The Theory of Assortative Matching Based on Costly Signals

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Abstract

We study two-sided markets with a finite numbers of agents on each side, and with two-sided incomplete information. Agents are matched assortatively on the basis of costly signals. We study how the signalling activity and welfare on each side of the market change when we vary the number of agents and the distribution of their attributes, thereby emphasizing new phenomena that cannot occur in large markets. We also identify conditions under which the potential increase in expected output due to assortative matching (relative to random matching) is completely offset if signalling is wasteful. Finally, we look at the continuous version of our two-sided market model, and establish the connections to the finite version. Technically, the paper is based on the very elegant theory about stochastic ordering of (normalized) spacings and other linear combinations of order statistics of distributions with monotone failure rates, pioneered by R. Barlow and F. Proschan (1966, 1975) in the framework of reliablity theory.

1 Introduction

We examine two-sided markets where a finite number of privately informed agents on each side of the market compete for potential matching partners on the other side. Examples include marriage markets, labor and education markets, markets for venture capital and new technologies. In these markets, agents typically differ in their attributes, and they gain from being matched with a better partner. If the matching surplus function is supermodular, as we assume here, total surplus from matching is maximized by matching the agents assortatively. But assortative matching cannot work (at least not directly) if types are private information - here signalling can fulfill a crucial role. By revealing private information about types, signalling can help determine who is to be matched with whom, thus increasing aggregate output. One of our main goals is to identify conditions under which the potential increase in output is completely offset by the costs of signalling.

The paper combines three main features:

1. We consider a finite number of agents on both sides of the market. More precisely, we "multiply" two tournament models with several agents and several prizes (as developed by Moldovanu and Sela 2001, 2005) by letting the agents on one side represent the prizes for which the agents on the other side compete. Thus, both sides are active here and the signalling behavior of each agent is affected by features (such as number of agents and distribution of characteristics) of both sides of the market.

2. We allow for incomplete information on both sides. Since there is a finite number of agents, no agent knows here for sure his/her rank in its own population, nor the quality of a prospective equilibrium partner. This should be contrasted with the situation in models with a continuum of agents, or with complete information, where knowledge of own attribute and of the distributions of attributes on both sides of the market completely determines own and equilibrium-partner rank, and the value of the equilibrium match. In our model values are interdependent, and agents need to form expectations about the attributes of other agents on both sides of the market.

3. We introduce a new mathematical methodology to the study of two-sided markets with a finite number of agents. This is based on the elegant work on stochastic orders among (normalized) spacings (e.g., differences) and other linear combinations of order statistics, pioneered by Richard Barlow

and Frank Proschan (1965, 1966, 1975) in the framework of reliability theory.[1] Roughly speaking, Barlow and Proschan show how the behavior of linear combinations of order statistics is controlled by monotonicity properties of the failure rates of underlying distributions. As mentioned above, our agents form expectations about the others' attributes, and their strategic behavior is determined by properties of the marginal gains from getting (stochastically) better partners. We can apply Barlow and Proschan's theory precisely because these marginal gains are represented here by spacings of order statistics, and because aggregate signalling and net welfare on each side of the market are linear combinations of so called *normalized spacings.*[2]

The paper is organized as follows:

In Section 2, we describe the matching model and introduce some useful definitions.

In Section 3 we derive a side-symmetric signalling equilibrium in strictly monotonic strategies. In this equilibrium, assortative matching based on the ranking of signals is equivalent (in terms of output) to assortative matching based on the ranking of true attributes.

The effects of increasing the number of agents (i.e., entry) in two-sided markets are analyzed in Section 4[3]. Entry affects the expected matching surplus, but also the agents' signalling activity. We show that the effects of entry (e.g., net effect on welfare on each side of the market) are determined by the failure rates of the underlying distributions of characteristics. In particular, we illustrate phenomena that are specific to relatively small markets. The entry results are also methodologically useful since some of our subsequent proofs proceed by considering a market with equal numbers of agents on each side to which we add agents in order to create a long side.

In Section 5 we study the effects (on both sides of the market) of changes in the distribution of attributes on one side. In Subsection 5.1 we first study the effects of increased heterogeneity on output, signalling, and welfare. While it is intuitive that more heterogeneity increases output in assortative matching, we show that the effect on signalling activity and welfare may be ambiguous

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[1] For basic texts on order statistics and stochastic orders, see David and Nagaraja, (2003), and Shaked and Shanthikumar (1994), respectively. Boland et al. (2002) is a good survey of the material most relevant for the present study.

[2] These are appropriately scaled spacings (see below).

[3] Our comparative statics results in this and the next section focus on aggregate measures of signalling and welfare. We briefly point out the implications for individual measures - these are governed by the same properties of failure rates.

(and that it depends, again, on failure rates). We also examine the conditions under which one side of the market necessarily incurs higher signalling costs than the other. In Subsection 5.2 we increase the distribution of one side of the market (say men) in the hazard rate order (which implies first order stochastic dominance). The effects on total output, men's total signalling, and women's total welfare, respectively are unambiguous: these always increase. Quite interestingly, women's total signalling and men's total welfare necessarily increase only under additional conditions on failure rates.

In Section 6 we compare random matching (without any signalling) with assortative matching based on costly and wasteful signalling. For distribution functions having an decreasing failure rate average (DFRA), assortative matching with signalling turns out to be welfare-superior, while for distribution functions having an increasing failure rate average (IFRA), random matching is superior. In the latter case, we also show that agents may be trapped: given that all others engage in signalling, signalling is indeed individually optimal, even though each agent may be better off under random matching, no matter what her (his) type is.

In Section 7 we look at the continuous version of our market model. Direct arguments in the continuous model can be used to yield similar results to the discrete version, but under weaker conditions. Instead of failure rates, these conditions involve now the coefficient of (co)variation of the distributions of types. A main insight is that total signalling effort in the continuous model equals exactly half of total output. We use this result to show that, in symmetric settings, assortative matching with signalling is welfare-superior (welfare inferior) to random matching if the coefficient of variation is larger (smaller) than unity. We also show that the discrete model analyzed in the previous sections converges to the continuous version by letting the number of agents go to infinity. In particular, some phenomena displayed in the finite version disappear in large populations, calling for some caution when making arguments (e.g., about welfare effects) in small markets.

Section 8 concludes. Appendix A contains several useful results from the statistical literature, while Appendix B contains all the proofs of our results.

Finally, we want to note here that many of our results have immediate implications for models with incomplete information on one side, or with complete information, as have been often used in the literature reviewed below. We give some examples in the text.

1.1   Related literature

The general insight that agents may choose costly signals to reveal private information is of course well-known: Spence (1973) has prominently shown how investment in education may serve as a signal to prospective employers even if the content of the education is itself negligible. A related idea appears in evolutionary biology where animals signal their fitness, i.e., their propensity to survive and reproduce, to potential mating partners. According to the *handicap principle*, put forward by Zahavi (1975), signals must be disadvantageous in order to be honest. The peacock's tail is a classical example.[4] The handicap principle is widely used to relate the evolution of some animal and human traits to *sexual selection*, i.e., the competition for mates,[5] but we are not aware of a full-fledged signalling-cum matching model in the biological literature (see the survey in Maynard-Smith and Harper, 2003).

The study of two-sided matching based on individual preferences was pioneered by Gale and Shapley (1962).[6] Becker (1973) focused on populations vertically differentiated by a unique, linearly ordered attribute, and stressed the implications of assortative matching.[7]

In Becker's framework (with a continuum of types and complete information), McAfee (2002) shows that, for a certain subset of distributions of characteristics,[8] a coarse matching involving only two distinct classes achieves at least as much output as the average of assortative matching and random matching.

Cole, Mailath and Postlewaite (1992) first emphasized an important variety of models (called *matching tournaments* by Hopkins, 2005) where agents get matched on the basis of some ex-ante costly actions (as the signals in our paper). We review below several relevant papers on matching tournaments - none of them contains a model with two active sides, incomplete information on both sides, and finite numbers of agents.

Cole, Mailath and Postlewaite (2001a,b) study complete information models with a continuum of agents and a finite number of agents, respectively, and with identical distributions of attributes. Agents can increase the value of a match by making costly investments, and the focus is on the possibility of achieving

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[4] Charles Darwin once remarked: "The sight of a peacock tail, whenever I gaze at it, makes me sick".

[5] See Zahavi and Zahavi, 1997, and Miller, 2001.

[6] Roth and Sotomayor (1990) is an excellent survey of the literature following Gale and Shapley's contribution.

[7] The classical contributions focus on centralized matching. In the framework of Becker's model, Shimer and Smith (2000) derive conditions under which a decentralized search equilibrium leads to assortative matching as search frictions become small.

[8] This is a subset of the class of distributions with increasing failure rates.

efficient ex-ante investments levels (thus overcoming the hold-up problem).

Peters (2004) studies the limit, as the number of agents goes to infinity, of mixed strategy equilibria arising in a model where a finite number of agents on each side of the market make costly investments prior to the match. There is complete information about the agents' types, and the utility of each agent depends on both her and her partner's investment. The limit need not correspond to the hedonic equilibrium in the market with a continuum of agents on each side.

Bulow and Levin (2005) look at the mixed-strategy equilibrium in a complete information model with a finite number of workers and firms. Only firms are active, and they make salary offers to workers, while workers choose firms in terms of their offers. The main results compare the resulting outcome with the competitive equilibrium in that market.

All models mentioned so far had complete information about attributes. Let us now review several models that allow for incomplete information on only one, active side.

Chao and Wilson (1987) and Wilson (1989) consider a seller facing a continuum of customers who differ in their valuations for service quality. Valuations are private information. They show how customers can be assortatively matched to service qualities by offering them price menus that induce them to reveal their type. They also derive asymptotic results about the relative efficiency loss of offering coarser quality classes.

Fernandez and Gali (1999) compare markets to matching tournaments in a model with a continuum of agents on each side. Again, only one side is active. The main result is that, in spite of the wasteful signalling, tournaments may be welfare superior to markets if the active agents have budget constraints.

Hopkins and Kornienko (2005) and Hopkins (2005) consider several versions of a labor markets with a continuum of workers and firms who differ in their quality levels. Only one side is active: while the quality of firms is observable, workers must exert an effort in order to signal their quality to firms. Their main results distinguish the effects on individual equilibrium behavior of workers caused by changes in the distribution of workers' attributes from those caused by changing the distribution of firms' attributes. Our analysis in Section 5 is, roughly speaking, similarly motivated, but the focus here is on aggregate levels of output, signalling, and welfare in markets where both sides are active.

In Nöldeke and Samuelson's (2003) biologically inspired model, several privately informed males compete for the attention of a unique female by sending costly signals. This is similar to an all-pay one-object auction, but the twist

is that the chosen male's fitness (and hence his ultimate attractiveness for the female) is reduced by the amount of the signal -this would correspond here to a corresponding decrease in the feasible output of a matched pair. The authors identify situations in which a signalling equilibrium may not exist.

Moldovanu and Sela (2001, 2005) study all-pay auctions with a finite number of privately informed bidders, and with a finite number of different prizes. Bids are submitted and ranked and then prizes are awarded accordingly (highest prize to highest bidder, etc...). The focus is on the revenue effects of changes in the number and size of the various prizes, in the bidding costs, and in the tournament's structure (e.g., one-stage or two-stage competition over prizes). As mentioned above, our present model is the extension of their analysis whereby "prizes come to life".

Kittsteiner and Moldovanu (2005) study an all-pay auction model where privately informed, randomly arriving (Poisson) customers bid for positions in a queue. In their model the value of a customer-position match depends also on attributes (e.g., processing times) of other customers. Both assortative and anti-assortative matching can occur in equilibrium, depending on the shape of the function measuring the cost of delay.

Damiano and Li (2004) allow for two-sided incomplete information in a model with a continuum of types on each side, extending the type of analysis performed on one side by Chao and Wilson (1987) and Wilson (1989) - we show here how this model arises as the limit of our model with finite numbers of agents on each side (see Section 7). Damiano and Li find conditions under which a revenue-maximizing match-maker finds it optimal to choose assortative matching rather than coarser schemes.

Hoppe, Moldovanu and Ozdenoren (2005) assess an intermediary's revenue loss from coarse matching à la McAfee (2002), and study the effects on revenue of various contractual agreements among matched partners.

The above two papers consider models with two-sided incomplete information. In this context it is also worth mentioning the relations to the literature on double auctions - see Perry and Reny (2005) for a recent model with interdependent values, and for a good survey of this strand of the literature. In contrast to the standard case in the double auction literature, our signals (that can be interpreted as bids) only determine who trades with whom, but not the terms of trade. On the other hand, in Perry and Reny's model (and in most of the literature) all traded units are identical (so that the optimal matching problem is fairly simple), while here they are heterogenous.

2 The matching model

There is a finite set $N = \{1, 2, ..., n\}$ of men, and a finite set $K = \{1, 2, ..., k\}$ of women, where $n \geq k$. Each man is characterized by an attribute x, each woman by an attribute y. If a man and a woman are matched, the utility of each is the product of their attributes. Thus, total output from a match between agents with types x and y is $2xy$.

Agent's i attribute is private information to i. Attributes are independently distributed over the interval $[0, \tau_F]$, $[0, \tau_G]$, τ_F, $\tau_G \leq \infty$, according to distributions F (men) and G (women), respectively. For all distributions mentioned in the paper we assume (without mentioning it again) that $F(0) = G(0) = 0$, that F and G have continuous densities, $f > 0$ and $g > 0$, respectively, and finite first and second moments - in particular, this last requirement will ensure that all integrals used below are well defined (e.g., all order statistics have finite expectations).

We study the following matching tournament: Each agent sends a costly signal b, and signals are submitted simultaneously. Agents on each side are ranked according to their signals, and are then matched assortatively. That is, the man with the highest signal is matched with the woman with the highest signal, the man with the second-highest signal is matched with woman with the second-highest signal, and so forth. Agents with same signals are randomly matched to the corresponding partners. The utility of a man with attribute x that is matched to a woman with attribute y after sending a signal b is given by $xy - b$ (and similarly for women). Thus, signals are costly. For the subsequent welfare comparisons we assume that signalling efforts are wasted from the point of view of our men and women[9]. In other variations, not explicitly considered here, these may accrue as rents to a third party. The equilibrium analysis is invariant to such alternative specifications.

Note that all our results can be extended to asymmetric production functions having the form $\delta(x)\rho(y)$, where δ and ρ are strictly increasing and differentiable (see Section 5.1 for an example with a Cobb-Douglas production function)

2.1 Order statistics and hazard rates

Let $X_{(1,n)} \leq X_{(2,n)} \leq \cdots \leq X_{(n,n)}$ and $Y_{(1,k)} \leq Y_{(2,k)} \leq \cdots \leq Y_{(k,k)}$, denote the order statistics of men and women, respectively. We define $X_{(0,n)} \equiv 0$ ($Y_{(0,k)} \equiv 0$).

Let $F_{(i,n)}$ ($G_{(i,k)}$) denote the distribution of $X_{(i,n)}$ ($Y_{(i,k)}$). The density of

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[9]That is, apart from their function enabling matching.

$X_{(i,n)}$ is given by :

$$f_{(i,n)}(x) = \frac{n!}{(i-1)!\,(n-i)!} F(x)^{i-1} \left[1 - F(x)\right]^{n-i} f(x),$$

and similarly for $Y_{(i,k)}$.

Let $F_i^n(s)$ be the probability that a man with type s meets $n-1$ competitors such that $i-1$ have a lower type and $n-i$ have a higher type. For $i = 2, \ldots n-1$, we obtain:

$$F_i^n(s) = F_{(i-1,n-1)}(s) - F_{(i,n-1)}(s) = \frac{(n-1)!}{(i-1)!\,(n-i)!} F(s)^{i-1} \left[1 - F(s)\right]^{n-i},$$

We let $F_n^n(s) = F_{(n-1,n-1)}(s)$, and $F_1^n(s) = 1 - F_{(1,n-1)}(s)$.

Similarly, we denote by $G_i^k(s)$ the probability that a woman with type s meets $k-1$ competitors such that $i-1$ have a lower type and $k-i$ have a higher type.

Let EX be the expectation of F, and let EY be the expectation of G. We denote by $EX_{(i,n)}$ $(EY_{(i,k)})$ the expected value of the order statistic $X_{(i,n)}$ $(Y_{(i,k)})$, and define $EX_{(0,n)} = EY_{(0,k)} = 0$. A useful identity, repeatedly used below, is :

$$\sum_{i=1}^{n} EX_{(i,n)} = nEX$$

Definition 1 *Let H be a distribution on $[0, \tau_H]$ with density f.*

1. *The* failure rate *of H is given by the function $\lambda(x) \equiv f(x) / \left[1 - H(x)\right]$, $x \in [0, \tau_H)$.*

2. *H is said to have an increasing (decreasing) failure rate (IFR) (DFR) if $\lambda(x)$ is increasing (decreasing) in x.*[10]

3. *H is said to have an increasing (decreasing) failure rate on average (IFRA) (DFRA) if $(\int_0^x \lambda(t)\,dt)/x$ is increasing (decreasing) in x.*

The exponential distribution has a constant failure rate, and it is the only distribution that is both *IFR* and *DFR*. Clearly, the family of *IFRA (DFRA)* distributions includes all *IFR (DFR)* distributions.

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[10] *IFR* distributions are also called *logconcave*. Examples are the exponential, uniform, normal, power (for $\alpha \geq 1$), Weibull (for $\alpha \geq 1$), gamma (for $\alpha \geq 1$). *DFR* distributions are also called *logconvex*. Examples are the exponential, Weibull (for $0 < \alpha \leq 1$), gamma (for $0 < \alpha \leq 1$). See Barlow and Proschan (1975).

3 Equilibrium analysis

We focus below on a symmetric equilibrium where all agents on one side of the market use the same strategy.

Assume that all men use the same, strictly monotonic and differentiable equilibrium signalling function β. Then, the maximization problem of a man with type x is

$$\max_s \quad \left\{ \sum_{i=n-k+1}^{n} x F_i^n (s) EY_{(k-(n-i),k)} - \beta (s) \right\}$$

The first order condition is

$$\begin{aligned} \beta' (s) &= \sum_{i=n-k+1}^{n-1} s \left[f_{(i-1,n-1)} (s) - f_{(i,n-1)} (s) \right] EY_{(k-(n-i),k)} \\ &\quad + f_{(n-1,n-1)} (s) s EY_{(k,k)} \end{aligned}$$

The man with the lowest type either never wins a woman (if $n > k$) or wins for sure the woman with the lowest type (if $n = k$). Hence, the optimal signal of this type is always zero, which yields the boundary condition $\beta (0) = 0$. The solution of the differential equation gives candidate equilibrium effort functions.

Proposition 1 *The profile of strategies where each man employs the strictly increasing signalling function*

$$\begin{aligned} \beta (x) &= \int_0^x s \left\{ \sum_{i=n-k+1}^{n-1} \left[f_{(i-1,n-1)} (s) - f_{(i,n-1)} (s) \right] EY_{(k-n+i,k)} \right\} ds \\ &\quad + \int_0^x s f_{(n-1,n-1)} (s) EY_{(k,k)} ds \end{aligned} \tag{1}$$

and each woman employs the analogously derived signalling function $\gamma (y)$ constitutes an equilibrium of the matching contest.

The next proposition reveals that the aggregate signalling effort (say women's) is a weighted sum of expectations of *normalized spacings* of order statistics on the men's side, $(n - i + 1)(X_{(i,n)} - X_{(i-1,n)})$, where the weights $EY_{(i-(n-k),k)}$ are expectations of the women's order statistics (and vice-versa for men). The weights are increasing in i.

The same observation holds for the net welfare terms of each side (note that the expression for the men's total welfare, (3), is similar to that of women's total signalling, while women's total welfare, (5), is similar to men's total signalling). Here we introduce the assumption of wasteful signalling.

Proposition 2 *For any F, G, n, k, it holds that:*

1. *Men's total signalling effort and (net) welfare are given by :*

$$S_m(n, k) = n \int_0^{\tau_F} \beta(x) f(x) \, dx$$
$$= \sum_{i=n-k+1}^{n} (n - i + 1) \left(EY_{(k-n+i,k)} - EY_{(k-n+i-1,k)} \right) EX_{(i-1,n)},$$
$$(2)$$

$$W_m(n, k) = \sum_{i=n-k+1}^{n} EX_{(i,n)} EY_{(k-(n-i),k)} - S_m(n, k)$$
$$= \sum_{i=n-k+1}^{n} (n - i + 1) \left(EX_{(i,n)} - EX_{(i-1,n)} \right) EY_{(i-(n-k),k)} \quad (3)$$

2. *Women's total signalling effort and (net) welfare are given by :*

$$S_w(n, k) = k \int_0^{\tau_G} \gamma(y) g(x) \, dx$$
$$= \sum_{i=n-k+1}^{n} (n - i + 1) \left(EX_{(i,n)} - EX_{(i-1,n)} \right) EY_{(k-n+i-1,k)}$$
$$(4)$$

$$W_w(n, k) = \sum_{i=n-k+1}^{n} EX_{(i,n)} EY_{(k-(n-i),k)} - S_w(n, k)$$
$$= \sum_{i=n-k+1}^{n} (n - i + 1) \left(EY_{(i-(n-k),k)} - EY_{(i-(n-k)-1,k)} \right) EX_{(i,n)}$$
$$(5)$$

3. *Total expected (net) welfare in assortative matching based on costly signalling is at least half the expected output (or, in other words, aggregate signalling efforts are less than half output).*

$$W(n, k) = 2 \sum_{i=n-k+1}^{n} EX_{(i,n)} EY_{(k-(n-i),k)} - S_m(n, k) - S_w(n, k) \quad (6)$$
$$\geq \sum_{i=n-k+1}^{n} EX_{(i,n)} EY_{(k-(n-i),k)} \quad (7)$$

The spacings of order statistics of types $(X_{(i,n)} - X_{(i-1,n)})$ represent the marginal gains from winning a stochastically better partner. For the exponential distribution H, it is well-known that the normalized spacings $(n - i + 1)(Z_{(i,n)} - Z_{(i-1,n)})$ are i.i.d. for $i = 1, 2, ..n$. Thus, it is to be expected that certain transformations of the exponential lead to some monotonicity properties of the spacings. Indeed, Barlow and Proschan (1966) have shown:

Theorem 1 *If F is IFR (DFR), then the corresponding normalized spacings $(n-i+1)\left(X_{(i,n)} - X_{(i-1,n)}\right)$ are stochastically decreasing (increasing) in $i = 1, 2, ..., n$ for fixed n, and stochastically increasing (decreasing) in $n \geq i$, for fixed i.*

Thus, if the normalized spacings are increasing in i (as is the case for DFR distributions), both total signalling effort and total welfare will be relatively high (we have "assortative matching" between weights and normalized spacings in the above expressions). The opposite holds for IFR distributions, since then the normalized spacings are decreasing in i.

In the following sections, we will obtain several kinds of comparative static results with respect to aggregate measures - as we will see, some of the effects are particular to relatively small markets and cannot occur in large populations (see Section 7). The difference stems from the fact that, in the model with a continuum of agents arising as the limit of the present one, total signalling is very tightly related to output (in ratio of one half), while here there is a leeway among output and signalling (caused by the strictly positive spacings) that may get larger or smaller in various circumstances.

4 The effects of entry

We now analyze the effects of changes in the number of agents on each side.

Additional agents unambiguously increase the expected matching output. If there is entry on the long side (i.e. entry by men), the number of matches remains unchanged, but the expected value of the i'th man is increased. If there is entry on the short side (i.e. entry by women), both the number of matches and the expected value of the i'th woman gets higher. On the other hand, entry also affects the agents' signalling activity. The next propositions show how the net effect on welfare depends on certain properties of the distribution of agents' types.

Proposition 3 *Suppose there is entry on the men's side. Then, for all G,*

1. *men's total signalling increases for all F,*

2. *women's total signalling increases (decreases) if F is DFR (IFR),*

3. *men's total welfare increases (decreases) if F is DFR (IFR),*

4. *women's total welfare increases for all F.*

Entry on the men's side leads to stiffer competition among men, and hence higher signalling efforts by men.[11] In contrast, the effect on women's signalling effort depends on whether the distribution of men's type F has an increasing or a decreasing failure rate. By Theorem 1, if F is DFR (IFR), $(n - i + 1) \left(X_{(i,n)} - X_{(i-1,n)} \right)$ is stochastically increasing (decreasing) in i, for fixed n. This implies that the marginal gains from winning a better man are relatively high (small) with respect to highly-ranked men if F is DFR (IFR). Moreover, by Theorem 6, if F is DFR (IFR), the difference of successive order statistics is stochastically increasing (decreasing) jointly in i and n. This implies that, for DFR (IFR), entry by men even further increases (reduces) the relatively high (small) marginal gains from winning a better men with respect to highly-ranked men. As a consequence, total signalling by women goes up if F is DFR, while the opposite holds if F is IFR.

Note that total output is always increasing in the number of men. It can be shown that this increase is larger if F is DFR than if F is IFR. On the women's side, we find that this output effect outweighs the increase in total signalling if F is DFR. Moreover, if F is IFR, women's total signalling goes down. Hence, women's total welfare is always increasing in the number of men.

On the men's side, if F is DFR, the output effect outweighs the increase in men's signalling, similarly as on the women's side. However, in contrast to women's total signalling, men's total signalling gets also higher if F is IFR. In fact, we find that in this case the signalling effect outweighs the output effect, leading to a reduction in men's total welfare. Combining these observations, we can conclude that overall total welfare is increased if F is DFR, but may be reduced if F is IFR. The following example illustrates a welfare loss due to an increase in the number of men.

Example 1 *Suppose $F = x^{10}$, $G = x$, and $\tau = 1$. Fix $k = 3$. Then: $W(3,3) \simeq 2.0779$, $W(4,3) \simeq 1.5202$, $W(5,3) \simeq 1.4994$, $W(6,3) \simeq 1.4944$, $W(7,3) \simeq 1.4929$, $W(8,3) \simeq 1.4926$; for $n > 8$ entry on the men's side is welfare increasing, and $\lim_{n \to \infty} W(n,3) = 1.5$*

The next proposition analyzes the effects of entry on the women's side.

Proposition 4 *Suppose there is entry on the women's side. Then, for all F,*

1. *men's total signalling increases if G is DFR,*

2. *women's total signalling increases for all G,*

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[11] In this case the effort of high types gets larger and the effort of low types gets smaller.

3. men's total welfare increases for all G,

4. women's total welfare increases if G is DFR.

Entry by women has similar effects to entry by men, except that it leads to a higher number of matches, and hence a higher number of prizes for men. This increase has, ceteris paribus, a positive effect on the men's signalling effort. Therefore, even if the distribution of women's types G is IFR, men's total signalling may increase due to the presence of an additional woman.

5 The effects of changes in the distributions of attributes

In this section we study how output, signalling, and welfare on both sides of the market are affected by changes in the distribution of agents' attributes on one side of the market. We also compare the signalling activity and welfare of men and women in a given, fixed setting.

5.1 The effects of increasing heterogeneity

Definition 2   *1. A function ϕ is star-shaped on $[0, \tau)$ if $\phi(x)/x$ is increasing in x.*

   2. Let $X(Z)$ have distributions $F(H)$ such that $F(0) = H(0) = 0$. Distribution F is star-shaped with respect to H if the function $H^{-1}F(x)$ is star-shaped (that is, $H^{-1}F(x)/x$ is increasing for $x \geq 0$) .

   3. Distribution F is convex with respect to H if the function $H^{-1}F(x)$ is convex on the support of F.

Consider two distributions F and H such that $F(0) = H(0) = 0$. Since convex functions ϕ on $[0, \tau)$ such that $\phi(0) \leq 0$ are star-shaped, we obtain that $H^{-1}F(x)$ is convex implies $H^{-1}F(x)$ is star-shaped. If H is the exponential distribution then $H^{-1}F(x)$ is convex (concave) is equivalent to F being *IFR (DFR)*, and $H^{-1}F(x)$ $(F^{-1}H(x))$ star-shaped is equivalent to F being *IFRA (DFRA)*.

A crucial property is *single crossing*: If $H^{-1}F(x)$ is star-shaped then $1 - F(x)$ crosses $1 - H(x)$ at most once, and then from above, as x increases from 0 to ∞. In particular, if F and H have the same mean, then a crossing must occur, and F has a smaller variance than H.

In the Appendix A we detail the consequences of single-crossing on order statistics - these are the mathematical results used in this part. We can now state:

Proposition 5 *Let H, F be two distributions of the men's attributes with the same expectation, and assume that $H^{-1}F(x)$ is star-shaped . Let G be the distribution of women's attributes. Then the following hold:*

1. *For any $n \geq k$, and for any G, total output in assortative matching under F is smaller than total output under H.*

2. *For $n = k$, and for G IFR, men's total signalling under F is higher than under H.*

3. *For any $n \geq k$, and for any G, women's total signalling under F is lower than under H.*

4. *For any $n \geq k$, and for any G, men's total welfare in the signalling equilibrium under F is smaller than men's total welfare under H.*

5. *For $n = k$, and for G IFR (DFR), women's total welfare under F is higher (lower) than under H. For any $n \geq k$ and for G DFR, women's total welfare under F is lower than under H.*

It is interesting to observe that increased heterogeneity on one side of the market (say men) always leads to higher expected output and to higher total welfare on the same side of market (point 1,4 above),[12] while this is not necessarily true for the other side (point 5). The reason is as follows: While expected output increases, total women signalling also increases (point 3).[13] But the increase in output is relatively large if the women's distribution G is *DFR*, thus offsetting the increase in women's signalling, while the increase in output is relatively small if G is *IFR*, in which case women's welfare may go down.

In many applications (e.g., biological studies of sexual selection, or development studies about marriage markets in rural societies[14]) it is of interest to compare the signalling activity on both sides of the market.

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[12] Men with high types face less intensive competition under H. If G is IFR, the marginal gains from winning a better women are small for these men. This reduces the signalling efforts for high types, and vice versa for low types.

[13] Both the expected quality of high-ranked men and the differences between high-ranked men and low-ranked men are higher under H. Therefore, under H, the effort of high-type women is larger and the the effort of low-type women is smaller, leading to higher total signalling by women.

[14] Consider, for example, an insightful excerpt taken from the empirical study of marriage in rural Ethiopia due Fafchamps and Quisumbing (2005): "if the difference between grooms is

Proposition 6   *1. Let $n = k$ and let $G^{-1}F$ be star-shaped. Then men's signalling effort is higher than women's. Thus, women are better-off than men.*

2. *Let either F or G be IFR, and let $G^{-1}F$ be convex. Then, for any $n \geq k$, men's signalling effort is higher than women's*

The above conditions involve the function $G^{-1}F$. This function has an important meaning here: it describes the matching function of assortative matching in the continuous version of our model (see Section 7).

A simple and more intuitive corollary is as follows:

Corollary 1 *Let F be convex and G be concave. Then, for any $n \geq k$, men's total signalling effort is higher than women's.*

If F has an increasing density while G has a decreasing density, then men with relatively high types face a stiffer competition than those with relatively low types, while the opposite holds for women. Note that individual efforts are monotonically increasing in types. Therefore, when F gets more convex and G more concave, total men's signalling tends to increase relative to women's total signalling. In addition, if F is convex and G concave, the differences in successive order statistics is decreasing on the men's side, and increasing on the women's side (see Boland et al., 2001). This implies that the marginal gains in terms of winning a better matching partner are larger with respect to highly-ranked men than highly-ranked women, which tends to further increase total men's effort relative to women's.

Remark 1 *We have mentioned in Section 2 that our results can be easily extended to models where the production function has the form $2\delta(x)\rho(y)$ where δ and ρ are strictly increasing, non-negative functions. Here is an example derived from the above observations: consider the Cobb-Douglas production $2\delta(x)\rho(y)$ $= 2x^c y^d$, $c, d > 0$. Let $n = k$, and assume that men's and women's attributes are uniformly distributed on $[0, 1]$. This model is equivalent to the one where the types are $\widetilde{x}$, $\widetilde{y}$, the production function is $2\widetilde{x}\widetilde{y}$, and the distributions of attributes are $\widetilde{F}(\widetilde{x}) = \widetilde{x}^{1/c}, \widetilde{G}(\widetilde{y}) = \widetilde{y}^{1/d}$. Thus, men signal more and are worse-off if $c \leq d$.*

---

large relative to the difference between brides, brides must bring more to fend off competition from lower ranked brides who wish to improve their ranking"

5.2 The effects of overall increases in quality

We now assume that the distribution of men's attributes F (random variable X) changes to another distribution F (random variable Z), such that $X \leq_{hr} Z$. In particular, $EX \leq EZ$, and $EX_{(i,n)} \leq EZ_{(i,n)}$, $i = 1, 2, ...n$. Thus, there is an unambiguous increase in the quality of men. The effects on expected output , men's total signalling, and women's total welfare are also unambiguous: all these measures are higher under H than under F because women receive better prizes and because competition among men is stronger.[15] The effect on men's total welfare and women total signalling are more subtle, and some of them cannot occur in large populations (see Section 7).

Proposition 7 *Let X, Z two random variables with distributions F and H, respectively, such that $X \leq_{hr} Z$. Let G be any distribution of women's attributes. Then the following hold:*[16]

1. *For any $n \geq k$, total output in assortative matching under F is smaller than total output under H.*

2. *For any $n \geq k$, total men's signalling under F is smaller than total men's signalling under H.*

3. *For any $n \geq k$, total men's welfare under F is smaller than total men's welfare under H if either F or H are DFR.*

4. *For any $n \geq k$, total women's signalling under F is smaller than total women's signalling under H if either F or H are DFR.*

5. *For any $n \geq k$, total women's welfare under F is smaller than total women's welfare under H.*

6 Assortative versus random matching

We now compare the equilibrium outcome of assortative matching with signalling to the outcome where agents are matched randomly. Random matching can also be seen as the outcome of a completely pooling equilibrium in our signalling model.

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[15] Under H, the men with high types face more intensive competition and therefore their effort is larger, while the effort of low types is smaller.

[16] Note that points 1,2,5 also hold for increases in the first-order (or standard) stochastic sense.

While the matching surplus generated through assortative matching is clearly larger than the one obtainable through random matching, assortative matching involves the cost of signalling efforts. The main questions are: 1) Under which conditions is the increase in total expected output achieved by assortative matching completely offset by the increased cost of signalling ? 2) Which types prefer random matching, and which types prefer assortative matching with signalling ?

6.1 Total welfare

Total welfare in random matching is given by:

$$W^r(n,k) = 2\min(n,k)EX \cdot EY \tag{8}$$

We obtain the following result:

Proposition 8 *1. Suppose that $n = k$. Then random matching is welfare superior (inferior) to assortative matching based on signalling if F and G are IFRA (DFRA).*

2. For any $n \geq k$, assortative matching based on signalling is welfare superior to random matching if F and G are DFR.

In particular, for $n = k$, random matching and assortative matching with signalling are welfare-equivalent if the distributions of agents' types are exponential. The results in Section 7 will provide some intuition for the above result in terms of a measure of the heterogeneity in the populations.

6.2 Individual welfare

We have compared above total welfare from assortative matching with the total welfare from random matching. We now make this comparison from each agent's point of view.

Obviously, agents with low types prefer random matching. To see this, consider a man with a very low type x. This man's expected utility under random matching is xEY. On the other hand, this man's expected utility from assortative matching is approximately $xEY_{(1,n)}$ minus his bid, since he is going to match almost surely with the woman with the lowest type. Since $EY > EY_{(1,n)}$, such a man prefers the random matching.

Lemma 1 *For any distributions F and G, and for any $n \geq k$, there exists at most one cutoff type $\hat{x} \in [0, \tau_F]$ such that all men $x < \hat{x}$ are better-off under random matching, while all men $x \geq \hat{x}$ are better-off under assortative matching based on signalling (and analogously for women).*

From Proposition 8 we know that assortative matching with signalling yields a higher total welfare than random matching if F and G are DFR. Together with the above Lemma, this implies that, if F and G are DFR, there must exist some types of agents that prefer assortative matching with signalling to random matching, i.e., the cutoff points are interior.

Suppose now that F and G are not DFR, such that the cutoff defined in Lemma 1 does not necessarily exist. The interesting question is now whether it is possible that all agents, including those with high types, are better-off under random matching ? The answer is affirmative, and is illustrated next:

Proposition 9 *Let $n = k$, and assume that $\tau_F < \infty$. If F stochastically dominates the uniform distribution on $[0, \tau_F]$, then all types of men prefer random matching to assortative matching based on signalling.*[17] *Analogous results hold for women.*

7 Large populations

We now consider the continuous version where there are measures of men and women, distributed according to F and G, respectively. Both measures are normalized to one.[18] Our analysis will focus on the connections between this model and the discrete model analyzed so far.

Under assortative matching, a man with attribute x is matched with a woman with attribute $y = \psi(x)$, where $\psi(x) = G^{-1}F(x)$.

Expected output (and welfare) under random matching is given by $2EXEY = 2(\int_0^{\tau_F} xf(x)dx)(\int_0^{\tau_G} yg(y)dy)$. Expected output under assortative matching is given by $2\int_0^{\tau_F} x\psi(x)f(x)dx$. The signalling activity that enable assortative matching is characterized in the next proposition:

Proposition 10 *1. In the continuous model, the equilibrium signalling function for men (women) in the side-symmetric signalling equilibrium is given by $\beta(x) = \int_0^x z\psi'(z)\,dz$ ($\gamma(y) = \int_0^y z\varphi'(z)\,dz$, where $\varphi = \psi^{-1}$).*

---

[17] If F is stochastically dominated by the uniform distribution, then some types of men prefer random matching to the assortative matching with signalling.

[18] This is for simplicity. The generalization should be clear.

2. *Aggregate signalling (men + women) is always equal to $\int_0^{\tau_F} x\psi(x) f(x)dx$.*
   That is, exactly half the surplus from assortative matching is wasted through
   signalling.

In the discrete case we showed that total signalling effort is less than one half output (see Proposition 2-4). Here perfect competition always drives signalling up to precisely half output. As a consequence, several phenomena that occurred in the discrete model cannot occur here. For example, recall that the effects of increasing the number of men on total welfare, or the effects of overall increases in the quality of men may be ambiguous if the distribution of men is not *DFR* (see Proposition 3, Example 1, Proposition 7). In the continuous limit, both these changes are equivalent,[19] but their effect is clear-cut: since total signalling is tightly proportional to output, and since output goes up when the overall quality of men increases, total welfare necessarily goes up. This observation emphasizes that some caution is necessary when making arguments about small markets.

An immediate application of the above result yields the comparison between assortative and random matching in the continuous version:

Proposition 11 *In the continuous model, assortative matching based on signalling is welfare superior (inferior) to random matching if*

$$\frac{Cov(X, \psi(X))}{EX \cdot E\psi(X)} \geq (\leq) \ 1$$

The above result can be more easily explained in the symmetric setting where $F = G$. Let $CV \equiv \sqrt{Var(X)}/EX$ be the coefficient of variation of $F = G$. A smaller CV means that types are less heterogeneous. Proposition 11 immediately yields:

Corollary 2 *Let $F = G$. In this symmetric continuous model, assortative matching based on signalling is welfare superior (inferior) to random matching if $CV \equiv \sqrt{Var(X)}/EX \geq (\leq) \ 1$. In particular, assortative matching based on signalling is welfare superior (inferior) to random matching if F is DFRA (IFRA).*

The last part of the corollary follows by Barlow and Proschan's (1975) result whereby $CV \geq (\leq) \ 1$ if F is DFRA (IFRA). Thus, the result of the continuous

---

[19]When men enter, and when men get stochastically better the expected values of the order statistics increase, while the additional "number effect" in the entry case is negligible in large markets.

model neatly fits, and is stronger than the one obtained for the discrete model. Note that $CV = 1$ for the exponential distribution. As in the discrete case with equal numbers of men and women, total welfare in random matching equals total welfare in assortative matching with signalling under this distribution.

It is intuitive that the difference in expected output between assortative matching and random matching gets smaller as the heterogeneity in the population gets smaller. By Proposition 10, total signalling efforts are proportional to output for any level of heterogeneity[20]. Therefore, net welfare in assortative matching eventually falls below the welfare level in random matching as the level of heterogeneity gets smaller.

Remark 2 *We have mentioned in the introduction that our insights delivers immediate results for other, less general models: Here are two examples: 1) Consider a setting where the attributes of one side of the market (say men) are known. Then, signalling is only performed by one side, and the waste from signalling is halved. Thus, assortative matching via signalling becomes more attractive relative to random matching, and, by an argument similar to the one in Corollary 2, we obtain that assortative matching is welfare superior (inferior) to random matching if $CV \equiv \sqrt{Var(X)}/EX \geq (\leq) 1/3$. In particular, the two alternatives are now equivalent for the distribution of attributes that is uniform on a bounded interval. 2) Consider now a model with complete information, without any signalling. Since signalling amounted to half output in the two-sided incomplete information model, Corollary 2 basically says that for IFR (DFR) distributions random matching yields an output that is more than half (less than half) the output from assortative matching. In particular, the "blunt" coarse matching analyzed by McAfee (2002) yields at least three-quarters of the output in assortative matching for the class of distributions studied in that paper[21].*

The above comparison of assortative and random matching in the continuous model was obtained by a direct argument. But, the result was clearly related to the one we previously obtained in the discrete version. What is the general relation between the discrete model and the continuous model? We now show how results in the continuous model can be obtained by considering the limit in the discrete model as the number of agents goes to infinity. We illustrate

---

[20]For example, consider a sequence of distribution functions converging to the Dirac distribution on $\tau < \infty$. Observe that, as the distributions become more concentrated, the limiting value is still half the limiting value of the expected output, and thus bounded away from zero. Even when the probability that a potential matching partner is worse than τ gets arbitrarily small, agents still engage in signalling in order to prevent being matched with a low type.

[21]This is a subclass of the IFR distributions.

this phenomenon by showing that average output, and average total signalling effort in the discrete model indeed converge to their continuous counterparts. We focus below on the symmetric case where $n = k$, and $F = G$. Recall that in the discrete case where $n = k$ and $F = G$, total welfare is given by :

$$W(n, n) = 2 \sum_{i=1}^{n} (n - i + 1) \left(EX_{(i,n)} - EX_{(i-1,n)} \right) EX_{(i,n)} \tag{9}$$

Proposition 12 *Assume $F = G$, and $n = k$ in the discrete model. For each n, consider the side-symmetric signalling equilibrium yielding assortative matching, and let n go to infinity. Then, average (i.e., per pair) expected output, average expected total signalling, and average expected total welfare in the discrete model converge to expected output, expected signalling, and expected total welfare in the continuous model.*

8 Conclusion

We have studied two-sided matching models where privately informed agents on each side are matched on the basis of costly signals. For the welfare analysis we have assumed that signals are wasted. Our study reveals how welfare on both sides of the market is affected by changes in primitives of the model such as the number of the agents, and the distributions of their attributes. In particular, our analysis suggests that one cannot blindly apply to small markets insights obtained from studies of large populations. We have also identified conditions under which assortative matching based on wasteful signalling is welfare superior (inferior) to random matching. Thus, the effects of policies that attempt to curb wasteful signalling need to be carefully examined in each particular situation[22]. The analysis of markets with finite numbers of agents on each side has been made possible by the application of elegant results and methods from mathematical statistics. We believe that the applications of these methods will be fruitful also in other areas (such as double auctions). Finally, we hope that our model (or some of its many possible variations) will be useful as a sound, theoretical basis around which to organize observations in a variety of empirical studies, e.g., of marriage, labor and education markets.

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[22] Alternatively, this holds for policies that attempt to manipulate the rent accruing to a third party, such as an intermediary (see Hoppe et al., 2005).

9 Appendix A: Order statistics and stochastic orders

Definition 3 *For any two non-negative random variables, X and Z, with distributions F and H and hazard rates λ_x and λ_z, respectively, X is said to be smaller than Z in the hazard rate order (denoted as $X \leq_{hr} Z$) if $\lambda_x(s) \geq \lambda_z(s)$, for all $s \geq 0$. X is said to be smaller than Z in the usual stochastic order (denoted as $X \leq_{st} Z$) if $F(s) \geq H(s)$ for all $s \geq 0$.*

Theorem 2 *(see Shaked and Shanthikumar ,1994):*

1. *If X and Z are two random variables such that $X \leq_{hr} Z$, then $X \leq_{st} Z$.*

2. *Let $X_1, X_2, ..., X_n$ be independent random variables. Then:*

   - *$X_{(i,n)} \leq_{hr} X_{(i+1,n)}$ for $i = 1, 2, ..., n-1$,*

   - *$X_{(i-1,n-1)} \leq_{hr} X_{(i,n)}$ for $i = 2, 3, ..., n$*

   - *$X_{(i,n-1)} \geq_{hr} X_{(i,n)}$ for $i = 2, 3, ..., n-1$.*

With respect to order statistics, the basic consequence of single crossing for random variables ordered by the star-shaped order is:

Theorem 3 *(see Barlow and Proschan, 1966) Let X, Z two random variables with distributions F, H respectively, such that $F(0) = H(0) = 0$, and such that $H^{-1}F$ is star-shaped. Then:*

1. *The function $\omega(i, n) = EX_{(i,n)} - EZ_{(i,n)}$ changes sign at most once when i (n) increases and then from positive to negative (negative to positive), if at all. If $EX = EZ$ then a change of sign when i increases must occur.*

2. *The ratio $EX_{(i,n)}/EZ_{(i,n)}$ is decreasing (increasing) in i (n).*

3. *The ratio $EX_{(n-i,n)}/EZ_{(n-i,n)}$ is decreasing in n.*

Many of our proofs rely on a conjunction of the above result with the following Lemma:

Theorem 4 *(see Barlow and Proschan, 1966): Consider $\alpha_i > 0$, $\beta_i \geq 0$, $i = 1, 2, , n$, such that β_i/α_i is increasing in i. Then $\sum_1^n a_i\beta_i / \sum_1^n \beta_i \leq \sum_1^n a_i\alpha_i / \sum_1^n \alpha_i$ for any $a_1 \geq a_2... \geq a_n$.*

Two simple, but important consequences are:

Theorem 5 *(see Barlow and Proschan, 1966)*

1. *If $H^{-1}F$ is star-shaped, and if $EX = EZ$ then*

$$\sum_{i=1}^{n} a_i[(n-i+1)(EX_{(i,n)}-EX_{(i-1,n)})] \geq \sum_{i=1}^{n} a_i[(n-i+1)(EZ_{(i,n)}-EZ_{(i-1,n)})]$$

2. *If F is IFRA (DFRA) then:*

$$\sum_{i=1}^{n} a_i[(n-i+1)(EX_{(i,n)} - EX_{(i-1,n)})] \geq (\leq)E\sum_{i=1}^{n} a_i$$

We will also use the following generalization of Barlow and Proschan's results:

Theorem 6 *(see Hu and Wei, 2001) Define $U_{(j,i,n)} \equiv X_{(j,n)} - X_{(i,n)}$ for $0 \leq i < j \leq n$. Let F be DFR (IFR). Then $U_{(j-1,i-1,n-1)} \leq_{hr} (\geq_{hr}) U_{(j,i,n)}$.*

Theorem 7 *(see Khaledi and Kochar, 1999):*

1. *If $X \leq_{hr} Z$ and either X or Z is DFR, then $(n-i+1)(X_{(i,n)}-X_{(i-1,n)}) \leq_{st} (n-i+1)(Z_{(i,n)} - Z_{(i-1,n)})$, $i = 1, 2, 3, ..., n$.*

10 Appendix B: Proofs

Proof of Proposition 1. We first show that the function β in (1) is strictly monotonically increasing. Note that (1) can be written as

$$
\begin{aligned}
\beta(x) &= \int_0^x \left\{ \sum_{i=n-k+1}^{n-1} f_{(i,n-1)}(s) \left[EY_{(k-n+i+1,k)} - EY_{(k-n+i,k)} \right] \right\} sds \\
&\quad + \int_0^x f_{(n-k,n-1)}(s) EY_{(1,k)} sds
\end{aligned}
$$

Taking the derivative with respect to x yields

$$
\begin{aligned}
\beta'(x) &= \sum_{i=n-k+1}^{n-1} f_{(i,n-1)}(x) \left[EY_{(k-n+i+1,k)} - EY_{(k-n+i,k)} \right] x \\
&\quad + f_{(n-k,n-1)}(x) EY_{(1,k)} x
\end{aligned}
$$

which is strictly positive because $Y_{(k-n+i+1,k)} \geq_{st} Y_{(k-n+i,k)}$.

Next, we check whether the second-order condition is satisfied. Integrating the RHS of (1) by parts, yields

$$\beta(x) = x \sum_{i=n-k+1}^{n-1} EY_{(k-(n-i),k)} \left[F_{(i-1,n-1)}(x) - F_{(i,n-1)}(x) \right]$$
$$+ x F_{(n-1,n-1)}(y) EY_{(k,k)}$$
$$- \int_0^x \left\{ \sum_{i=n-k+1}^{n-1} EY_{(k-(n-i),k)} \left[F_{(i-1,n-1)}(s) - F_{(i,n-1)}(s) \right] \right\} ds$$
$$- \int_0^y F_{(n-1,n-1)}(s) EY_{(k,k)} ds$$
$$= x \sum_{i=n-k+1}^{n} F_i^n(x) EY_{(k-(n-i),k)} - \int_0^x \sum_{i=n-k+1}^{n} EY_{(k-(n-i),k)} F_i^n(s) ds$$

Let $z = \beta^{-1}(b)$ be the type for which the equilibrium effort is b. The expected payoff of a man with type x from exerting effort $\beta(z)$ is thus given by:

$$U(b,x) = \sum_{i=n-k+1}^{n} \left[F_{(i-1,n-1)}(z) - F_{(i,n-1)}(z) \right] x EY_{(k-(n-i),k)} - \beta(z)$$
$$= \sum_{i=n-k+1}^{n} F_i^n(z) EY_{(k-(n-i),k)} (x-z)$$
$$+ \int_0^z \sum_{i=n-k+1}^{n} EY_{(k-(n-i),k)} F_i^n(s) ds$$

Hence, the difference between the expected payoffs of type x when he exerts efforts of $\beta(x)$ and $\beta(z)$ is:

$$U(\beta(x),x) - U(\beta(z),x) = \sum_{i=n-k+1}^{n} F_i^n(z) EY_{(k-(n-i),k)} (z-x) \qquad (10)$$
$$- \int_x^z \sum_{i=n-k+1}^{n} EY_{(k-(n-i),k)} F_i^n(s) ds$$

Since β is strictly increasing, the function $H(s) = \sum_{i=n-k+1}^{n} F_i^n(s) EY_{(k-(n-i),k)}$ increases in s and therefore the difference in (10) is always positive. $\blacksquare$

Proof of Proposition 2. **1)** Substituting (1) into (2) yields:

$$S_m(n,k) = n \int_0^{\tau_F} \int_0^x \sum_{i=n-k+1}^{n-1} f_{(i-1,n-1)}(s) EY_{(k-(n-i),k)} s ds f(x) dx$$
$$- n \int_0^{\tau_F} \int_0^x \sum_{i=n-k+1}^{n-1} f_{(i,n-1)}(s) EY_{(k-(n-i),k)} s ds f(x) dx$$
$$+ n \int_0^{\tau_F} \int_0^x f_{(n-1,n-1)}(s) EY_{(k,k)} s ds f(x) dx \qquad (11)$$

25

Integrating the first plus the third terms of (11) by parts and rearranging terms, we obtain

$$
n \int_0^{\tau_F} \int_0^x \sum_{i=n-k+1}^{n} f_{(i-1,n-1)}(s) EY_{(k-(n-i),k)} s ds f(x) dx
$$

$$
= n \int_0^{\tau_F} [1 - F(x)] \sum_{i=n-k+1}^{n} x f_{(i-1,n-1)}(x) EY_{(k-(n-i),k)} dx
$$

$$
= n \int_0^{\tau_F} \sum_{i=n-k+1}^{n} \frac{n-i+1}{n} x f_{(i-1,n)}(x) EY_{(k-(n-i),k)} dx
$$

$$
= \sum_{i=n-k+1}^{n} (n-i+1) EX_{(i-1,n)} EY_{(k-(n-i),k)}
$$

Similarly, integrating the second term of (11) by parts, we obtain

$$
-n \int_0^{\tau_F} \int_0^x \sum_{i=n-k+1}^{n-1} f_{(i,n-1)}(s) EY_{(k-(n-i),k)} s ds f(x) dx
$$

$$
= -n \int_0^{\tau_F} [1 - F(x)] \sum_{i=n-k+1}^{n-1} x f_{(i,n-1)}(x) EY_{(k-(n-i),k)} dx
$$

$$
= -n \int_0^{\tau_F} \sum_{i=n-k+1}^{n-1} \frac{n-i}{n} x f_{(i,n)}(x) EY_{(k-(n-i),k)} dx
$$

$$
= \sum_{i=n-k+1}^{n} (n-i) EX_{(i,n)} EY_{(k-(n-i),k)}
$$

Collecting terms, yields:

$$
S_m(n,k)
$$
$$
= \sum_{i=n-k+1}^{n} \left[(n-i+1) EX_{(i-1,n)} - (n-i) EX_{(i,n)} \right] EY_{(k-(n-i),k)} \quad (12)
$$
$$
= \sum_{i=n-k+1}^{n} (n-i+1) EX_{(i-1,n)} \left(EY_{(k-n+i+1,k)} - EY_{(k-n+i,k)} \right)
$$

2) Analogous to the above.

3) Follows from the definition of gross surplus and points 1, 2 above.

4) Note that the only difference in the expressions for $W(n,k)$ on the one hand, and $S_m(n,k) + S_w(n,k)$ on the other, is that the normalized spacings appearing in $W(n,k)$ are multiplied by a higher weight, corresponding to the

expectation of a higher order statistic. Thus

$$
\begin{aligned}
S_m\left(n,k\right) + S_w\left(n,k\right) &\leq W\left(n,k\right) \Leftrightarrow \\
W(n,k) + S_m\left(n,k\right) + S_w\left(n,k\right) &\leq 2W\left(n,k\right) \Leftrightarrow \\
2\sum_{i=n-k+1}^{n} EX_{(i,n)}EY_{(k-(n-i),k)} &\leq 2W\left(n,k\right) \Leftrightarrow \\
\sum_{i=n-k+1}^{n} EX_{(i,n)}EY_{(k-(n-i),k)} &\leq W\left(n,k\right) \quad (13)
\end{aligned}
$$

as desired. ∎

Proof of Proposition 3. **1)** We rewrite total men's signalling, $j = i - (n-k)$, as:

$$
S_m\left(n,k\right) = \sum_{j=1}^{k}\left(k-j+1\right)\left(EY_{(j,k)} - EY_{(j-1,k)}\right)EX_{(j+n-k-1,n)} \quad (14)
$$

Note that $EY_{(j,k)} \geq_{st} EY_{(j-1,k)}$. Note further that by Theorem 2, $EX_{(j+n-k-1,n)}$ is stochastically increasing in n.

2) We rewrite total women's signalling, $j = i - (n-k)$, as:

$$
S_w\left(n,k\right) = \sum_{j=1}^{k}\left(k-j+1\right)\left(EX_{(j+n-k,n)} - EX_{(j+n-k-1,n)}\right)EY_{(j-1,k)} \quad (15)
$$

Thus, applying Theorem 6 yields statement 2.

3) Men's total welfare can be written as:

$$
W_m\left(n,k\right) = \sum_{j=1}^{k}\left(k-j+1\right)\left(EX_{(j+n-k,n)} - EX_{(j+n-k-1,n)}\right)EY_{(j,k)} \quad (16)
$$

which is is similar to men's signalling (expression (15)). The proof is analogous to that at point 2 above, and we omit it here.

4) Women's total welfare can be written as:

$$
W_w\left(n,k\right) = \sum_{j=1}^{k}\left(k-j+1\right)\left(EY_{(j,k)} - EY_{(j-1,k)}\right)EX_{(j+n-k,n)} \quad (17)
$$

which is is similar to men's signalling (expression (14)). The proof is analogous to that at point 1 above, and we omit it here. ∎

Proof of Proposition 4. **1)** Men's total signalling is given by (14). Thus,

$$S_m(n,k) = \sum_{j=1}^{k}(k-j+1)\left(EY_{(j,k)}-EY_{(j-1,k)}\right)EX_{(j+n-k-1,n)}$$

$$= \sum_{i=2}^{k+1}(k-i+2)\left(EY_{(i-1,k)}-EY_{(i-2,k)}\right)EX_{(i+n-k-2,n)}$$

$$S_m(n,k+1) = \sum_{j=1}^{k+1}(k-j+2)\left(EY_{(j,k+1)}-EY_{(j-1,k+1)}\right)EX_{(j+n-k-2,n)}$$

$$= \sum_{j=2}^{k+1}(k-j+2)\left(EY_{(j,k+1)}-EY_{(j-1,k+1)}\right)EX_{(j+n-k-2,n)}$$
$$+(k+1)EY_{(1,k+1)}EX_{(n-k-1,n)}$$

Define $EV_{(i,k)} \equiv EY_{(i,k)}-EY_{(i-1,k)}$. Then we get:

$$S_m(n,k+1)-S_m(n,k)$$
$$= \sum_{j=2}^{k+1}(k-j+2)\left(EV_{(j,k+1)}-EV_{(j-1,k)}\right)EX_{(j+n-k-2,n)}$$
$$+(k+1)EY_{(1,k+1)}EX_{(n-k-1,n)}$$

Note that the first term of the RHS is positive if G is DFR and negative if G is IFR, which follows from Theorem 6. This yields the result as stated.

2) Women's total signalling is given by (4). The stated result follows immediately from Theorem 2.

3) Men's total welfare (expression (3)) is similar to women's total signalling (expression (4). The proof is analogous to the one for women's total signalling at point 2 above, and we omit it here.

4) Women's total welfare (expression (5)) is similar to men's total signalling (expression (2). The proof is analogous to the one for men's total signalling at point 1 above, and is omitted. ∎

Proof of Proposition 5 **1)** Let $a_i = -EY_{(k-n+i,k)}$ for $i = n, n-1, ..n-k+1$, and $a_i = 0$ for $i = n-k, n-k-1, ..., 1$. Then $a_1 \geq a_2... \geq a_n$, and output under $F(H)$ is given by $\sum_{i=1}^{n} a_i EX_{(i,n)}$ $\left(\sum_{i=1}^{n} a_i EZ_{(i,n)}\right)$. By Theorem A2-(2), we know that $EZ_{(i,n)}/EX_{(i,n)}$ is increasing in i. By assumption, $EX = EZ$. This implies $\sum_{i=1}^{n} EX_{(i,n)} = \sum_{i=1}^{n} EZ_{(i,n)}$. Setting $\beta_i = Z_{(i,n)}$ and $\alpha_i = X_{(i,n)}$ in Theorem A3 we get $\sum_{i=1}^{n} a_i EZ_{(i,n)} \leq \sum_{i=1}^{n} a_i EX_{(i,n)}$. Since for all i, $a_i \leq 0$, the wished result follows.

2) Men's total signalling is given by (2). Consider first the case $n = k$. Set $\beta_i = EZ_{(i,n)}$ and $\alpha_i = EX_{(i,n)}$, $i = 1, 2, ..n$, and $a_i = (n-i)(EY_{(i+1,n)}-EY_{(i,n)})$

, $i = 1, 2, ..n - 1$, and $a_n = 0$. If the distribution of women is *IFR,* we obtain that $a_1 \geq a_2.. \geq a_n$, and the result follows from Theorem A4-(1).

3) Women's total signalling (expression (2)) is similar to men's total welfare (expression (3)) . The proof is analogous to the one for men's welfare at point 4 below, and we omit it here.

4) Men's total welfare is given in (3). The result follows directly from Theorem A4-(1) by setting a_i as in point 1 above.

5) Women's total welfare is given in (5). Assume first that $n \geq k$. Set α_i and β_i as in point 1 above, and set $a_i = -(n - i + 1)(EY_{(k-n+i,k)} - EY_{(k-n+i-1,k)})$ If G is *DFR*, then $a_1 \geq a_2... \geq a_n$, and the result follows from Theorem A3.

Assume now $n = k$. Set α_i and β_i as in point 1 above, and set $a_i = (n - i + 1)(EY_{(i,n)} - EY_{(i-1,n)})$, $i = 1, 2, ..n$. If G is *IFR*, then $a_1 \geq a_2... \geq a_n$, and the result follows from Theorem A3.

Proof of Proposition 6. **1)** Using (2) and (4), we get:

$$S_m(n,n) - S_w(n,n) = \sum_{i=1}^{n} (n - i + 1) \left(EY_{(i,n)} EX_{(i-1,n)} - EY_{(i-1,n)} EX_{(i,n)} \right) \geq 0$$

The last inequality follows from Theorem A2-(2).

2) From Proposition 3 we know that:

(i) for any $n \geq k$, and any F, G, $S_m(n,k) \geq S_m(k,k)$,

(ii) for any $n \geq k$, for any G, and for F IFR, $S_w(n,k) \leq S_w(k,k)$

Since $G^{-1}F$ convex implies $G^{-1}F$ star-shaped, the result follows directly from **1)** and (i), (ii) if F is IFR.

Assume now that G is IFR. This means that $H^{-1}G$ convex, where H is the exponential distribution. Thus, $H^1GG^{-1}F = H^{-1}F$ is convex (since it is a composition of increasing convex functions), which means that F is IFR. The result follows as above. ∎

Proof of Proposition 7: Points 1, 2 , 5 follow immediately by inspection of the relevant expressions in Proposition2. Points 3,4 follow by these expressions and Theorem A6 in Appendix A. ∎

Proof of Proposition 8 **1)** Welfare in random matching can be written as:

$$W^r(n,n) = 2nEX \cdot EY = nEX \frac{\sum_{i=1}^{n} EY_{(i,n)}}{n} + nEY \frac{\sum_{i=1}^{n} EX_{(i,n)}}{n} \quad (18)$$

$$= EX \sum_{i=1}^{n} EY_{(i,n)} + EY \sum_{i=1}^{n} EX_{(i,n)} \quad (19)$$

Welfare in assortative matching is given by (6). Let $a_i = -EY_{(i,n)}$, and note that a_i is decreasing in i. [23] Applying Theorem A4-(2) yields: if F is IFRA (DFRA), then

$$-\sum_{i=1}^{n} EY_{(i,n)}[(n-i+1)\left(EX_{(i,n)} - EX_{(i-1,n)}\right)] \geq (\leq) - EX \sum_{i=1}^{n} EY_{(i,n)} \quad (20)$$

Multiplying by (-1), we obtain: if F is IFRA (DFRA), then

$$\sum_{i=1}^{n} EY_{(i,n)}[(n-i+1)\left(EX_{(i,n)} - EX_{(i-1,n)}\right)] \leq (\geq) EX \sum_{i=1}^{n} EY_{(i,n)} \quad (21)$$

Similarly, we obtain: if G is IFRA (DFRA), then

$$\sum_{i=1}^{n} EX_{(i,n)}[(n-i+1)\left(EY_{(i,n)} - EY_{(i-1,n)}\right)] \leq (\geq) EX \sum_{i=1}^{n} EX_{(i,n)} \quad (22)$$

The combination of (21) and (22) completes the proof.

2) The result for the general case $n \geq k$ follows by applying the entry results of Proposition 3: Recall that, by Proposition 3, entry by men increases welfare in assortative matching based on signalling if F is DFR (and hence DFRA). The result follows by noting that entry on the long side does not affect welfare from random matching since the number of matched pairs remains constant. ∎

Proof of Lemma 1 Let $U^a(x)$, $U^r(x)$ denote the expected utility of type x under assortative matching with signalling, and under random matching, respectively.

Note that $U^a(x) = \max_s \left\{ \sum_{i=n-k+1}^{n} F_i^n(s) xEY_{(k-n+i,k)} - \beta(s) \right\}$ is an increasing convex function (since it is the maximum of linear increasing functions), while U^r is an increasing linear function with slope EY. Thus, these functions can cross at most once. Note further that the derivative of $U^a(x)$ at $x = 0$ is

$$\left.\frac{dU^a(x)}{dx}\right|_{x=0} = \sum_{i=n-k+1}^{n} F_i^n(0) EY_{(k-n+i,k)} \leq EY_{(1,k)} < EY$$

where the first inequality follows either by $\sum_{i=n-k+1}^{n} F_i^n(0) EY_{(k-n+i,k)} = 0$ if $n > k$, (since $F_i^n(0) = 0$ if $i > 1$) or by $\sum_{i=n-k+1}^{n} F_i^n(0) EY_{(k-n+i,k)} \leq EY_{(1,n)}$ for $n = k$, (since $F_1^n(0) = \lim_{\varepsilon \to 0} F(\varepsilon)^\varepsilon \leq 1$). Thus, $U^a(x) \leq U^r(x)$ in a neighborhood of zero, and the wished result follows. ∎

_____

[23] Please note that Barlow and Proschan (1966) contains a crucial typo here, and they mix (only at their point (iii)) IFRA and DFRA.

Proof of Proposition 9 By Lemma 1, it is clear that if the man with the highest type prefers random matching, then all other types of men prefer random matching as well (and analogously for women). Under assortative matching based on signalling, the expected utility of the type τ man is

$$U^a(\tau) = \tau EY_{(n,n)} - \sum_{i=1}^{n-1} EX_{(i,n-1)} \left(EY_{(i+1,n)} - EY_{(i,n)} \right)$$

The expected utility of this type under random matching is

$$U^r(\tau) = \tau \cdot EY = \frac{\tau}{n} \left(\sum_{i=1}^{n} EY_{(i,n)} \right)$$

If F stochastically dominates (is stochastically dominated by) the uniform distribution, we obtain that $EX_{(i,n-1)} \geq (\leq) \tau \frac{i}{n}$. Then

$$
\begin{aligned}
U^a(\tau) &\leq \ (\geq) \ \tau EY_{(n,n)} - \frac{\tau}{n} \sum_{i=1}^{n-1} i \left(EY_{(i+1,n)} - EY_{(i,n)} \right) \\
&= \frac{\tau}{n} \left(\sum_{i=1}^{n} EY_{(i,n)} \right) = U^r(\tau)
\end{aligned}
$$

∎

Proof of Proposition 10   1) Consider men's types $x, \hat{x}$, $x > \hat{x}$, with equilibrium bids $\beta(x)$, $\beta(\hat{x})$. In equilibrium, type x is assortatively matched with type $\psi(x)$, and $\hat{x}$ is matched with $\psi(\hat{x})$. Type x should not pretend that he is $\hat{x}$ (thus being matched with $\psi(\hat{x})$ and paying $\beta(\hat{x})$), and vice-versa for type $\hat{x}$. This yields:

$$
\begin{aligned}
x\psi(x) - \beta(x) &\geq x\psi(\hat{x}) - \beta(\hat{x}) \\
\hat{x}\psi(\hat{x}) - \beta(\hat{x}) &\geq \hat{x}\psi(x) - \beta(x)
\end{aligned}
$$

Combining the above and dividing by $x - \hat{x}$, gives:

$$\frac{\hat{x}\psi(x) - \hat{x}\psi(\hat{x})}{x - \hat{x}} \leq \frac{\beta(x) - \beta(\hat{x})}{x - \hat{x}} \leq \frac{x\psi(x) - x\psi(\hat{x})}{x - \hat{x}}$$

Taking the limit $\hat{x} \to x$ gives $\beta'(x) = x\psi'(x)$. Together with $\beta(0) = 0$, this yields $\beta(x) = \int_0^x z\psi'(z)\, dz$. Letting $\varphi = \psi^{-1}$, we obtain $\gamma(y) = \int_0^y z\varphi'(z)\, dz$ analogously.

2) Total signalling effort by men and women is given by:

$$S_m + S_w = \int_0^{\tau_F} \int_0^x z\psi'(z)\,dz\,f(x)dx + \int_0^{\tau_G} \int_0^y z\varphi'(z)\,dz\,g(y)\,dy$$

$$= \int_0^{\tau_F} \left(x\psi(x) - \int_0^x \psi(z)\,dz \right) f(x)dx$$
$$+ \int_0^{\tau_G} \left(\varphi(y)y - \int_0^y \varphi(z)\,dz \right) g(y)\,dy$$
$$= \int_0^{\tau_F} \left(x\psi(x) - \psi(x)\frac{1-F(x)}{f(x)} \right) f(x)dx$$
$$+ \int_0^{\tau_G} \left(\varphi(y)y - \varphi(y)\frac{1-G(y)}{g(y)} \right) g(y)\,dy$$

where the last equality follows by integration by parts.

Using $y = \psi(x)$, $\varphi(\psi(x)) = x$ and $G(\psi(x)) = F(x)$ we obtain:

$$S_m + S_w$$
$$= \int_0^{\tau_F} \left(x\psi(x) - \psi(x)\frac{1-F(x)}{f(x)} + x\psi(x) - x\psi'(x)\frac{1-F(x)}{f(x)} \right) f(x)dx$$
$$= 2\int_0^{\tau_F} x\psi(x) f(x)dx - \int_0^{\tau_F} \left(\psi(x) + x\psi'(x) \right)\frac{1-F(x)}{f(x)}f(x)dx$$
$$= \int_0^{\tau_F} x\psi(x) f(x)dx$$

where the last equality follows from integrating $\int_0^{\tau_F} x\psi(x) f(x)dx$ by parts. The last term in the chain of equalities equals of course half gross output in the continuum model. ∎

Proof of Proposition 11 By Proposition 10, total welfare in assortative matching base don signalling is $\int_0^\tau x\psi(x) f(x)dx$. Thus, assortative matching with signalling is welfare superior (inferior) to random matching if:

$$\int_0^\tau x\psi(x) f(x)dx \geq [\leq] 2\int_0^\tau xf(x)\,dx \int_0^\tau yg(y)dy \Leftrightarrow$$
$$E(X\psi(X)) \geq [\leq] 2EX \cdot EY \Leftrightarrow$$
$$E(X\psi(X)) \geq [\leq] 2EX \cdot E\psi(X) \Leftrightarrow$$
$$\frac{Cov(X\psi(X))}{EX \cdot E\psi(X)} \geq [\leq] 1$$

(Note that $EY = E\psi(X)$; the proof uses the well-known fact that for any random variable Z with cumulative distribution H, $EZ = \int_0^1 H^{-1}(z)\,dz$.) ∎

Proof of Proposition 12. **1)** Consider first gross total output. In the discrete model, gross output is then given by:

$$
\begin{aligned}
2\sum_{i=1}^{n}[(EX_{(i,n)})]^{2} &= 2(\sum_{i=1}^{n}EX_{(i,n)}^{2}) - \sum_{i=1}^{n}Var\left(X_{(i,n)}\right)) \\
&= 2(E\left(\sum_{i=1}^{n}X_{(i,n)}^{2}\right) - \sum_{i=1}^{n}Var\left(X_{(i,n)}\right)) \\
&= 2(E\left(\sum_{i=1}^{n}X_{i}^{2}\right) - \sum_{i=1}^{n}Var\left(X_{(i,n)}\right)) \\
&= 2(nEX^{2} - \sum_{i=1}^{n}Var\left(X_{(i,n)}\right))
\end{aligned}
$$

where the second equality follows from the independence of attributes, the third is a rearrangement of summands, and last one follows by the well-known Wald's Identity (see David and Nagaraja, 2003). Thus, for average gross output, we obtain that:

$$
\begin{aligned}
\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}[(EX_{(i,n)})^{2} &= 2EX^{2} - \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}Var\left(X_{(i,n)}\right) \\
&= 2EX^{2} = 2\int_{0}^{\tau_{F}}x^{2}f(x)dx
\end{aligned}
$$

The second to last equality follows since $\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}Var\left(X_{(i,n)}\right) = 0$, which follows, for example, by David and Johnson's approximation (see Arnold and Balakrishnan, 1989):

$$
Var\left(X_{(i,n)}\right) \simeq \frac{\frac{i(n+1-i)}{(n+1)^{2}}[dF^{-1}(\frac{i}{n+1})]^{2}}{n+2} + \frac{c_{2}}{(n+2)^{2}} + \frac{c_{3}}{(n+2)^{3}}...
$$

where c_i are uniformly bounded constants.

2) For the other parts it is enough to show that net welfare in the discrete model (output less bids) converges to $\int_{0}^{\tau_{F}}x^{2}f(x)dx$. Net average welfare is given by:

$$
\begin{aligned}
W^{a} &= \frac{2}{n}\sum_{1}^{n}(n-i+1)EX_{(i,n)}(EX_{(i,n)} - EX_{(i-1,n)}) \\
&= \frac{2(n+1)}{n}\sum_{1}^{n}(1 - \frac{i}{n+1})EX_{(i,n)}(EX_{(i,n)} - EX_{(i-1,n)})
\end{aligned}
$$

It is well-known that, for large n, $EX_{(i,n)}$ is approximated by $F^{-1}(\frac{i}{n+1})$ (see David and Nagaraja, 2003). Thus, we obtain:

$$
W^{a} \simeq \frac{2(n+1)}{n}\sum_{1}^{n}(1 - \frac{i}{n+1})(F^{-1}(\frac{i}{n+1})[F^{-1}(\frac{i}{n+1}) - F^{-1}(\frac{i-1}{n+1})]
$$

Without the term $\frac{2(n+1)}{n}$, the RHS is precisely a Riemann-Stieltjes sum of the type $\sigma_d[(1-x)F^{-1}(x)\ ,\ F^{-1}(x)]$ where d is the partition of $[0,1]$ given by $\frac{1}{n+1}, \frac{2}{n+1}, \dots \frac{n}{n+1}$. Since F is strictly increasing, F and F^{-1} have bounded variation. Thus, we obtain:

$$\lim_{n\to\infty}\{\frac{2(n+1)}{n}\sum_1^n(1-\frac{i}{n+1})(F^{-1}(\frac{i}{n+1})[F^{-1}(\frac{i}{n+1})-F^{-1}(\frac{i-1}{n+1})]\}$$

$$=\ \int_0^1(1-x)F^{-1}(x)dF^{-1}(x)$$

$$=\ 2\int_0^{\tau_F}((1-F(u))udu$$

where the last line follows by the change of variable $x = F(u)$.

Finally we have :

$$
\begin{aligned}
2\int_0^{\tau_F}(1-F(u)udu &= 2[\int_0^{\tau_F}udu-\int_0^{\tau_F}F(u)udu]\\
&= 2\{\frac{\tau^2}{2}-[\frac{u^2}{2}F(u)\mid_0^{\tau_F}-\int_0^{\tau_F}\frac{u^2}{2}f(u)du]\}\\
&= 2[\frac{\tau^2}{2}-\frac{\tau^2}{2}+\frac{1}{2}\int_0^{\tau_F}u^2f(u)du]\\
&= \int_0^{\tau_F}u^2f(u)du
\end{aligned}
$$

∎

References

[1] Arnold, B.C. and Balakrishnan, N. (1989) *Relations, Bounds and Approximations for Order Statistics*, Springer Verlag, New York.

[2] Barlow, R. E. and Proschan, F. (1965) *Mathematical Theory of Reliability*, Wiley, New York.

[3] Barlow, R. E. and Proschan, F. (1966) "Inequalities for linear combinations of order statistics from restricted families" *Ann. Math. Statistic.* 37, 1593-1601.

[4] Barlow, R. E. and Proschan, F. (1975) *Statistical Theory of Reliability and Life Testing*, McArdle Press, Silver Spring.

[5] Becker, G. S. (1973) "A theory of marriage: part 1", *Journal of Political Economy* 81, 813-846.

[6] Boland, Philip J., Taizhong Hu, Moshe Shaked and J.George Shanthiku-mar (2002), "Stochastic ordering of order statistics II," in *Modeling Uncertainty: An Examination of Stochastic Theory, Methods, and Applications,* M. Dror, P. L'Ecuyer, and F. Szidarovszky (eds), Boston: Kluwer.

[7] Bulow, J. and Levin, J. (2005): "Matching and price competition", *American Economic Review* forthcoming.

[8] Chao, H. and Wilson, R. (1987): "Priority service:pricing, investment, and market organization" *American Economic Review* 77, 899-916.

[9] Cole, H.J, Mailath, G. and Postlewaite, A. (1992): "Social norms, savings behavior, and growth", *Journal of Political Economy* 100, 1092-1125.

[10] Cole, H.J, Mailath, G. and Postlewaite, A. (2001a): "Efficient non-contractible investments in large economies", *Journal of Economic Theory* 101, 333-373.

[11] Cole, H.J., Mailath, G. and Postlewaite, A. (2001b): "Efficient non-contractible investments in finite economies", *Advances in Theoretical Economics* 1, Article 2.

[12] Damiano, E. and Li, H. (2004): "Price discrimination and efficient matching", *Economic Theory*, forthcoming.

[13] David, H. A. and Nagaraja, H. N. (2003) *Order Statistics*, Wiley & Sons, New Jersey.

[14] Fafchamps, M. and Quisumbing, A. (2005) "Assets at marriage in rural Ethiopia" *Journal of Development Economics* 77, 1-25.

[15] Fernandez, R. and Gali, J. (1999): "To each according to...? Markets, tournaments and the matching problem with borrowing constraints" *Review of Economic Studies* 66, 799-824.

[16] Gale, D. and Shapley, L. (1962) "College admissions and the stability of marriage" *American Mathematical Monthly* 69, 9-15.

[17] Hopkins, E. (2005) "Job market signalling of relative position, or Becker married to Spence" Discussion paper, University of Edinburgh.

[18] Hopkins, E. and Kornienko, T. (2005) "Which inequality ? The inequality of resources versus the inequality of rewards" , discussion paper, University of Edinburgh.

[19] Hoppe, H.C., Moldovanu, B., and Ozdenoren, E. (2005) "Intermediation and Matching", discussion paper, University of Bonn.

[20] Hu, T. and Wei, Y. (2001) "Stochastic comparisons of spacings from restricted families of distributions", *Statistics and Probability Letters* 52, 91-99.

[21] Khaledi, B.E. and Kochar, S.C. (1999) "Stochastic orderings between distributions and their sample spacings -II" *Statistics and Probability Letters* 44, 161-166.

[22] Kittsteiner, T. and Moldovanu, B. (2005) "Priority auctions and queue disciplines that depend on processing time", *Management Science* 51, 236-248.

[23] McAfee, R. P. (2002) "Coarse matching" *Econometrica* 70, 2025-2034.

[24] Maynard Smith, J. and D. Harper (2003) *Animal Signals,* Oxford University Press, Oxford.

[25] Miller, G. (2001), *The Mating Mind*, Anchor Books, New York.

[26] Moldovanu, B. and Sela, A. (2001) "The optimal allocation of prizes in contests" *American Economic Review* 91, 542-558.

[27] Moldovanu, B. and Sela, A. (2005) "Contest architecture" *Journal of Economic Theory*, forthcoming.

[28] Nöldeke, G. and Samuelson, L. (2003) "Strategic choice handicaps when females seek high male net viability", *Journal of Theoretical Biology* 221, 53-59.

[29] Peters, M. (2004) "The pre-marital investment game", Discussion paper, University of British Columbia.

[30] Perry, M. and Reny, P. (2005) "Towards a strategic foundation for rational expectations equilibrium", discussion paper, Hebrew University, Jerusalem

[31] Roth, A. E. and Sotomayor, M. A. (1990), *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*, Cambridge University Press, Cambridge.

[32] Shaked, M. and Shanthikumar, J. G. (1994) *Stochastic Orders and their Applications*, Academic Press, Boston.

[33] Shimer, R. and Smith, L. (2000) "Assortative matching and search" *Econometrica* 68, 343-369.

[34] Spence, M. (1973) "Job market signaling" *Quarterly Journal of Economics* 87, 296-332.

[35] Wilson, R. (1989) "Efficient and competitive rationing, *Econometrica* 57, 1-40.

[36] Zahavi, A. (1975) "Mate selection - a selection for a handicap" *Journal of Theoretical Biology* 53, 205-214.

[37] Zahavi, A. and Zahavi, A. (1997), *The Handicap Principle: A Missing Piece of Darwin's Puzzle*, Oxford University Press, New York.