Identifying Risk Preferences

Pierre-André Chiappori, Amit Gandhi, Bernard Salanié and Francois Salanié

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What Do We Know About Risk Preferences?

Not that much:

- intuition drawn from theory + casual observation (Arrow 65):
  - $\text{ARA}(x) = -u''(x)/u'(x)$ should be decreasing, since richer people buy more risk;
  - $\text{RRA}(x) = -xu''(x)/u'(x)$ should be close to constant, as the proportion of wealth invested in risky assets is fairly constant across wealth levels (?).
- but this completely neglects composition effects, inter alia.
- financial and insurance evidence: points (or used to point) to very high risk-aversion, $\text{RRA} \simeq 30$.

Experimental evidence

Points to violations of expected utility, since Allais 1953, at least "close to the edges of the triangle" (where some probabilities are small).

Also suggests that (generalized) risk aversions are very heterogeneous:

Barsky et al (QJE 1997) use survey questions, linked to actual behavior;

they report D1=2 and D9=25 for RRA, poorly explained by demographics.

Guiso-Paiella (2003) report similar findings ("massive unexplained heterogeneity").

Yet much of economics does not take this heterogeneity very seriously.

Can we document this heterogeneity on "actual" data?

Using Horse Bets: The Pros

- very simple set up: a “win bet” at odds R on horse i buys an Arrow-Debreu asset for state “i wins” with net return R;
- lots of data is available (more than 100,000 races every year)

Cf Jullien-Salanié (JPE 2000): the representative bettor violates expected utility as (s)he overweighs small probabilities of losses.

Using Horse Bets: The Cons

- only weirdoes bet on horses (or at least gamblers); (but Gandhi (2006) shows that risk-averse agents may find it profitable to bet "against" risk-lovers)
- odds are only known ex ante on bookmaker markets (the UK), but then market shares are hard to get—which is why Jullien-Salanié (2000) could only estimate a representative risk preference;
- market shares are known (since they determine odds) on parimutuel markets, but then odds are imperfectly known ex ante—although estimates are published;
- we "must" assume that state probabilities are perfectly anticipated;
- we do not observe the size of the bets, so we estimate "reduced" utility functions.

The Identification Question

Assume a population of bettors, stable in time (given some observed characteristics)—participation is for future work

Take one of them: (s)he values a \$1 bet that

- wins (net) $\$R$ with probability p
- loses \$1 with probability $(1-p)$

as $V(p,R)$.

e.g., with expected utility theory (EUT), u rebased at current wealth:

$$V(p,R) = pu(R) + (1-p)u(-1).$$

or, for Cumulative Prospect Theory (CPT)

$$V(p,R) = G(p)pu_{+}(R) + H(1-p)u_{-}(-1).$$

Can we recover uniquely the distribution of V in the population?

The Data

Our data is a large number of races $m = 1, \ldots, M$
A race m consists of

- a number of horses n^m
- a vector of odds R_i^m for $i = 1, \ldots, n^m$
- the index f^m of the horse that won race m.

Identification

The data gives us directly *market shares:* in race m for each horse i

$$s_i^m(R_i^m + 1) = 1 - t$$

where s_i is market share of i and t is "track take"
so

$$s_i^m = \frac{\frac{1}{R_i^m+1}}{\sum_{j=1}^{n^m} \frac{1}{R_j^m+1}}.$$

which we denote $S_i(R^m)$.
But we also know that

$$s_i^m = \Pr(V(p_i^m, R_i^m) \geq V(p_j^m, R_j^m) \ \ \forall j = 1, \ldots, n^m),$$

where the probability is over V in the population of bettors
(*at this stage, behavioral errors also enter* V)
We also have one realization of a draw in $(p_1^m, \ldots, p_{n^m}^m)$, since we know that f^m won.

Intuitive Identification

Suppose all races have exactly n horses
and we observe an infinity of races, so that
for every possible vector of odds $R = (R_1, \ldots, R_{n-1})$

- we can estimate $p_i(R)$ for $i = 1, \ldots, n-1$ by the proportion of such races won by horse i

$$p_i(R) \simeq \frac{\sum_{R^m = R}(f^m = i)}{\sum_{R^m = R} 1}.$$

- we know that by definition,

$$S_i(R) = \Pr(V(p_i(R), R_i) \geq V(p_j(R), R_j) \ \ \forall j). \qquad (E)$$

Counting Equations

We have $(n-1)$ functions $p_i(R)$
"Therefore" we can identify the distribution of V in an $(n-1)$-dimensional space:
assume that there exists a parameterization $V(p, R) = W(p, R, \theta)$ where

- the "master function" W is known to us
- θ is a vector of parameters in some subset Θ of $\mathbb{R}^{n-1}$ that describes preferences + behavioral errors

then barring misspecification (i.e. if no bettor has preferences outside of $W(.,.,\Theta)$)
we can recover the true distribution of preferences uniquely.
Not so obvious: e.g., in fact $p_i(R) = P(R_i, (R_{-i}))$ must be symmetric in the $R_j, j \neq i$.

A Digression on Equilibrium

A tangential question: is there an equilibrium, is it unique? I.e. what is the set of solutions to (E)?

Theorem: fix any $(p_1, \ldots, p_{n-1})$ (all positive) and any distribution of preferences V that

- is atomless
- only contains increasing preferences
- is such that any horse that may win is *desirable*: if its odds go to infinity its market share eventually will be positive

Then the system $S_i(R) = \Pr(V(p_i, R_i) \geq V(p_j, R_j) \ \ \forall j)$ has a unique solution $(R_1, \ldots, R_n) = R(p)$.

Where We Are Stuck

For any $R = (R_1, \ldots, R_n)$, denote $\Theta(i, R)$ the subset of Θ such that

$$\forall j = 1, \ldots, n, \quad V(p_i(R), R_i, \theta) \geq V(p_j(R), R_j, \theta).$$

Then in fact $\Theta(i, R)$ is a function of R_i and symmetrically of (R_{-i}):

$$\Theta(R_i, (R_{-i})).$$

and we know the probability of all such sets when R_i and (R_{-i}) vary freely

Is it enough? I.e. is this a *probability-determining family* for Θ so that it identifies the distribution of θ?

Conjecture

Yes, it is "usually" enough to insure that the following two condition holds:

The Many-Races Assumption: for any subset A of Θ, there exists a race R and a horse i such that $\Theta_i(R) \subset A$.

Then assume two candidate probabilities on Θ with pdfs f and g.

Say they differ on a subset A, with $f(\theta) > g(\theta)$ on A.

By MRA, take $\Theta_i(R) \subset A$; then f puts greater probability than g on $\Theta_i(R)$, which contradicts

$$\int_{\Theta_i(R)} f(\theta) d\theta = \int_{\Theta_i(R)} g(\theta) d\theta = S_i(R).$$

Note that MRA implies **separability:** for any $\theta \neq \theta'$, there exists a race R and horse i such that $\theta \in \Theta_i(R)$ but not θ'.

And separability cannot hold if Θ is more than $(n-1)$-dimensional.

Assume that Θ is a subset of $\mathbb{R}$, and that $n \geq 4$. We need a single-crossing condition:

Condition (SC): each $W(.,.,\theta)$ is increasing in p and R, and the marginal rate of substitution W'_R/W'_p increases in θ.

(SC) means that larger θ's prefer longer odds; e.g. if all preferences are EUT-CARA, we need the ARA index to decrease in θ.

It more or less excludes behavioral errors not perfectly correlated with heterogeneity.

What We Can Prove

Theorem: let F_0 be the true cdf of θ on an interval Θ of $\mathbb{R}$; then

- the data uniquely identify F_0;
- the assumption that all preferences belong to $W(.,.,\Theta)$ is testable.

From now on, look at the equivalent problem: F_0 known (we take it to be uniform on $[0,1]$), we look for the master function W.

Intuition

Given (SC), if we order odds as $R_1 \leq \ldots \leq R_n$ then the set of θ's who bet on horse i is some interval

$$\Theta(R_i, (R_{-i})) = [\theta_{i-1}(R), \theta_i(R)]$$

where $\theta_0(R) = 0, \theta_n(R) = 1$ and for $i = 1, \ldots, n-1$,

$$W(p_i(R), R_i, \theta_i(R)) = W(p_{i+1}(R), R_{i+1}, \theta_i(R)) \qquad (I_i).$$

With F_0 uniform on $[0,1]$, we can estimate the $\theta_i(R)$'s using

$$S_i(R) = \theta_i(R) - \theta_{i-1}(R)$$

But intervals are probability-determining sets on $\mathbb{R}$... so we are done and there is nothing to test?
Not quite: symmetry + our assumptions on derivatives have consequences.

Some Notation

First define $\Gamma(v, R, \theta)$ by

$$\Gamma(W(p, R, \theta), R, \theta) \equiv p :$$

Γ increases in v, decreases in R, and $\Gamma''_{R\theta} < 0$ by (SC).
Then use change of variables:

$$\phi_1 = p_i(R);\ \phi_2 = \theta_i(R);\ \phi_3 = R_i;\ \phi_4 = R_{i+1};$$

$$\text{complete with } R_1, \ldots, R_{i-3} and R_{i+2}, \ldots, R_n \text{ if non-empty}$$

and define $\pi_{i+1}(\phi) = p_{i+1}(R)$.

The New Indifference Condition

$$W(p_i(R), R_i, \theta_i(R)) = W(p_{i+1}(R), R_{i+1}, \theta_i(R)) \qquad (I_i).$$

becomes

$$\pi_{i+1}(\phi) = \Gamma(W(\phi_1, \phi_3, \phi_2), \phi_4, \phi_2) \qquad (J_i).$$

Immediate consequence:

$$\pi_{i+1} \text{ does not depend on i,} \quad \text{and } \frac{\partial \pi_{i+1}}{\partial \phi_k} = 0 \text{ for } k > 4. \qquad (IC)$$

Testable by "regressing" $p_{i+1}(R)$ on

$$p_i(R), \theta_i(R), R_i, R_{i+1} \text{ and } R_1, \ldots, R_{i-3}, R_{i+2}, \ldots, R_n, \text{ and } i,$$

and testing that "the coefficients in the second group are all zero".

Another Equality Condition

The "marginal rate of substitution" between ϕ_1 and ϕ_3, i.e.

$$\frac{\frac{\partial \pi_{i+1}}{\partial \phi_1}}{\frac{\partial \pi_{i+1}}{\partial \phi_3}}$$

does not depend on ϕ_4; call it (MRS).

Sufficiency?

If (IC) and (MRS) hold then we can write

$$\pi_{i+1}(\phi) = G(H(\phi_1, \phi_3, \phi_2), \phi_4, \phi_2)$$

for some functions G and H.
We would like to identify H to W and G to Γ;
But we also need to check that

$$H'_p > 0,\ H'_R > 0,\ H'_R/H'_p \text{ increases in } \theta,$$

and

$$G \text{ increases in } H \text{ and decreases in } \phi_4.$$

Sufficiency!

These additional conditions turn out to boil down to:

$$\pi_{i+1} \text{ increases in } \phi_1 \text{ and in } \phi_3; \qquad (V_1)$$

$$\pi_{i+1} \text{ decreases in } \phi_4; \qquad (V_2)$$

and the MRS of π_{i+1} in (ϕ_1, ϕ_3), i.e.

$$\frac{\frac{\partial \pi_{i+1}}{\partial \phi_3}}{\frac{\partial \pi_{i+1}}{\partial \phi_1}}$$

increases in ϕ_2 (call this (V_3)).
Adding these conditions $(V_1), (V_2), (V_3)$ to (IC) and (MRS) yields a set of necessary and sufficient conditions for identification (up to an increasing transformation $w(p, R, \theta) = F(W(p, R, \theta), \theta)$)
If the model is well-specified.

Constructing the Indifference Curves

Back

Given the estimated $\pi_{i+1}(\phi_1, \phi_2, \phi_3, \phi_4)$ function, we fix $\phi_2 = \theta$; for any point in the $(\phi_1, \phi_3) = (p, R)$ plane we know that the indifference curve of any representation of $W(p, R, \theta)$ has slope

$$\frac{\frac{\partial \pi_{i+1}}{\partial \phi_1}}{\frac{\partial \pi_{i+1}}{\partial \phi_3}}(\phi_1, \phi_2, \phi_3, \phi_4)$$

(for any value of ϕ_4).

This gives the last condition, a **test for misspecification:** Once the indifference curve for θ that goes through (p, R) is constructed, choose some odds R' and compute

$p' = \pi_{i+1}(p, \theta, R, R')$;

then (p', R') should lie on that same indifference curve.

Going Further: CPT

CPT is equivalent to

$$\frac{\partial^2 \log \frac{\partial W}{\partial R}}{\partial p \partial R} = 0$$

for *one* representation of W.
Not straightforward to test (nonparametrically). Expected utility is easier:
Assume $W(p, R, \theta) = F(pu(R, \theta), \theta)$; then we get

$$\pi_{i+1}(\phi) = \phi_1 \frac{u(\phi_3, \phi_2)}{u(\phi_4, \phi_2)}$$

Going Further: Expected Utility

Thus EUT yields three additional conditions; define $\psi_{i+1}(\phi) = \log\left(\pi_{i+1}(\phi)/\phi_1\right)$:

$$\psi_{i+1}(\phi) \text{ only depends on } \phi_2, \phi_3 \text{ and } \phi_4 \qquad (EU_1)$$

$$\frac{\partial^2 \psi_{i+1}}{\partial \phi_3 \partial \phi_4} = 0 \qquad (EU_2)$$

and

$$\psi_{i+1}(\phi) = 1 \text{ if } \phi_3 = \phi_4. \qquad (EU_3)$$

$(EU_1), (EU_2), (EU_3)$ complete the set of necessary and sufficient conditions under expected utility
(Visually: fix θ and R and plot $p \longrightarrow W(p, R, \theta)$).

Testing Homogeneous Risk Preferences

An easy one: just add

$$\frac{\partial \pi_{i+1}(\phi)}{\partial \phi_2} = 0.$$

(Visually: just plot the indifference curves through some (p, R) for various θ's).

Empirical Strategy: Estimating Probabilities

First specify a flexible functional form for $p_i(R) = P(R_i, (R_{-i}))$:

$$p_i = \frac{e^{q_i}}{\sum_{j=1}^{n} e^{q_j}}$$

with, e.g.

$$q_i(R) = \sum_{k=1}^{K} a_k(R_i, \alpha) T_k(R_{-i})$$

and the T_k's are symmetric polynomials
Then maximize over α the log-likelihood

$$\sum_{m=1}^{M} \log p_{f^m}(R^m, \alpha).$$

Let $\hat{\alpha}$ be the estimate; at this stage we take it to be the true parameter vector (we do not use its estimated variance in the tests.)

Empirical Strategy: Testing

For any race m we plug the $\hat{p}_i(R) = P(R_i, (R_{-i}), \alpha$ or, better, the corresponding odds ratio

$$\hat{O}_i(R) = \log \frac{p_i(R)}{1 - p_i(R)}$$

in the identification conditions.

Testing Equality Conditions

E.g. for the equality condition $\frac{\partial \pi_4}{\partial R_5}(\phi) = 0$:
take all races $m \in M_5$ with at least 5 horses; on this subsample

1. we regress (flexibly) $\hat{O}_4(R^m)$ on

$$\hat{p}_3(R^m), \theta_3(R^m), R_3^m \text{ and } R_4^m$$

2. we add R_5^m to the regression
3. we evaluate the increase in (generalized) R^2.

If this R^2 is (economically!) significantly positive, it signals a violation.

Testing Inequality Conditions

E.g. for the inequality condition $\frac{\partial \pi_6}{\partial \phi_1}(\phi) > 0$:
take all races m with at least 6 horses; on this subsample

1. we regress (flexibly) $\hat{O}_6(R^m)$ on

$$\hat{p}_5(R^m), \theta_5(R^m), R_5^m \text{ and } R_6^m$$

2. for each such race m we evaluate the derivative wrt $\hat{p}_5(R^m)$, call it A^m.

A large enough percentage of negative A^m's signals a violation.

Empirical Strategy: Estimation

(At least) three possible methods:

1. given the estimated $P(R_i, (R_{-i}), \hat{\alpha})$, use the nonparametric construction of indifference curves as above
2. given the estimated $P(R_i, (R_{-i}), \hat{\alpha})$ and a flexible functional form for $W(p, R, \theta, \beta)$,
   for any β generate the $p_j(\beta)$, $j \geq 2$ from $P(R_1, (R_{-1}), \hat{\alpha})$ using the recursive conditions
   $$W(p_i, R_i, \theta_i, \beta) = W(p_{i+1}, R_{i+1}, \theta_i, \beta)$$
   then match them to the $P(R_j, (R_{-j}), \hat{\alpha})$
3. define $P_i^m(\alpha) = P(R_i^m, (R_{-i}^m), \alpha)$; use GMM to estimate jointly α and β, using the moment conditions $E(h(R^m, \alpha_0, \beta_0)|R^m) = 0$ where h can be
   $$(i = f^m) - P_i^m(\alpha)$$
   for $i = 1, \ldots, n^m$ and
   $$W(P_i^m(\alpha), R_i^m, \theta_i(R^m), \beta) - W(P_{i+1}^m(\alpha), R_{i+1}^m, \theta_i(R^m), \beta)$$