a social cost of $p(0)$ is assigned to the commodities. The calculation of W_{-i}^{**} calculates social surplus absent bidder *i*, but it discards the supply constraint and instead assumes that commodities are available in arbitrary supply at a social cost of $p(0)$. Similar to the regular VCG mechanism, the modified mechanism awards bidder *i* exactly the difference between these two surplus calculations. As will be emphasized in Theorem 3 below, if $p(0)$ happens to be chosen such that the market absent bidder *i* clears (i.e., $\sum_{j \neq i} q_j(p(0)) = S$), then bidder *i*'s modified VCG payoff coincides with her regular VCG payoff.

3. "CLINCHING" VERSUS "CREDITING AND DEBITING"

In Ausubel (1997, 2000), I introduced the notion of "clinching" for auctions of homogeneous goods. In the current notation, this corresponds to the case of $K = 1$, and so our various quantity and price vectors temporarily reduce to scalars. Let the auction start at $p(0) = 0$ and clear at $p(T) = T$. Let:

$$x_{-i}(t) = \sum\nolimits_{j \neq i} x_j(t) \,,\; \bar{c}_i(t) = \max\{0\,, S - x_{-i}(t)\} \text{ and } c_i(t) = \sup_{\bar{t} \in [0,t]} \bar{c}_i(\bar{t}) \,, \qquad (4)$$

and define the payment, $y_i(T)$, of bidder *i* by the following Stieltjes integral:

[9] If we strengthen Assumption (A4) to assume strictly concave utilities (as in Sections 5 and 6), then the Walrasian and VCG assignments of goods are each unique.

$$y_i(T) = \int_0^T p(t)\, dc_i(t)\,. \tag{5}$$

Eq. (5) has the simple interpretation that, every time t it becomes a foregone conclusion that bidder i will win additional units of the homogeneous good, she wins them at the current price $p(t)$. If bidder i is assigned $x_i(T)$ units and is assessed a payment $y_i(T)$ determined by Eqs. (4) and (5), while bidders $j \neq i$ are bidding sincerely, then bidder i receives the same outcome as in the Vickrey (1961) auction. As a result, sincere bidding by every bidder is an (efficient) equilibrium of this ascending auction.

It is also possible to modify Eq. (5) in a relatively innocuous way. Define instead the payment, $a_i(T)$, of bidder i by the following Stieltjes integral plus a constant term:

$$a_i(T) = p(0)\left[S - x_{-i}(0)\right] - \int_0^T p(t)\, dx_{-i}(t)\,. \tag{6}$$

The principal difference between the original notion of "clinching" in Eqs. (4)–(5) and the extended notion in Eq. (6) occurs when $x_{-i}(t)$ is non-monotonic. In Eqs. (4)–(5), price is integrated against $dc_i(t)$, so units are won when opponents' demands decrease, but units are *not* lost when opponents' demands correspondingly increase. By contrast, in Eq. (6), price is integrated against $dx_{-i}(t)$, so decreases and increases in opponents' demands are treated completely symmetrically, allowing both "crediting" and "debiting" to occur. Nevertheless, the differences should not be overstated. Suppose that, in an ascending auction of homogeneous goods, we replace Eqs. (4)–(5) with Eq. (6), and we choose $p(0)$ so that $x_{-i}(0) \equiv \sum_{j \neq i} x_j(0) = S$. Then the central result from Ausubel (1997) continues to hold: sincere bidding by all bidders is an equilibrium. Thus, reformulating the payoffs need not disturb incentive compatibility.

4. THE EXTENSION TO K HETEROGENEOUS COMMODITIES

The most naïve way that one might think about generalizing the homogeneous goods procedure to the case of K heterogeneous commodities is to run K price clocks (one for each commodity) simultaneously, to compute the "credits" and "debits" for each independently, and to sum them up. Let the movement of the K price clocks be described by a continuous, piecewise smooth,[10] vector-valued function $p(t) = (p^1(t), \ldots, p^K(t))$ from $[0,T]$ to $\mathbb{R}^K$. Further suppose that each bidder i bids according to the vector-valued function $x_i(t) = \left(x_i^1(t), \ldots, x_i^K(t)\right)$ from $[0,T]$ to X_i, which is constrained to be of bounded variation in each component k. Then the naïve extension of Eq. (6) would be to define $x_{-i}^k(t) = \sum_{j \neq i} x_j^k(t)$, for $k = 1, \ldots, K$, and to define payments by:

$$a_i(T) = p(0) \cdot [S - x_{-i}(0)] - \int_0^T p(t) \cdot dx_{-i}(t) \equiv \sum_{k=1}^K \left\{ p^k(0)\,[S^k - x_{-i}^k(0)] - \int_0^T p^k(t)\, dx_{-i}^k(t) \right\}, \tag{7}$$

[10] The (vector-valued) continuous function p is said to be piecewise smooth if each component p^k has a bounded derivative which is continuous everywhere in $[0,T]$, except (possibly) at a finite number of points. At these exceptional points it is required that both right- and left-hand derivatives exist. A curve Γ is said to be piecewise smooth if it can be described by a piecewise smooth function (Apostol, 1957, Definition 9–61).

where the integrals of Eq. (7) are calculated as Stieltjes integrals. We begin by observing:

LEMMA 1. *If* $p(\bullet)$ *is continuous and if* $x_j^k(\bullet)$ *is of bounded variation for every bidder j* ≠ *i and commodity k, then the payment* $a_i(T)$ *of Eq.* (7) *is well defined.*

PROOF. By Theorem 9–26 of Apostol (1957), since p^k is continuous on [0,*T*] and each x_j^k (and, hence, x_{-i}^k) is of bounded variation on [0,*T*], each Stieltjes integral $\int_0^T p^k(t)\, dx_{-i}^k(t)$ exists. ■

Next, in order for the payment formula of Eq. (7) to serve its intended purpose, it is critical for us to establish the property of *path independence*. Suppose that bidders $j \neq i$ bid $q_j(\bullet)$, sincere bids relative to their utility functions. Consider two different price paths, $p^1(\bullet)$ and $p^2(\bullet)$, which originate at the same price vector and conclude at the same price vector. We need to show that the line integrals calculated along the two paths are equal, i.e., $\int_0^T p^1(t)\cdot dq_{-i}(t) = \int_0^T p^2(t)\cdot dq_{-i}(t)$. Otherwise, bidder *i* would have the incentive to manipulate her demand reports so as to alter the price adjustment path to her advantage.

Recall from the first year of graduate school that the same issue of path independence arose in the classical consumer theory problem of *integrability*. There, the question was: what are necessary and sufficient conditions on a vector-valued function $x(\bullet)$ satisfying Walras' law so that it is sure to be the demand function generated by some utility function.[11] Recall that the answer was: $x(\bullet)$ must satisfy the symmetry condition, $\partial x_i / \partial p_j = \partial x_j / \partial p_i$. The reason was that, for path independence, $x(\bullet)$ needs to be the gradient of a *potential function* (i.e., there must exist a function $\phi(\bullet)$ such that $\nabla\phi(\bullet) = x(\bullet)$). Furthermore, the symmetry condition on derivatives is essentially necessary and sufficient for the existence of such a potential function (e.g., see Apostol, 1957, Theorems 10-38, 10-45 and 10-48).

In the classical case of integrability, the potential function has the interpretation of an expenditure function. Path independence is the requirement that the incremental expenditure needed for an agent to attain a fixed level of utility, as the price vector changes from p^A to p^B, must not depend on the particular price adjustment path taken from p^A to p^B. Outside of economics, a better-known example of path independence is the analysis of a gravitational field in Newtonian mechanics. In a frictionless world, the amount of work required to move an object from point A to point B is the same along any possible path.

Thus, in the current auction context, path independence of the payment formula of Eq. (7) requires the existence of a potential function. With sincere bidding, the following important lemma shows that the potential function, $\sum_{j\neq i} U_j(q_j(p(\bullet)))$, is associated with the crediting/debiting formula, implying

[11] In addition, the function $x(\bullet)$ is required to satisfy a negative semidefiniteness condition and the utility function is required to satisfy quasiconcavity, monotonicity and continuity.

path independence. For greater generality, the result is obtained using subgradients rather than gradients, so that Lemma 2 applies to both the continuous and discrete models:

LEMMA 2. *Suppose that* $p(\bullet)$ *is a continuous, piecewise smooth function from* $[0,T]$ *to* $\mathbb{R}^K$. *Also suppose that* $x_j(\bullet)$ *is a measurable selection from* $Q_j(p(\bullet))$, *the demand correspondence from a concave, continuous utility function* $U_j(\bullet)$, *and that* $x_j^k(\bullet)$ *is of bounded variation, for every bidder* $j \neq i$ *and commodity k. Then the integral* $\int_0^T p(t)\cdot dx_{-i}(p(t))$ *of Eq.* (7) *is independent of the path from* $p(0)$ *to* $p(T)$ *and equals:*

$$\int_0^T p(t)\cdot dx_{-i}(p(t)) \;=\; U_{-i}(x_{-i}(p(T)))-U_{-i}(x_{-i}(p(0))) \equiv \sum_{j\neq i}\big[U_j(x_j(p(T)))-U_j(x_j(p(0)))\big]. \tag{8}$$

PROOF. The Stieltjes integral $\int_0^T p^k(t)\,dx_j^k(p(t))$ exists if and only if the Stieltjes integral $\int_0^T x_j^k(p(t))\,dp^k(t)$ exists. Consequently, by integration by parts:

$$\int_0^T p(t)\cdot dx_j(p(t)) = p(T)\cdot x_j(p(T)) - p(0)\cdot x_j(p(0)) - \int_0^T x_j(p(t))\cdot dp(t)\,,\ \text{ for all } j\in N. \tag{9}$$

Let Γ denote the (piecewise smooth) curve in $\mathbb{R}^K$ described by $p(t)$, $t\in[0,T]$. The integral on the right side of Eq. (9) may be rewritten as the line integral $\int_\Gamma x_j\cdot dp$. (For a formal definition of the line integral, see Apostol, 1957, Definition 10–32.) Since $V_j(\bullet)$ is a convex function and $-x_j(\bullet)$ is a measurable selection from its subdifferential (see Eq. (3), above, and the surrounding text), Theorem 1 of Krishna and Maenner (2001) guarantees that the line integral is independent of path and equals $\int_\Gamma x_j\cdot dp = -V_j(p(T))+V_j(p(0))$. Noting that $V_j(p(t)) = U_j(x_j(p(t))) - p(t)\cdot x_j(p(t))$:

$$\int_0^T p(t)\cdot dx_j(p(t)) = U_j(x_j(p(T))) - U_j(x_j(p(0)))\,, \tag{10}$$

and summing over all $j\neq i$ yields Eq. (8). ■

5. THE DYNAMIC AUCTION GAME FOR DIVISIBLE COMMODITIES

The auction is modeled as a dynamic game in continuous time. There are *n* players. To simplify matters, we henceforth assume for the model of divisible commodities that utility functions, $U_i(\bullet)$, are *strictly* concave, making the sincere demand correspondences single-valued at all prices. At each time $t\in[0,\infty)$, a price vector $p(t)$ is announced to the players. Each player *i* then demands a consumption bundle $x_i(t)$. The law of motion for the price vector is any continuous, sign-preserving transformation of the Walrasian tâtonnement price adjustment process (as formalized by Samuelson, 1941):

Walrasian tâtonnement. Let $Z(t) = -S + \sum_{i=1}^n x_i(t)$ denote the excess demand vector. Let $h(\bullet):\mathbb{R}^K\to\mathbb{R}^K$ denote any continuous function that is sign preserving in the sense that $h^k(\vec{z})>0 \Leftrightarrow z^k>0$ and $h^k(\vec{z})<0 \Leftrightarrow z^k<0$. Price adjusts according to:

$$\dot{p}^k(t) = h^k\left(Z(t)\right),\ \text{for } k = 1, \ldots, K. \tag{11}$$

Given the initial price, *p*(0), and suitable restrictions on $\{x_i(s)\}_{i=1}^n$, Eq. (11) determines the evolution of the price vector, *p*(*t*), at all times $t \in [0,\infty)$.

Let H_i^t denote the part of the history of play prior to time *t* that is observable to player *i* at time *t*. One sensible specification is that H_i^t comprises the history of aggregate excess demand and player *i*'s own actions, i.e., $H_i^t = \{Z(s) \text{ and } x_i(s) : s \in [0,t)\}$. Observe that, given Eq. (11), this observable history conveys *p*(*s*) for all $s \in [0,t]$.[12] The *strategy* $\sigma_i(t, H_i^t)$ of player *i* (*i* = 1, …, *n*) is a function associating times and observable histories with elements of X_i. The *strategy spaces* Σ_i may be any sets of functions $\sigma_i(t, H_i^t)$ which: (a) include sincere bidding; and (b) induce actions $x_i(t)$ by bidder *i* that are piecewise continuous and of bounded variation, for each bidder *i* and each commodity *k*.[13] The following theorems may be proven for many possible choices of strategy spaces; for specificity, we will use the following:

Piecewise Lipschitz-continuous functions. The strategy space of each player *i* is given by: $\Sigma_i = \{\sigma_i : [0,\infty) \times \mathbb{R}^K \to X_i \text{ such that } \sigma_i \text{ is a piecewise Lipschitz-continuous function of } (t,p)\}$.[14]

The strategies in Σ_i are similar in spirit to "Markovian" strategies in the sense that a player has full knowledge of the history of aggregate excess demands (or even of individual demands), yet the player chooses to base her demands only on the current time and price. This strategy space is restrictive enough to induce actions satisfying the hypothesis of Lemma 2, while general enough to include sincere bidding.

The auction is said to terminate at time *T* if $\sum_{i=1}^n x_i(T) = S$, i.e., the aggregate demand equals the supply for every commodity. It is said to terminate at time *T* = ∞ if $\lim_{T\to\infty} \sum_{i=1}^n x_i(T) = S$. Following termination of the auction, players receive their quantities demanded at the termination time, $x_i(T)$ (or $\lim_{T\to\infty} x_i(T)$), and payments are assessed according to Eq. (7). If the auction fails to terminate, i.e., if $\lim_{T\to\infty} \sum_{i=1}^n x_i(T) \neq S$ or if $\lim_{T\to\infty} \sum_{i=1}^n x_i(T)$ fails to exist, then every player is assigned a payoff of −∞.

The next lemma shows that, if all bidders bid sincerely, then starting from any history the auction converges to a Walrasian price vector. The proof is little more than the classical argument (see, for

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[12] Another sensible specification is that the observable history comprises the complete history of individual demands, i.e., $H_i^t = \{x_j(s) : s \in [0,t) \text{ and } j = 1, \ldots, n\}$.

[13] More precisely, we assume that for every player *i*, there exists a partitioning $0 = t_i^0 < t_i^1 < \ldots < t_i^l < \ldots$ of the time interval [0,∞) by the points t_i^l without finite points of accumulation, such that within each piece $[t_i^l, t_i^{l+1})$ of the domain, the function $x_i(\bullet)$ is continuous, and each $x_i^k(\bullet)$ is required to be a function of bounded variation on every finite time interval [0,*T*].

[14] More precisely, we assume that for every player *i*, there exists a partitioning $0 = t_i^0 < t_i^1 < \ldots < t_i^l < \ldots$ of the time interval [0,∞) by the points t_i^l without finite points of accumulation, and there exists a constant $C > 0$, such that within each piece $[t_i^l, t_i^{l+1}) \times \mathbb{R}^K$ of the domain, the function $x_i(t,p)$ is Lipschitz-continuous in (t,p), i.e., for $(t_1,p_1),(t_2,p_2) \in [t_i^l, t_i^{l+1}) \times \mathbb{R}^K$, we have $|x_i^k(t_2,p_2) - x_i^k(t_1,p_1)| \leq C(|t_2 - t_1| + |p_2 - p_1|)$.

example, Varian, 1981, pp. 104–106) that, with price-taking agents, the price is globally convergent to a Walrasian equilibrium price vector. For completeness, a proof is provided in the Appendix:

LEMMA 3. *With divisible goods and strictly concave utility functions for all bidders, and after any history, sincere bidding by every bidder i induces convergence to a Walrasian equilibrium price vector.*

The information structure of the auction game may be one of complete or incomplete information regarding opposing bidders' valuations. With complete information, each bidder is fully informed of the functions $\{U_j(\bullet)\}_{j=1}^n$, and the obvious equilibrium concept is subgame perfect equilibrium. With incomplete information, each bidder i is informed only of her own utility function $U_i(\bullet)$ and of the joint probability distribution $F(\bullet)$ from which the profile $\{U_j(\bullet)\}_{j=1}^n$ is drawn. The equilibrium concept that will then be used is ex post perfect equilibrium, which requires that the strategy specified for each bidder at each node of the auction game would remain optimal if the bidder were to learn her opponents' types:

Ex Post Perfect Equilibrium. The strategy n-tuple $\{\sigma_i\}_{i=1}^n$ is said to comprise an ex post perfect equilibrium if for every time t, following any history H_i^t, and for every realization $\{U_i\}_{i=1}^n$ of private information, the n-tuple of continuation strategies $\{\sigma_i(\bullet,\bullet\,|\,t,H_i^t,U_i)\}_{i=1}^n$ constitutes a Nash equilibrium of the game in which the realization of $\{U_i\}_{i=1}^n$ is common knowledge.

Alternatively, we could have explicitly defined beliefs for each bidder, and stated the theorems of this paper in terms of the perfect Bayesian equilibrium concept.[15] However, stating the results in their current form gives them a number of additional desirable properties, e.g., the results are independent of the underlying distribution of bidders' types (see Maskin, 1992, and Perry and Reny, 2001, 2002). The results as stated also encompass the complete-information version of the model, as ex post perfect equilibrium reduces under complete information to the familiar equilibrium concept of subgame perfect equilibrium.

Our assumptions above, which assure that the strategies $\sigma_i \in \Sigma_i$ induce demands $x_i(t)$ by each bidder i that are piecewise continuous and of bounded variation in each component k, and that the price adjustment process $\dot{p}(t)$ is continuous in $x_i(t)$, guarantee that the technical requirements for Lemma 2 hold. In light of the path independence established by Lemma 2, if the bidders $j \neq i$ bid sincerely relative to strictly concave utility functions, then the strategic choice by bidder i reduces from an optimization problem over *price paths* in $\mathbb{R}^K$ to one over *endpoints* in $\mathbb{R}^K$, implying:

[15] To state the results in terms of perfect Bayesian equilibrium, we would begin by specifying that, after any history, each player i has posterior *beliefs* over opponents' utility functions, $U_{-i}(\bullet) \equiv \{U_j(\bullet)\}_{j\neq i}$. The beliefs of player i are denoted $\mu_i(\bullet\,|\,t,H_i^t,U_i)$. The n-tuple $\{\sigma_i,\mu_i\}_{i=1}^n$ is then defined to comprise a *perfect Bayesian equilibrium* if the strategies $\sigma_i \in \Sigma_i$, the beliefs μ_i are updated by Bayes' rule whenever possible, and if following any history H^t of play prior to time t, σ_i is a best response for player i in the continuation game against $\{\sigma_j\}_{j\neq i}$ given beliefs $\mu_i(\bullet\,|\,t,H_i^t,U_i)$, and for every $i = 1, \ldots, n$.

THEOREM 1. *With divisible goods, if each opposing bidder $j \neq i$ bids $q_j(p(\bullet))$, a sincere bid relative to a strictly concave utility function $U_j(\bullet)$, then bidder i with strictly concave utility function $U_i(\bullet)$ maximizes her payoff by bidding sincerely. By bidding sincerely, bidder i selects a Walrasian price vector as the endpoint of the price path and maximizes social surplus, $U_i(\bullet)+\sum_{j\neq i} U_j(\bullet)$, over all feasible allocations. This result holds at every time t and after every history H_i^t.*

PROOF. By Lemma 2, bidder *i*'s payoff from any bidding strategy $\sigma_i \in \Sigma_i$ that causes the auction to terminate at time *T* with price vector *p*(*T*) and with bidder *i* receiving quantity $x_i(T)$ is:

$$\begin{aligned} U_i(x_i(T)) - a_i(T) &= U_i(x_i(T)) - p(0)\cdot[S - q_{-i}(p(0))] + \int_0^T p(t)\cdot dq_{-i}(p(t)) \\ &= U_i(x_i(T)) - p(0)\cdot[S - q_{-i}(p(0))] + U_{-i}(q_{-i}(p(T))) - U_{-i}(q_{-i}(p(0))). \end{aligned} \tag{12}$$

In order for the auction to terminate at time *T*, given the opposing bidders' strategies of $q_j(p(\bullet))$, bidder *i* must bid $x_i(T) = S - q_{-i}(p(T))$. Consequently, bidder *i* receives payoff of:

$$U_i\left(S - q_{-i}(p(T))\right) + U_{-i}\left(q_{-i}(p(T))\right) - \left\{p(0)\cdot[S - q_{-i}(p(0))] + U_{-i}\left(q_{-i}(p(0))\right)\right\}. \tag{13}$$

Since the expression within braces in Expression (13) — determined only by the starting price and the other bidders' starting actions — is a constant, bidder *i* maximizes Expression (13) by maximizing the first two terms. These first two terms coincide with social surplus for the allocation associated with *p*(*T*).

Moreover, given the quasilinearity of utility, the Fundamental Theorems of Welfare Economics imply that any Walrasian equilibrium is associated with a surplus-maximizing allocation, and vice versa. Consequently, bidder *i*'s payoff is maximized if and only if a Walrasian equilibrium price vector is the endpoint. By Lemma 3, bidder *i* can attain this maximum by bidding sincerely. ■

This immediately implies that sincere bidding is an ex post perfect equilibrium of the new auction game:

THEOREM 2. *With divisible goods, strictly concave utility functions, mandatory participation, and any arbitrary initial price vector of p*(0):

- (i) *sincere bidding by every bidder is an ex post perfect equilibrium of the auction game;*
- (ii) *with sincere bidding, the price vector converges to a Walrasian equilibrium price vector; and*
- (iii) *with sincere bidding, the outcome is that of the modified VCG mechanism with price of p*(0).

PROOF. Suppose that all opposing bidders $j \neq i$ bid sincerely in the auction game. One available strategy for bidder *i* is also to bid sincerely. By Lemma 3, price then converges to a Walrasian equilibrium price vector and, by Theorem 1, the strategy is a best response for bidder *i*. Furthermore, the payoff in expression (13) then evaluates to bidder *i*'s payoff from the modified VCG mechanism with price of *p*(0). This holds for every bidder *i* = 1, …, *n*, proving the theorem. ■

6. RELATIONSHIP WITH THE VICKREY-CLARKE-GROVES MECHANISM

In Theorem 2, each bidder i received her payoff from the modified VCG mechanism with price of $p(0)$. It coincides with bidder i's VCG payoff if the starting price $p(0)$ is chosen appropriately. Moreover, since the VCG payoff is nonnegative, we no longer need to assume mandatory participation. We have:

THEOREM 3. *With divisible goods and strictly concave utility functions, if the initial price $p(0)$ is chosen such that the market without bidder i clears at $p(0)$ (i.e., $\sum_{j \neq i} q_j(p(0)) = S$) and if each bidder $j \neq i$ bids sincerely, then bidder i maximizes her payoff by bidding sincerely and thereby receives her VCG payoff.*

PROOF. Following the proof of Theorem 1, observe that if the initial price is chosen such that the market without bidder i clears at $p(0)$, then the term $p(0) \cdot [S - q_{-i}(p(0))]$ in expression (13) equals zero. The remaining payoff term, when maximized, is exactly bidder i's payoff from the VCG mechanism. ■

Furthermore, the hypothesis of Theorem 3 is trivial to satisfy if all bidders are identical, providing a simple procedure to generate the VCG outcome. We immediately have the following corollary:

COROLLARY 1. *With identical bidders, divisible goods and strictly concave utility functions, if the initial price $p(0)$ is chosen such that the market without one bidder clears (i.e., $\sum_{j \neq i} q_j(p(0)) = S$), then the sincere bidding equilibrium of the auction game gives the same outcome as the VCG mechanism.*

However, without identical bidders, it obviously is not generally possible to select an initial price $p(0)$ such that every bidder receives her VCG payoff. Theorem 3 nevertheless suggests a more intricate, *parallel auction* procedure that could be followed so that *every* bidder receives *exactly* her VCG payoff.

PARALLEL AUCTION GAME. Begin with any initial price $p(0) \in \mathbb{R}^K$. We first perform the following n steps, which may be done in any order or may be run in parallel:

Step 1: Run the auction procedure of announcing a price $p(t)$, allowing bidders $j \neq 1$ to respond with quantities $x_j(t)$ while imposing $x_1(t) = 0$, and adjusting price according to Eq. (11) until a price p_{-1} is determined at which the market (absent bidder 1) clears.

…

Step n: Run the auction procedure of announcing a price $p(t)$, allowing bidders $j \neq n$ to respond with quantities $x_j(t)$ while imposing $x_n(t) = 0$, and adjusting price according to Eq. (11) until a price p_{-n} is determined at which the market (absent bidder n) clears.

Then, the auction is completed by continuing the auction *from price* p_{-n} (and/or any of the other generated prices p_{-i}) with all bidders' demands now included:

Step n+1: Run the auction procedure of announcing a price $p(t)$, allowing all bidders $i = 1, \dots, n$ to respond with quantities $x_i(t)$, and adjusting price according to Eq. (11) until a price p^* is determined at which the market (with all bidders included) clears.

Finally, payoffs are computed as follows. The payment of bidder n is given by the crediting/debiting formula of Eq. (7), calculated along the path from p_{-n} to p^*. The payment of bidder i ($1 \le i \le n-1$) is also given by the crediting/debiting formula of Eq. (7), but is calculated along the union of *three* paths: the path from p_{-i} to $p(0)$; the path from p(0) to p_{-n}; and the path from p_{-n} to p^*. Note that the first path (as well as all necessary demands) was generated in Step i, the second path was generated in Step n, and the third path was generated in Step n+1. We have:

THEOREM 4. *With divisible goods, strictly concave utility functions, and any initial price vector, sincere bidding by every bidder is an ex post perfect equilibrium of the parallel auction game, price converges to a Walrasian equilibrium price vector, and the outcome is exactly that of the VCG mechanism.*

PROOF. By Theorem 2, sincere bidding at each step $i = 1, \dots, n$ is an equilibrium yielding a Walrasian price vector p_{-i} for the economy without bidder i, while sincere bidding at step n+1 is an equilibrium yielding a Walrasian price vector p^* for the economy with all bidders. By Lemma 2, the payment of each bidder i is independent of the path from p_{-i} to p^* and, as in the proof of Theorem 3, it equals exactly her VCG payment. ■

7. THE DYNAMIC AUCTION GAME FOR DISCRETE GOODS

We now turn to an environment of indivisible goods. A seller wishes to allocate units of each of K types of *discrete* heterogeneous commodities among n bidders. The seller's available supply of commodities is denoted by $S = (S^1, \dots, S^K) \in \mathbb{Z}_+^K$. Bidder i's consumption set is the set $X_i = \left\{x_i \in \mathbb{Z}^K : 0 \le x_i^k \le \bar{x}_i^k \text{ for all } k = 1, \dots, K\right\}$ bounded below by zero and bounded above by $(\bar{x}_i^1, \dots, \bar{x}_i^K) \in \mathbb{Z}_+^K$. Bidder i's consumption bundle is denoted by $x_i = (x_i^1, \dots, x_i^K) \in X_i$.

In order to treat the case of discrete commodities, it will now be necessary for us to impose the *substitutes condition*. This condition, often known as "gross substitutes,"[16] requires that if the prices of some commodities are increased while the prices of the remaining commodities are held constant, then a bidder's sincere demand weakly increases for each of the commodities whose prices were held constant.

[16] The substitutes condition is often referred to as *gross substitutes* (as opposed to *net substitutes*, which would be the case if the analogous condition held for compensated demands). However, in the current context of quasilinear utility, there is no distinction between gross substitutes and net substitutes, so in this paper, the condition will simply be called the *substitutes* condition.

The reason for requiring the substitutes condition in the discrete case is to assure the existence of Walrasian equilibrium.[17] We define:

DEFINITION 4. *Consider an economy with* $\overline{S}$ *indivisible commodities, each of which is available in a supply of one.* $U_i(\bullet)$ *is said to satisfy the* ***substitutes condition*** *if, for any two price vectors p and p′ such that* $p' \geq p$ *and demand is single valued at p and p′,* $q_i^k(p') \geq q_i^k(p)$ *for any commodity k such that* $p'^k = p^k$.

The assumption in Definition 4 that each commodity is available in a supply of one is without loss of generality, since if there are multiple units of some commodities, one can expand the commodity space by treating each unit of a commodity as a unique item (Bikhchandani and Mamer, 1997, Section 2). The substitutes condition is defined with respect to this "unique items" formulation, since it is then a "sufficient and almost necessary" condition for the existence of Walrasian equilibrium.[18] The condition can be conveniently characterized in terms of the net indirect utility function: commodities are substitutes for bidder *i* if and only if $v_i(\bullet)$ (as written in the unique items formulation) is submodular.[19] However, the auction procedure itself will be specified to allow multiple units of each discrete commodity. This provides a more compact way for bidders to communicate information, and is more naturally connected with both the traditional Walrasian auctioneer procedure and the divisible commodities treatment, above.

The substitutes condition is stronger than the concave utility assumption that we made above. Substitutes preferences imply concave utility. However, let $u_i : \mathbb{R} \to \mathbb{R}$ be concave and consider the utility function $U_i(x_i) = u_i(\min_{k=1,\ldots,K} x_i^k)$. Then $U_i(\bullet)$ is a concave utility function, but starting from a consumption vector $x_i = (\alpha, \ldots, \alpha)$, the various commodities are complements for bidder *i*. The following assumptions are made for the discrete case:

(A1') *Integer pure private values*: The utility function $U_i : X_i \to \mathbb{Z}$ takes integer values.

(A2') *Quasilinearity*: The same as (A2).

(A3') *Monotonicity*: The function $U_i : X_i \to \mathbb{Z}$ is increasing.

(A4') *Substitutes condition*: The preferences derived from $U_i : X_i \to \mathbb{Z}$ satisfy Definition 4.[20]

[17] Indeed, in the case of discrete items, given any one bidder with preferences violating the substitutes condition, it is possible to specify another bidder with additive preferences and an endowment of goods such that the resulting economy has no Walrasian equilibrium (Milgrom, 2000, Theorem 4).

[18] Kelso and Crawford (1982), Milgrom (2000), and Gul and Stacchetti (2000).

[19] Ausubel and Milgrom (2002, Theorem 10).

[20] Assumption (A4') replaces (and strengthens) Assumption (A4) from the divisible goods case. Meanwhile, Assumption (A5) from the divisible goods case is now unnecessary, since the consumption set is discrete.

As before, the price vector will be denoted by $p = (p^1, \ldots, p^K) \in \mathbb{R}^K$, but it will soon be restricted to take nonnegative integer values at integer times. The *demand correspondence*, $Q_i(p)$, is defined as before.

The auction is modeled as a dynamic game in discrete time. At each time $t = 0, 1, 2, \ldots$, the variable, $p(t)$, representing a price vector is announced to the n players. Each player i responds by demanding a *set* $x_i(t) \subset X_i$ of *one or more* consumption bundles. The law of motion for the price variable will be specified later in this section.

Let H_i^t denote the part of the history of play prior to time t that is observable to player i at time t. One sensible possibility is that the observable history comprises the complete history of demand profiles, i.e., $H^t = \{x_h(s) : 0 \le s < t \text{ and } h = 1, \ldots, n\}$. Given the price adjustment procedure that will be specified, this history conveys $p(s)$ for all $0 \le s \le t$. The *strategy* $\sigma_i(t, H^t)$ of a player i ($i = 1, \ldots, n$) is a *set-valued* function of times and observable histories $\sigma_i : \mathbb{Z}_+ \times H_i^t \to 2^{X_i}$. The *strategy space* Σ_i is the set of all such functions $\sigma_i(t, H_i^t)$. As in the divisible goods game of Sections 5 and 6, the equilibrium concept will be *ex post perfect equilibrium*. The equilibrium that we will construct utilizes sincere bidding, which now means reporting the entire demand correspondence. This sincere bidding strategy will simply be denoted by $Q_i(p)$, the same notation as for the demand correspondence.

While the logic behind classical Walrasian tâtonnement remains sound in a discrete environment, the Walrasian tâtonnement process of differential equation (11) encounters technical difficulties, as the following example demonstrates. Suppose that there are two indivisible goods, each available in a quantity of one, and three bidders, each with utility functions given by: $U(0,0) = 0$; $U(1,0) = c$; $U(0,1) = c$; and $U(1,1) = c$. At a price vector of (p, p), where $0 < p < c$, each bidder would demand one of the goods, and so one good is in excess demand. Without loss of generality, say it is the first good, so its price must be increased. However, at a price vector of $(p+\varepsilon, p)$, the aggregate demand for the first good drops to zero and for the second good jumps to three. Eq. (11) now requires the price of the second good instead to rise. The choice of good whose price is required to increase may oscillate back and forth arbitrarily frequently.

In a recent article, Gul and Stacchetti (2000) provide an elegant procedure that circumvents this difficulty and thereby extends classical Walrasian tâtonnement to environments with discrete goods and substitute preferences. If agents bid sincerely by announcing their true demand *correspondences* at each price, the procedure is guaranteed to converge in finitely many steps to the lowest Walrasian equilibrium, and to get there via an ascending price path. While the Gul-Stacchetti procedure remains subject to the same critique as classical Walrasian tâtonnement — that if bidders possess any market power then the posited bidder behavior fails to be incentive compatible — it provides a needed input to adapt the auction framework of the current article to discrete goods.

The Appendix describes in detail a simplified version of Gul and Stacchetti's price adjustment procedure. It draws heavily from Gul and Stacchetti's work, but builds upon it to yield an improved procedure for the problem at hand.[21] At each time T = 0, 1, 2, … , the auctioneer announces an integer-valued price vector $p(T)$, and bidders respond by reporting demand correspondences $\{x_i(p(T))\}_{i=1}^{n}$. , The auctioneer then determines whether the set, $\{1, \ldots, K\}$, of all commodity types is overdemanded at $p(T)$. If it is, the auctioneer next calculates a *minimal* overdemanded set, $E \subset \{1, \ldots, K\}$, and the price adjustment process for times $t \in [T, T+1)$ is then given by:

$$\dot{p}^k(t) = \begin{cases} 1, \text{if } k \in E, \\ 0, \text{if } k \notin E. \end{cases} \tag{14}$$

and bidders are next queried at time T+1. Meanwhile, if the set, $\{1, \ldots, K\}$, is not overdemanded at $p(T)$, then the auctioneer is guaranteed to be able to choose a feasible selection from $\{x_i(p(T))\}_{i=1}^{n}$, and therefore the auction is deemed to end at $p(T)$.

The price adjustment process specified in Eq. (14) avoids the technical difficulty described three paragraphs above. In the example there, at price vectors (p, p), where $0 < p < c$, the minimal overdemanded set is E = {1,2}, meaning that if either commodity is increased in price by itself, it does not remain in excess demand. Consequently, Eq. (14) has both commodities' prices increased by 1. We have the following results for the discrete model (paralleling the above results for the continuous model):

THEOREM 1′. *With discrete goods and current price vector p, if each opposing bidder $j \neq i$ bids sincerely relative to a substitutes utility function $U_j(\bullet)$, then bidder i with substitutes utility function $U_i(\bullet)$ maximizes her payoff by bidding sincerely, which has the effect of maximizing social surplus, $U_i(\bullet) + \sum_{j \neq i} U_j(\bullet)$, over all allocations associated with $\{\hat{p} : \hat{p} \geq p\}$.*

The proof of Theorem 1′ is similar to the proof of Theorem 1; the only novelty occurs in two places. First, the price adjustment procedure specified in Eq. (14) yields a continuous, piecewise smooth price path, guaranteeing that Lemmas 1 and 2 continue to hold. Second, Observations 2 and 3 in the Appendix assure us that, despite the fact that the auctioneer receives demand reports from bidders at only a discrete sequence of price vectors, the auctioneer can nevertheless reconstruct a well-behaved profile of

[21] The procedure in the Appendix departs from Gul and Stacchetti's (2000) procedure in several respects. First, the description and proof outlined in the Appendix of the current paper are elementary and do not require any knowledge of matroid theory. Second, whereas Gul and Stacchetti define their procedure for the "unique items" formulation, the current procedure is specified for the useful generalization where there may be multiple units of each "type" of commodity. Third, in step 2 of their procedure, Gul and Stacchetti (2000, p. 78) find it necessary to require the auction to end without trade if bidders ever make (insincere) reports in such a way that the excess demand set is not well defined. By contrast, the current procedure operates by making use of any minimal overdemanded set—which always exists—so there is never a need for such a punishment. Fourth, and probably most importantly, the current procedure is specified in such a way as to yield a continuous price path along which bidders' demands can be inferred at almost all points. Demand information along an entire price path is needed so that Eq. (7) can be used to calculate the bidders' payments.

optimal demands for all of the bidders, everywhere on the continuous price path, enabling Eq. (7) to be calculated. This result then implies:

THEOREM 2′. *With discrete goods, the substitutes condition, and mandatory participation:*

(i) *sincere bidding by every bidder is an ex post perfect equilibrium of the auction game;*

(ii) *with sincere bidding and any arbitrary initial price vector* $p(0)$ *below the lowest Walrasian price vector, price converges to the lowest Walrasian price vector in finite time; and*

(iii) *with sincere bidding and any arbitrary initial price vector* $p(0)$ *below the lowest Walrasian price vector, the outcome is that of the modified VCG mechanism with price of* $p(0)$.

PROOF. The tâtonnement procedure will never select a commodity k whose price has already reached the lowest Walrasian price, since k can then never be an element of a minimal overdemanded set. Since the price vector is integer valued and increases by only one at each step, and given Observation 1 in the Appendix, the price vector will never overshoot the lowest Walrasian price vector. Since the price adjustment process is increasing and at least one component of price increases by one at each step, any current price vector below the lowest Walrasian price vector will lead to the lowest Walrasian price vector in finite time. Given Observations 2 and 3 in the Appendix, the auctioneer always possesses sufficient information to carry out the procedure.

The proof concludes similarly to the proof of Theorem 2. ■

Analogous to the results in Section 6, we also have:

THEOREM 3′. *With discrete goods and the substitutes condition, if the initial price vector is chosen to be a Walrasian price vector of the market without bidder i and if each bidder* $j \neq i$ *bids sincerely, then bidder i maximizes her payoff by bidding sincerely and thereby receives exactly her VCG payoff.*

and:

THEOREM 4′. *With discrete goods, the substitutes condition, and any initial price vector below the lowest Walrasian price vectors of the markets without any one bidder, sincere bidding by every bidder is an ex post perfect equilibrium of the parallel auction game, price converges to the lowest Walrasian equilibrium price vector, and the outcome is exactly that of the VCG mechanism.*[22]

[22] The (n+1)-step procedure of Section 6 is obviously restated so that p_{-1} is the smallest Walrasian price vector of the market (absent bidder 1), … , p_{-n} is the smallest Walrasian price vector of the market (absent bidder n), and p^* is the smallest Walrasian price vector of the market (with all bidders).

Finally, we will argue that, with the substitutes condition, iterated weak dominance yields sincere bidding as the unique solution to the current auction. For brevity, we restrict attention to a version of the dynamic auction for *discrete* commodities of this section.[23] In addition to (A1') – (A4'), we assume:

(A5') *Bounded Values*: The utility function $U_i : X_i \to \mathbb{Z}$ takes values within the interval $[0, \overline{v}]$, where $\overline{v}$ is a positive integer.

(A6') *Full Support*: For each bidder *i*, there is a positive probability of realizing any utility function $U_i(\bullet)$ satisfying (A1') – (A5').

It facilitates the analysis to make the following modifications to the description of the dynamic auction. First, we will henceforth assume that there exists a price $\overline{p}\,(\overline{p} \geq \overline{v})$ such that the price clock for every commodity *k* stops rising when $p^k = \overline{p}$. Second, the auction closes at the first time *T* that the price increment rule calls only for prices $p^k = \overline{p}$ to rise (i.e., when either no commodities are overdemanded, or when only commodity types *k* already priced at $p^k = \overline{p}$ are overdemanded). Third, if a commodity type *k* is overdemanded at the closing time *T*, then the commodity type is rationed according to a rule that is increasing in each component of the final demand vector, $\left(x_1^k(p(T)), \ldots, x_n^k(p(T))\right)$. We have:

THEOREM 5′. *With discrete goods, the substitutes condition and mandatory participation:*

(a) *under assumption* (A5')*, sincere bidding by all bidders is an outcome of iterated elimination of weakly dominated strategies; and*

(b) *under incomplete information and assumptions* (A5') *and* (A6')*, sincere bidding by all bidders is the unique outcome of iterated elimination of weakly dominated strategies.*

The complete proof of Theorem 5′ is omitted, since it is a straightforward generalization of the argument for homogeneous goods included in the revision of Ausubel (1997). Consider any history at which the price equals $(\overline{p}, \ldots, \overline{p})$. The auction must end after this period, so sincere bidding weakly dominates all other strategies. Next, suppose that after *t* iterations of weak dominance, all strategies other than sincere bidding have been eliminated for histories at which price satisfies $K\overline{p} - t < \sum_{k=1}^{K} p^k \leq K\overline{p}$. Consider any history at which $\sum_{k=1}^{K} p^k = K\overline{p} - t$. The auction must either end after this period or proceed to a history at which only sincere bidding is possible, so sincere bidding weakly dominates all other strategies. Conclusion (a) then follows by induction. Furthermore, with Assumption (A6'), sincere bidding is never eliminated, making sincere bidding the unique outcome of iterated weak dominance.

[23] However, a similar conclusion obtains in the continuous model, when it is assumed that there exists a time interval $\Delta > 0$ of delay which elapses before a bidder learns of the opposing bidders' actions.

8. APPROXIMATIONS

One critique that can be made of Theorem 4 (and 4′) is that the exact result — a dynamic procedure that exactly replicates the VCG mechanism — generally requires the auctioneer to run a parallel auction procedure that generates n price paths (one corresponding to each bidder). Moreover, it may be feared that bidders will find it easy to manipulate the parallel auction, since bidder i's own payoff is not affected by steps $j = 1, \ldots, n$, where $j \neq i$. This short section will outline two approximations to the full procedure needed for Theorem 4. If applied in an environment with substitutes preferences, each approximation will generate only a single ascending price path, which is therefore payoff relevant to all bidders. However, there will be no attempt to prove that the outcomes of these two approximating procedures are approximately efficient; such an examination will be left for future laboratory experimentation or computer simulations.

The reader should recall that the purpose of the full parallel auction procedure was to determine the exact, theoretically correct price vector, p_{-i}, at which the crediting/debiting calculation for each bidder i should commence. In principle, p_{-i} should be the price vector at which the market clears, absent bidder i. The key facet of each approximating procedure is to specify an alternative price vector at which credits and debits should begin to be computed involving bidder i ($i = 1, \ldots, n$) and commodity k ($k = 1, \ldots, K$).

Both approximation approaches harken back to the "clinching" approach in Ausubel (1997). In the first approximation, credits and debits continue to be calculated for bidder i using payment Eq. (7), but only in situations where the opposing bidders' aggregate demand for commodity k is less than the available supply, S^k. Very simply, we have:

APPROXIMATION 1. Define $y_{-i}^k = \min\{x_{-i}^k, S^k\}$ and modify the payment formula of Eq. (7) to be:

$$a_i(T) = -\int_0^T p(t) \cdot dy_{-i}(t) \equiv -\sum_{k=1}^K \left\{ \int_0^T p^k(t)\, dy_{-i}^k(t) \right\}. \tag{15}$$

The second approximation looks at all of the commodities together, and asks when it has become a forgone conclusion that bidder i will win some commodities. To give this second approximation, we need first to specify an *activity rule*, a function which constrains a bidder's future bidding activity based on past bidding activity:

DEFINITION 5. *An activity rule is a constraint,* $g_i(\bullet) \leq 0$, *where* $g_i : H^t \times X_i \to \mathbb{R}^K$, *which constrains a bidder's future bidding activity based on her past bidding activity. Three examples of useful activity rules are:*

Individual Monotonic Activity Rule:[24] *For all* $t' > t$, and for all $k = 1, \ldots, K$, $x_i^k(t') \leq x_i^k(t)$.

[24] See Ausubel (1997, 2000). Such an activity rule may make sense to impose on bidders in auctions of homogeneous items, or in auctions of heterogeneous items without significant substitution possibilities.

Aggregate Monotonic Activity Rule:[25] *For all t′ > t,* $\sum_{k=1}^{K} x_i^k(t') \le \sum_{k=1}^{K} x_i^k(t)$.

Revealed Preference Activity Rule:[26] *For all t′ > t,* $\sum_{k=1}^{K} \left(p^k(t') - p^k(t)\right)\left(x_i^k(t') - x_i^k(t)\right) \le 0$.

Furthermore, given an activity rule for bidders, we say that bidder i has clinched in the aggregate if it is not possible to assign all of the items to bidder i's opponents without violating some bidder's activity rule:

DEFINITION 6. Bidder i is said to have *clinched in the aggregate* after history H^t at time t if, for every assignment $\{x_j\}_{j \ne i}$ such that $\sum_{j \ne i} x_j^k \ge S^k$ for all $k = 1, \ldots, K$, there exists some bidder $j \ne i$ such that $g_j(H^t, x_j) > 0$.

APPROXIMATION 2. For bidder i and given any sequence of bids, define T_i to be the infimum of all times such that bidder i has clinched in the aggregate, and modify the payment formula of Eq. (7) to be:

$$a_i(T) = p(T_i)\cdot[S - x_{-i}(T_i)] - \int_{T_i}^{T} p(t)\cdot dx_{-i}(t) \equiv \sum_{k=1}^{K}\left\{p^k(T_i)\,[S^k - x_{-i}^k(T_i)] - \int_{T_i}^{T} p^k(t)\,dx_{-i}^k(t)\right\}. \quad (16)$$

One must be very cautious about implementing either of these two approximations. Only the full parallel auction procedure is non-manipulable; any approximating procedure is, in principle, manipulable. The way in which these approximations are likely to be manipulable is that a bidder may have incentive to bid insincerely along the price adjustment path, in order to influence the relative prices at which crediting and debiting begin to be applied to her own winnings. Still, the potential gain from such manipulation is likely to be relatively small and relatively difficult for a bidder to identify. Moreover, once an "endgame" is reached in which crediting and debiting have begun to be applied to all bidders, the theorems of this paper become applicable, and sincere bidding is an equilibrium strategy thereafter. Thus, the approximations may be effective for real-world auction applications, and are worthy of further exploration.

9. CONCLUSION

This paper has considered economic environments with K types of heterogeneous commodities, containing consumers with quasilinear utilities and pure private values, and has presented an auction procedure in which sincere bidding is an equilibrium that yields efficient outcomes. This is shown both for a divisible commodities environment (with strictly concave utility functions) and a discrete commodities environment (with substitutes preferences). In the latter case, and under incomplete information and a "full support" assumption, sincere bidding is the unique outcome of iterated weak dominance. Using a single ascending price trajectory, a "modified" VCG mechanism is implemented; and using n ascending price trajectories that may be elicited in parallel, the full VCG mechanism is

[25] Such an activity rule is sometimes imposed on bidders in real-world ascending clock auctions, for example, in the quarterly EDF generation capacity auction.

[26] See Ausubel and Milgrom (2001).

implemented. Finally, two approximations to the parallel auction procedure are proposed that require only a single ascending price trajectory.

One immediate question is whether the auction design herein can be generalized to treat the case where bidders have interdependent values. Perry and Reny (2001) provide an affirmative answer to this question for my earlier efficient auction design treating homogeneous goods. They consider a model where each bidder receives a one-dimensional signal and where each bidder's valuation depends on the signals received by all n bidders. They show that: (1) with two bidders, the homogeneous goods auction leads to efficient outcomes with interdependent values; and (2) by allowing bidders to submit *directed demands* (one against each other bidder), it is possible to obtain efficient outcomes with interdependent values and n bidders.

I conjecture that essentially the same two steps can be replicated for the auction design herein. That is, with interdependent values and two bidders, the efficient dynamic auction for heterogeneous commodities should also lead to efficient outcomes; and, again, by allowing bidders to submit directed demands (one against each other bidder), it should be possible to obtain efficient outcomes with interdependent values and n bidders.

At the same time, such a complication of the current auction design is not entirely in the spirit of the current paper. Even the n parallel auctions version of the current design has been critiqued as requiring excessive communication and for requesting bids that may be of minimal payoff relevance to the bidders making them. Avoiding these complications is the rationale behind the two single-trajectory approximations proposed in Section 8. Introducing directed demands into the design would be a further step in the direction of increasing the required communication and complicating the auction.

Despite their theoretical limitations, simple clean auction designs have distinct advantages over the more complicated mechanisms that are required for achieving full efficiency. These advantages are difficult to model formally and tend not to be treated in the existing literature. Nevertheless, the relatively simple designs of the predecessor and current papers seem likely to fare well on matters of cognitive simplicity and robustness. If bidders are easily able to understand the auction design, they seem more likely to bid consistently with equilibrium behavior. And if an auction design cleanly reflects intuitive first principles, it is more likely to perform robustly in environments somewhat different from stylized economic models.

The viewpoint of the predecessor paper for homogeneous goods has been that a good compromise between these competing considerations is to utilize an auction design that is dynamic (so as to give some recognition of value interdependencies) while still simple at every step. The heterogeneous commodities design of the current paper attempts to adhere to this philosophy as much as possible. By so doing, it aspires to introduce efficient auction procedures sufficiently transparent and robust that they might actually find themselves adopted into practical usage someday.

References

APOSTOL, TOM M. (1957), *Mathematical Analysis: A Modern Approach to Advanced Calculus*, Reading, Massachusetts: Addison-Wesley.

ARROW, KENNETH J. (1959), "Toward a Theory of Price Adjustment," in Moses Abramovitz, ed., *The Allocation of Economic Resources*, Stanford: Stanford University Press, pp. 41–51.

ARROW, KENNETH J., H. D. BLOCK AND LEONID HURWICZ (1959), "On the Stability of the Competitive Equilibrium, II," *Econometrica*, January, 27(1), pp. 82–109.

ARROW, KENNETH J. AND FRANK H. HAHN (1971), *General Competitive Analysis*, San Francisco: Holden-Day.

AUSUBEL, LAWRENCE M. (1996), "An Efficient Ascending-Bid Auction for Dissimilar Objects," mimeo, University of Maryland, Department of Economics, January.

AUSUBEL, LAWRENCE M. (1997), "An Efficient Ascending-Bid Auction for Multiple Objects," Working Paper No. 97–06, University of Maryland, Department of Economics, June.

AUSUBEL, LAWRENCE M. (1999), "A Mechanism Generalizing the Vickrey Auction," mimeo, University of Maryland, Department of Economics, September.

AUSUBEL, LAWRENCE M. (2000), "System and Method for an Efficient Dynamic Auction for Multiple Objects," U.S. Patent Number 6,026,383, filed January 4, 1996; issued February 15, 2000.

AUSUBEL, LAWRENCE M. AND PETER C. CRAMTON (1996), "Demand Reduction and Inefficiency in Multi-Unit Auctions," Working Paper No. 96–07, University of Maryland, Department of Economics, July.

AUSUBEL, LAWRENCE M. AND PETER C. CRAMTON (1999), "Vickrey Auctions with Reserve Pricing," mimeo, University of Maryland, Department of Economics, June.

AUSUBEL, LAWRENCE M. AND PAUL MILGROM (2001), "System and Method for a Dynamic Auction with Package Bidding," Patent Cooperation Treaty patent application, November.

AUSUBEL, LAWRENCE M. AND PAUL MILGROM (2002), "Ascending Auctions with Package Bidding," working paper, University of Maryland and Stanford University, April.

BANKS, JEFFREY S., JOHN O. LEDYARD AND DAVID P. PORTER (1989), "Allocating Uncertain and Unresponsive Resources: An Experimental Approach," *Rand Journal of Economics*, 20, pp. 1–25.

BIKHCHANDANI, SUSHIL, SVEN DE VRIES, JAMES SCHUMMER AND RAKESH V. VOHRA (2001), "Linear Programming and Vickrey Auctions," mimeo, UCLA and Northwestern University, May.

BIKHCHANDANI, SUSHIL AND JOHN W. MAMER (1997), "Competitive Equilibrium in an Exchange Economy with Indivisibilities," *Journal of Economic Theory*, June, 74(2), pp. 385–413.

BIKHCHANDANI, SUSHIL AND JOSEPH M. OSTROY (2001a), "The Package Assignment Model," forthcoming, *Journal of Economic Theory*.

BIKHCHANDANI, SUSHIL AND JOSEPH M. OSTROY (2001b), "Ascending Price Vickrey Auctions," mimeo, UCLA, September.

CLARKE, EDWARD H. (1971), "Multipart Pricing of Public Goods," *Public Choice*, 11, pp. 17–33.

DASGUPTA, PARTHA AND ERIC S. MASKIN (2000), "Efficient Auctions," *Quarterly Journal of Economics*, May, 115(2), pp. 341–388.

GROVES, THEODORE (1973), "Incentives in Teams," *Econometrica*, July, 41(4), pp. 617–631.

GUL, FARUK AND ENNIO STACCHETTI (1999), "Walrasian Equilibrium with Gross Substitutes," *Journal of Economic Theory*, July, 87(1), pp. 95–124.

GUL, FARUK AND ENNIO STACCHETTI (2000), "The English Auction with Differentiated Commodities," *Journal of Economic Theory*, May, 92(1), pp. 66–95.

HAHN, FRANK H. (1982), "Stability," in Kenneth J. Arrow and Michael D. Intriligator, eds., *Handbook of Mathematical Economics, Volume II*, Amsterdam: North-Holland, chapter 16.

JEHIEL, PHILIPPE AND BENNY MOLDOVANU (2001), "Efficient Design with Interdependent Valuations," *Econometrica*, September, 69(5), pp. 1237–1259.

KELSO, ALEXANDER S., JR. AND VINCENT P. CRAWFORD (1982), "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, November, 50(6), pp. 1483–1504.

KRISHNA, VIJAY AND ELIOT MAENNER (2001), "Convex Potentials with an Application to Mechanism Design," *Econometrica*, July, 69(4), pp. 1113–1119.

MASKIN, ERIC S. (1992), "Auctions and Privatization," in Horst Siebert, ed., *Privatization: Symposium in Honor of Herbert Giersch*, Tubingen: Mohr (Siebeck), pp. 115−136.

MILGROM, PAUL R. (2000), "Putting Auction Theory to Work: The Simultaneous Ascending Auction," *Journal of Political Economy*, April, 108(2), pp. 245−272.

MILGROM, PAUL R. AND ROBERT J. WEBER (1982), "A Theory of Auctions and Competitive Bidding," *Econometrica*, September, 50(5), pp. 1089−1122.

PARKES, DAVID C. AND LYLE H. UNGAR (2002), "An Ascending-Price Generalized Vickrey Auction," working paper, Harvard University and University of Pennsylvania, June.

PERRY, MOTTY AND PHILIP J. RENY (2001), "An Efficient Multi-Unit Ascending Auction," Working Paper, Hebrew University and University of Chicago, May.

PERRY, MOTTY AND PHILIP J. RENY (2002), "An Efficient Auction," *Econometrica*, May, 70(3), pp. 1199−1212.

ROCKAFELLAR, R. TYRRELL (1970), *Convex Analysis*, Princeton: Princeton University Press.

SAMUELSON, PAUL A. (1941), "The Stability of Equilibrium: Comparative Statics and Dynamics," *Econometrica*, April, 9(2), pp. 97−120.

SCARF, HERBERT (1960), "Some Examples of Global Instability of the Competitive Equilibrium," *International Economic Review*, September, 1(3), pp. 157−172.

VARIAN, HAL R. (1981), "Dynamical Systems with Applications to Economics," in Kenneth J. Arrow and Michael D. Intriligator, eds., *Handbook of Mathematical Economics, Volume I*, Amsterdam: North-Holland, chapter 3.

VICKREY, WILLIAM (1961), "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance*, March, 16(1), pp. 8−37.

WALRAS, LÉON (1874), *Éléments d'Économie Politique Pure*, Lausanne: Corbaz.

Appendix

PROOF OF LEMMA 3. With strictly concave utility functions and compact, convex consumption sets, observe that the demand correspondences are single-valued and continuous in price. Given that Eq. (11) describing Walrasian tâtonnement is uniformly bounded, all price paths $p(\bullet)$ are Lipschitz-continuous in *t*, and sincere bidding induces continuous actions $x_i(\bullet)$ by each bidder that are of bounded variation in *t* for each component *k*. Hence, Lemmas 1 and 2 apply. We define the following Lyapunov function:

$$V(p) = p \cdot S + \sum_{i=1}^{n} V_i(p), \tag{17}$$

where $V_i(p)$ is the net indirect utility function of Eq. (1). (This Lyapunov function is selected on account that its subgradient at p is $S - \sum_{i=1}^{n} q_i(p)$, the excess demand vector.) Using Eq. (3), we find that:

$$\dot{V} = \frac{dV(p(t))}{dt} = \left(S - \sum_{i=1}^{n} q_i(p(t))\right) \cdot \dot{p}(t) \text{, almost everywhere in } t. \tag{18}$$

Observe that, in the adjustment process of Eq. (11), $\dot{p}^k$ has the opposite sign as $S^k - \sum_{i=1}^{n} q_i^k(p)$. Hence, Eq. (18) implies that $\dot{V} = 0$ at all Walrasian equilibrium price vectors and $\dot{V} < 0$ at all other price vectors. Note that $V(\bullet)$ as defined by Eq. (17) is convex, and so any local minimum is also a global minimum. Letting $V^* = \lim_{t\to\infty} V(p(t))$, we conclude that V^* minimizes *V*(*p*), and p^* associated with V^* is a Walrasian equilibrium price vector. ■

TÂTONNEMENT PROCESS FOR DISCRETE GOODS

In order to facilitate the description of the efficient dynamic auction for the discrete goods case, we make a series of three observations, all relying on the assumption (included in A1') that each bidder has an integer valuation for every consumption bundle as well as on the substitutes condition (A4'):

OBSERVATION 1. *The lowest Walrasian price vector,* $\underline{p}$, *consists of integers.*

OBSERVATION 2. *Given bidder i's demand correspondence* $Q_i(p)$ *at integer-valued price vector p, the auctioneer can deduce the demand correspondence* $Q_i(\bullet)$ *over the region* $\{p + \Delta : \Delta \geq 0 \text{ and } \Delta \cdot \overline{S} < 1\}$, *where* $\overline{S} = \sum_{k=1}^{K} S^k$ *is the total supply of items.*

OBSERVATION 3. *Let* $E \subset \{1, \ldots, K\}$ *be any arbitrary set of items and let p be any arbitrary integer-valued price vector. Let* $p + \Delta 1^E$ *be notation for a price vector whose* k^{th} *coordinate equals* $p^k + \Delta$, for $k \in E$ *and equals* p^k, for $k \notin E$. *Then there exists* $x = (x_1, \ldots, x_n)$, *a profile of demand vectors for every bidder, such that* $x_i \in Q_i(\hat{p})$ *for all prices in the set* $\{\hat{p} \in \mathbb{R}^K : \hat{p} = p + \Delta 1^E, \text{ for some } \Delta \in [0,1]\}$.

Observation 1 exploits an elegant characterization of Walrasian equilibrium for substitutes goods provided by Gul and Stacchetti (1999). Let $W(S)$ denote the social value from an efficient allocation of the available supply of goods. Let $S + 1^k \equiv (S^1, \ldots, S^{k-1}, S^k + 1, S^{k+1}, \ldots, S^K)$. Thus, $W(S + 1^k) - W(S)$

represents the incremental social value of making one more unit of commodity k available. Theorem 4 of Gul and Stacchetti (1999) states that $\underline{p}$ defined by $\underline{p}^k = W(S+1^k) - W(S)$, for k = 1, … ,K, is the lowest Walrasian price vector. Given that every bidder has an integer valuation for every commodity bundle, the function $W(\bullet)$, and hence $\underline{p}$, is integer valued.

Observation 2 establishes that, if bidders report their demand correspondences at an integer price vector, the report contains sufficient information to infer a demand correspondence evaluated in a surrounding neighborhood. It is argued by supposing that $x \in Q_i(p)$ and $y \notin Q_i(p)$. Again using the assumption that bidders have integer valuations for commodity bundles, $U_i(x) - p \cdot x \geq U_i(y) - p \cdot y + 1$. Therefore, the inequality $U_i(x) - p \cdot x > U_i(y) - p \cdot y$ continues to hold everywhere in the region $\left\{p + \Delta : \Delta \geq 0 \text{ and } \Delta \cdot \overline{S} < 1\right\}$, so $y \notin Q_i(\bullet)$ in this region. Meanwhile, consider any $x, x' \in Q_i(p)$. Since the auctioneer knows that $U_i(x) - p \cdot x = U_i(x') - p \cdot x'$, the auctioneer has sufficient information to deduce which of $U_i(x) - \hat{p} \cdot x$ or $U_i(x') - \hat{p} \cdot x'$ is greater at *any* $\hat{p}$, enabling the complete determination of $Q_i(\bullet)$.

Observation 3 establishes that a constant profile of demand vectors may be selected along the straight line from p to $p + 1^E$. This follows directly from Observation 2, for the set of prices $\left\{\hat{p} \in \mathbb{R}^K : \hat{p} = p + \Delta 1^E, \text{for some } \Delta \in (0, 1/\overline{S})\right\}$. To extend this to all $\Delta \in [0,1]$, we heavily exploit the substitutes condition and suppose otherwise. Then there exists $\overline{\Delta} \in (0,1)$ and profiles of demand vectors x and y such that $x \in Q_i(p + \Delta 1^E)$ but $y \notin Q_i(p + \Delta 1^E)$ for $\Delta \uparrow \overline{\Delta}$, while $y \in Q_i(p + \Delta 1^E)$ but $x \notin Q_i(p + \Delta 1^E)$ for $\Delta \downarrow \overline{\Delta}$. At the price vector $p + \overline{\Delta} 1^E$, bidder i is indifferent between x and y. Using the substitutes condition, it is possible to show that y can be constructed so that $y^k = x^k - 1$ for some commodity k and $y^l \geq x^l$ for all $l \neq k$. But then, bidder i cannot prefer y to x at price vector $p + \overline{\Delta} 1^E$, unless $\overline{\Delta} \geq 1$, a contradiction.

We now provide a minor variation on Gul and Stacchetti's price adjustment procedure. Given Observations 1–3, above, the auctioneer proceeds iteratively, as follows. At each time T = 0, 1, 2, … , the auctioneer announces an integer-valued price vector p, receives demand correspondence reports $\{x_i(p)\}_{i=1}^n$, and determines a set of commodities E whose prices are incremented by 1 at time T+1. The set E is determined by a procedure somewhat simplifying that specified by Gul and Stacchetti (2000).

DEFINITION A. For integer-valued price vector p and scalar $\Delta > 0$, define the following neighborhood of prices above p:

$$R(p, \Delta) = \left\{\hat{p} \in \mathbb{R}^K : p^k < \hat{p}^k < p^k + \Delta \text{ for every } k \text{ and } Q_i(\hat{p}) \text{ is single valued for every } i\right\}. \quad (19)$$

The nonempty set E $(E \subset \{1, \ldots, K\})$ of commodities will be said to be *overdemanded* at price p if:

$$\text{There exists } \Delta > 0 \text{ s.t. for every } \hat{p} \in R(p, \Delta), \text{ there exists } k \in E \text{ satisfying } \textstyle\sum_{i=1}^n q_i^k(\hat{p}) > S^k. \quad (20)$$

Furthermore, E will be said to be a *minimal overdemanded set* at p if E is overdemanded at p but no subset $E' \subset E \; (E' \neq E)$ is overdemanded at p.

Observe that, if E is overdemanded, then all supersets of E are also overdemanded. Hence, it is sufficient to check all subsets E' formed by removing one element of E. This immediately giving us the following quick procedure, which also guarantees the existence of a minimal overdemanded set:

PROPOSITION A. For any report of demand correspondences by bidders in response to an integer price vector p, either the set $\{1, \ldots, K\}$ is not overdemanded or there exists at least one minimal overdemanded set.

PROOF. For any demand reports, begin with the set $\{1, \ldots, K\}$. If it is overdemanded, then consider each subset of $\{1, \ldots, K\}$ formed by removing one element. If none of these subsets is overdemanded, then $\{1, \ldots, K\}$ is a minimal overdemanded set. Otherwise, select an overdemanded subset and iteratively repeat this process. Since the empty set $\varnothing$ is not overdemanded, the process must terminate with the discovery of a minimal overdemanded set. ■

Section 7 of the paper goes on to show that, using the price adjustment equation (14) for a minimal overdemanded set E, the new auction procedure of this paper has analogous properties as we established in Sections 5 and 6 for the case of divisible commodities using Walrasian tâtonnement. To assist the reader in understanding the results, it may be helpful to make a distinction between where continuous time versus discrete time is being utilized. As Eq. (14) suggests, the price path and all payoff calculations continue to be specified in continuous time $t \in [0, \infty)$. This is done so that the payoff Eq. (7) and Lemmas 1 and 2 continue literally to hold. However, the auctioneer's price announcement and bidders' demand reports now only need to occur at discrete times $t = 0, 1, 2, \ldots$. This simplification is possible in light of Observations 2 and 3, which establish, first, that sufficient information is reported at integer time T to guide price adjustment for the interval $[T, T+1)$, and second, that sufficient information is reported at integer time T to deduce bidders' demands for the interval $[T, T+1)$. Finally, Observation 1 assures that proceeding in integer steps will never lead us to overshoot the lowest Walrasian equilibrium price vector.