Semiparametric identification and estimation in multi-object, sequential, English auctions *

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Abstract

Within the independent private-values paradigm, we derive the data-generating process of the winning bids for two different objects sold sequentially at English auction assuming the valuations across objects for a particular bidder are potentially dependent. We then illustrate that, in general, the model is unidentified, but that a model within the Archimedean family of copulas is identified. Under the copula assumption, using just the observed winning bids, we propose a semiparametric estimation strategy to recover the joint distribution of valuations. We implement our methods using data from a fish auction held in Grenå, Denmark.

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1. Motivation and Introduction

During the last four decades, economists have made considerable progress in understanding the theoretical structure of strategic behaviour under market mechanisms, such as auctions, when a small number of potential participants exists; see Krishna (2002) for a comprehensive presentation and evaluation.

One analytic device, commonly used to describe bidder motivation at single-object auctions, is a continuous random variable which represents individual-specific heterogeneity in valuations. The conceptual experiment involves each potential bidder's receiving an independent draw from a distribution of valuations. Conditional on his draw, a bidder is then assumed to act purposefully, maximizing either the expected profit or the expected utility of profit from winning the auction. Another, frequently-made assumption is that the bidders are *ex ante* symmetric, their independent draws coming from the same distribution of valuations; this is

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often referred to as the *symmetric independent private-values paradigm* (symmetric IPVP). Under this assumption, the researcher can then focus on a representative agent's decision rule when describing equilibrium behaviour.

At many real-world auctions, several goods are often for sale. Investigating equilibrium behaviour at auctions when several goods are sold, either sequentially or simultaneously, has challenged researchers for some time; see Weber (1983) for a discussion. One key issue is whether the goods for sale are identical or different. In fact, economic theorists often make a distinction between multi-object and multi-unit auctions. At multi-unit auctions, the objects for sale are assumed identical, so it matters not which unit a bidder wins but rather the aggregate number of units he wins. At multi-object auctions, on the other hand, the objects for sale are assumed different, so it matters to a bidder which specific objects he wins. Thus, an example of a multi-object auction would involve the sale of an apple and an orange, while an example of a multi-unit auction would involve the sale of two identical apples.

In models of multi-object auctions, researchers are often interested in deciding whether to bundle the objects for sale. For example, Chakraborty (1999) has derived some sufficient conditions concerning when it will be optimal for an auctioneer to bundle two goods and sell them as one. Under a particular regularity condition, he finds that it will be optimal to bundle the goods if and only if the number of buyers is less than some threshold. This condition depends on the joint distribution of valuations for the objects.

Most structural econometric research devoted to investigating equilibrium behaviour at auctions has involved single-object auctions within the symmetric IPVP. Examples include Paarsch (1992,1997); Donald and Paarsch (1993,1996,2002); Laffont, Ossard, and Vuong (1995); Guerre, Perrigne, and Vuong (2000); Haile and Tamer (2003); and Li (2005). Only recently have researchers begun to investigate multi-unit auctions. In particular, Donald, Paarsch, and Robert (forthcoming) as well as Brendstrup (2002) have investigated sequential, English auctions within the symmetric IPVP, while Jofre-Bonet and Pesendorfer (2003) have investigated the effects of capacity constraints at sequential, low-price, sealed-bid procurement auctions with symmetric, independent private costs. Hortaçsu (2002) has investigated share auctions within the symmetric IPVP. Brendstrup and Paarsch (forthcoming) have investigated sequential, English auctions within the asymmetric IPVP, when the valuation draws of bidders are from different distributions, while Brendstrup and Paarsch (2004) have investigated sequential, Dutch auctions within the asymmetric IPVP.

In this paper, we take a first step toward implementing structural-econometric methods for multi-object auction models. We develop an empirical, private-values framework within which dependence in valuations at multi-object, sequential, English auctions can be identified. We propose a semiparametric structural-econometric strategy, based on Archimedean copulas, to identify and to estimate the joint distribution of latent valuations for several objects when only the winning bids are observed. Our methods allows us to decide whether the regularity condition derived by Chakraborty hold in a specific application. We implement these methods using daily data from a fish auction held in Grenå, Denmark.

Our paper has four remaining parts. In the next section, we outline a notation, introduce some known results, and develop the intuition behind our approach. In particular, we construct a simple theoretical model of bidder behaviour at single-object, English auctions within the symmetric IPVP. In section 3, we extend this model to two objects. In this case, the valuation-pair draws of bidders are independent and identically-distributed, but dependence may exist between the valuations for the two objects on sale. We then derive the data-generating process of the winning bids for the two objects and illustrate why the model is nonparametrically unidentified when only winning bids are observed, a case typically encountered in practice. Within the family of Archimedean copulas, we demonstrate semiparametric identification of the model. Subsequently, we propose a semiparametric estimation strategy based on the empirical distribution functions of winning bids as well as the family of Archimedean copulas. We demonstrate that semiparametric methods provide consistent estimates of population quantities and we describe their asymptotic distribution. Because samples of data concerning auctions are often small, we conducted several Monte Carlo experiments to gauge the small-sample behaviour of our estimator. In section 4, we demonstrate the feasibility of our methods, applying them to data from a sample of "two-object" fish auctions held in Grenå, Denmark, which was often spelt "Grenaa" in old Danish, finding that Chakraborty's condition for bundling holds when fewer than five potential bidders exists. At the *Grenaa Fiskeauktion*, seven bidders typically attend, so we conclude that selling the objects individually is optimal. We summarize and conclude in the final section of the paper. In an appendix, we document the creation of the data set used.

2. Identification and Estimation in Single-Object, English Auctions

In this section, we first develop a notation, then introduce some known results, and finally demonstrate how existing methods work within a well-understood environment: a model of single-object, English auctions within the symmetric IPVP. Subsequently, in section 3, we develop a model of sequential, two-object, English auctions within the symmetric IPVP, where the valuations of the two objects may be dependent.

To begin, we consider the sale of a single object at English auction assuming that each of the $n(\geq 2)$ potential bidders draws his valuation V independently from the same cumulative distribution function $F(v)$ which has support on $(0, \infty)$.

We model the English auction using the Milgrom and Weber (1982) *clock model*. Specifically, the clock is set initially at some minimum (reserve) price and then proceeds to rise continuously. As the price rises, bidders signal their exit from the auction. For our purposes, it is unnecessary to be specific concerning this signalling. Suffice it to say that, when all but one of the bidders have dropped out, the remaining bidder is the winner and the price he pays is the last bid his last opponent was willing to pay.

At English auctions within the IPVP, it is a dominant strategy for nonwinners to bid up to their true valuations. Hence, the winner will be the bidder with the highest valuation and the winning bid will be the second-highest valuation. We denote the true population cumulative distribution of the second-highest order-statistic Y by $G^0(y)$ and note that it is related to the true population cumulative distribution of valuations $F^0(v)$ via

$$\begin{aligned} G^0(y) &= n(n-1)\int_0^{F^0(y)} u^{n-2}(1-u)\ du \\ &= nF^0(y)^{n-1} - (n-1)F^0(y)^n, \end{aligned}$$

so

$$F^0(v) = \varphi[G^0(v)]$$

where $\varphi(\cdot)$ is a known, monotonic function of its argument. Thus, $F^0(v)$ is identified by $G^0(y)$; for a general discussion of identification, see Prakasa Rao (1992), while for a specific discussion of identification in the case of single-object auction models, see Athey and Haile (2002).

Consider a sample of independent, single-object auctions of T identical goods where n bidders participate at each auction. Given the winning bids $\{Y_t\}_{t=1}^T$, a natural estimator of $G^0(y)$ is the empirical distribution function

$$\hat{G}(y) = \frac{1}{T}\sum_{t=1}^{T} \mathbf{1}(Y_t \leq y)$$

where $\mathbf{1}(\mathsf{A})$ is one when the event A obtains, and zero otherwise. Evaluating the transformation $\varphi(\cdot)$ at $\hat{G}(v)$ yields an estimator

$$\hat{F}(v) = \varphi[\hat{G}(v)]$$

of $F^0(v)$. Under the conditions maintained above, the distribution theory for the empirical distribution function is well-known. Also, because the function $\varphi(\cdot)$ is continuous and differentiable, standard application of the continous mapping theorem as well as the delta method can deliver the point-wise asymptotic distribution of $\hat{F}(v)$.

3. Sequential, English Auctions of Two Objects

We remain within the symmetric IPVP, but now assume that each of the n potential bidders draws a pair of valuations, V_1 for object A for sale and V_2 for object B, independently from the joint distribution $H(v_1, v_2)$ where $F_1(v_1)$ and $F_2(v_2)$ denote the marginal cumulative distribution functions of V_1 and V_2, respectively. We assume that the payoff to a winner of the two objects is the sum of his valuations for the two objects minus the price paid for each object. As such, ours is the model of Chakraborty (1999). Note that, within this setting, cases where the utility of winning the second object depends on the outcome of the first auction are not admitted. In fact, in our model, the objects are substitutes.

The two objects are then sold sequentially at English auction. As in the previous section, we assume the Milgrom–Weber clock model. Note that, regardless of the outcome at the first auction, it remains a dominant strategy to bid ones valuation at the second auction. This then implies that it is also a dominant strategy to bid up to ones valuation for the object on sale at the first auction. Thus, in this model, the order of sale is irrelevant. We make the following assumption concerning the data available.

Assumption 1. At each auction, we observe the winning bids as well as the number of bidders.

Now, within this framework, an important policy decision faced by the auctioneer is whether to bundle the two objects and then to sell them as one. In fact, in the model considered above, Chakraborty (1999) has derived a regularity condition concerning $H(v_1, v_2)$ that needs to be verified. In order to define this condition, we introduce some additional notation.

Specifically, let $Q_1(\tau)$ denote the τ-quantile of V_1 and $Q_2(\tau)$ denote the τ-quantile of V_2. Thus, $Q_i(\tau)$ equals $F_i^{-1}(\tau)$. Consider, too, the sum S of V_1 and V_2. Define

$$Q(\tau) = Q_1(\tau) + Q_2(\tau) - Q_S(\tau).$$

Now,

$$f_S(s) = \int_0^\infty h(v_1, s - v_1)\ dv_1,$$

and

$$F_S(s) = \int_0^s f_S(u)\ du = \int_0^s \int_0^\infty h(v_1, u - v_1)\ dv_1 du,$$

so the quantiles of S depend on $h(\cdot, \cdot)$. Chakraborty (1999) has stated the following:

Regularity Condition. There exist a τ_0 such that

$$\begin{aligned} Q(\tau) < 0 \quad & \text{if } \tau \in (0, \tau_0) \\ Q(\tau) > 0 \quad & \text{if } \tau \in (\tau_0, 1). \end{aligned}$$

As Chakraborty has mentioned, the regularity condition is a condition on the tails of the distributions. When the distributions of V_1 and V_2 are identical, the distribution their sum S is more concentrated than either of the marginal distributions: a random draw from the sum is more likely to yield a *central* value than would random draws from the marginal distributions. In an auction application, bundling — selling two objects together — will only be more profitable in expected-revenue terms when Chakraborty's regularity condition holds for a particular amount of competition. If this condition is met, then bundling is optimal, otherwise the two objects should be sold individually. Specifically, Chakraborty had proven the following proposition concerning how to find the threshold point of competition:

Proposition. Suppose that $h(v_1, v_2)$ satisfies the regularity condition, then there exists a ν such that

$$\int_0^1 Q(\tau)\tau^{n-2}(1-\tau)\ d\tau > 0$$

for all n greater than ν.

Of interest to us, is the following theorem from Chakraborty:

Theorem. Suppose a distribution satisfies the regularity condition. There is a $\nu > 2$ such that selling in a single bundle is strictly better than separate sales if $n < \nu$ and selling in a bundle is strictly worse than selling separately if $n > \nu$.

We would like to decide whether bundling is, in fact, optimal in a particular application, assuming the above structure.

We make another assumption concerning the data available.

Assumption 2. We observe an independent sequence of T identical pairs of auctioned objects.

Now, if we had a sample of T sequential, second-price, sealed-bid (Vickrey) auctions, then we could identify and estimate the true joint distribution function $H^0(v_1, v_2)$ nonparametrically from the submitted bid pairs $\{[(B_{1,i,t}, B_{2,i,t}]_{i=1}^n\}_{t=1}^T$ of all participants because

$$B_{1,i,t} = V_{1,i,t}$$

$$B_{2,i,t} = V_{2,i,t}.$$

From this estimated joint distribution function $\hat{H}(v_1, v_2)$, we could then decide whether Chakraborty's condition is met.

The problem is that at English auctions we typically never observe each bidder's valuation for each object. Instead, most often, we just observe the winning bids at the auctions, which we denote in this case by Y_1 and Y_2, respectively. Now, under the Milgrom–Weber clock model, Y_1 is $V_1^{(2:n)}$ and Y_2 is $V_2^{(2:n)}$, the second-highest order-statistics of valuations, but $V_1^{(2:n)}$ might be the valuation of bidder i, while $V_2^{(2:n)}$ could be the valuation for bidder j. Only occasionally will one bidder win both objects. But even in such a circumstance, it is unclear whether one can obtain identification. At English auctions, the winners are the bidders with the highest valuations, but they pay what their nearest opponents were willing to pay.

Our task is to recover the joint distribution $H^0(v_1, v_2)$ using just the true marginal distributions of observed winning bids $G_1^0(y_1)$ and $G_2^0(y_2)$. But without additional structure, we are unable to identify $H^0(\cdot, \cdot)$ from $G_1^0(\cdot)$ and $G_2^0(\cdot)$, except when V_1 and V_2 are independent. What to do?

3.1. Some Results concerning Copulas

We employ the *copula*. Nelsen (1999) has provided a detailed introduction to the theory of copulas; we next repeat some basic facts which are relevant to our later work. For two variables, U_1 and U_2, a bivariate copula $C(u_1, u_2)$ is a continuous function having the following properties:

1. Domain$(C) = [0, 1]^2$;
2. $C(u_1, 0) = 0 = C(0, u_2)$;
3. $C(u_1, 1) = u_1$ and $C(1, u_2) = u_2$;
4. C is a twice-increasing function, so for any rectangle $[u_1^0, u_2^0] \times [u_1^1, u_2^1]$

$$C(u_2^0, u_2^1) - C(u_1^0, u_1^1) - C(u_2^0, u_1^1) + C(u_1^0, u_1^1) \geq 0.$$

Because U_1 and U_2 are both defined on the unit interval, they can be viewed as uniform random variables with $C(u_1, u_2)$ being their joint distribution function. Alternatively, U_1 and U_2 can be viewed as the cumulative distribution functions of two random variables V_1 and V_2. In this case, their marginal distribution functions $F_1(v_1)$ and $F_2(v_2)$ are linked to their joint distribution $H(v_1, v_2)$ by

$$H(v_1, v_2) = C[F_1(v_1), F_2(v_2)].$$

One attractive feature of copulas is that the marginal cumulative distribution functions do not depend on the choice of the dependence function for the two random variables in question. When one is interested in the association between random variables, copulas are a useful device because the dependence structure is easily separated from the marginal cumulative distribution functions.

Often, at multi-object auctions, sufficient data exist to obtain nonparametric estimates of the marginal distributions of winning bids, but not the joint distribution of valuations for all bidders. Hence, it is convenient to adopt a specific form for the dependence function, the copula.

From Sklar's Theorem, we know that a unique function C, the copula, exists such that

$$H(v_1, v_2) = C[F_1(v_1), F_2(v_2)].$$

Also, if we introduce the copulas

$$M(u_1, u_2) = \min(u_1, u_2)$$

and

$$W(u_1, u_2) = u_1 + u_2 - 1,$$

then the following inequalities hold:

$$W(u_1, u_2) = u_1 + u_2 - 1 \leq C(u_1, u_2) \leq \min(u_1, u_2) = M(u_1, u_2),$$

also known as the *Fréchet–Hoeffding bounds.* Note, however, that the Fréchet–Hoeffding bounds are quite wide, so do not restrict the shape of the copula very much.

3.2. Identification within the Archimedean Family of Copulas

We seek to identify and to estimate the underlying distribution of valuations which, as argued above, is a copula, and then characterize the shape of the copula. At Vickrey auctions, the copula $C(\cdot)$ is nonparametrically identified and can be estimated nonparameterically using well-known methods. However, at English auctions, insufficient data exist to follow such a strategy. Thus, we constrain ourselves to the family of *Archimedean* copulas. The Archimedean family of copulas is defined by

$$C_\zeta(u_1, u_2) = \zeta^{-1}\left[\zeta(u_1) + \zeta(u_2)\right]$$

where the *generating function*, $\zeta(\cdot)$, is a convex, decreasing function. Note that $\zeta(1)$ must equal zero and $\zeta^{-1}(u)$ must be zero for any u exceeding $\zeta(0)$. These conditions are both necessary and sufficient for C_ζ to be a distribution function. Copulas within the Archimedean family have joint density functions of the following form:

$$c_\zeta(u_1, u_2) = -\frac{\zeta''(H)\zeta'(u_1)\zeta'(u_2)}{\left[\zeta'(H)\right]^3}.$$

Members of the Archimedean family are both symmetric and associative, so

$$C_\zeta(u_1, u_2) = C_\zeta(u_2, u_1) \quad \forall \quad u_1, u_2 \in [0, 1],$$

and

$$C_\zeta[C_\zeta(u_1, u_2), u_3] = C_\zeta[u_1, C_\zeta(u_2, u_3)] \quad \forall \quad u_1, u_2, u_3 \in [0, 1].$$

Another attractive feature of the Archimedean family is that it easy to identify whether it has points of singularity, mass points; see Genest and MacKay (1986).

Well-known members of the Archimedean family, which depend on a single parameter θ, are presented in Table 1. Note that independence obtains for the Clayton and Gumbel members when θ is zero and, in the case of Frank copulas, when θ is one. For these members of the Archimedean family, the joint density functions are easy to compute. For example, in the case of Frank copulas,

$$c_\zeta(u_1, u_2) = \frac{(\theta - 1)\log(\theta)\theta^{u_1+u_2}}{\left[(\theta - 1) + (\theta^{u_1} - 1)(\theta^{u_2} - 1)\right]^2}$$

where it is easy to show that this density converges to one as θ tends to one.

The reader may well ask why we have constrained ourselves to the Archimedean family and not some other one. First, the Archimedean family is a commonly-used family among scientists, presumably because it fits data well. Second, because the Archimedean family has been used quite often, its features appear to be well-understood by scholars. Had we chosen another, obscure, family of copulas readers would certainly have questioned that choice too. Above, we have demonstrated that, given the data typically available at sequential English auctions, nonparametric identification appears impossible. Thus, in order to obtain identification,

Table 1.
Well-Known Members of Archimedean Family of Copulas.

Member	$C(u_1, u_2)$	$\zeta(u)$	Range of θ
Clayton	$(u_1^{-\theta} + u_2^{-\theta} - 1)^{\frac{1}{\theta}}$	$\frac{u^{-\theta}-1}{\theta}$	$[0, \infty)$
Frank	$\log_\theta \left[1 + \frac{(\theta^{u_1}-1)(\theta^{u_2}-1)}{(\theta-1)}\right]$	$\log_\theta \left(\frac{1-\theta}{1-\theta^u}\right)$	$[0, \infty)$
Gumbel	$\exp\left(\left\{[-\log(u_1)]^{\theta+1} + [-\log(u_2)]^{\theta+1}\right\}^{\frac{1}{\theta+1}}\right)$	$[-\log(u)]^{\theta+1}$	$[0, \infty)$

some additional structure is required. It is quite possible that one can obtain identication within other families of copulas. We simply have not investigated this question.

How does constraining $H^0(\cdot, \cdot)$ to lie within the Archimedean family of copulas help us obtain identification when just the winning bids are observed at English auctions? Suppose we have n random vectors $\{\boldsymbol{V}^i\}_{i=1}^n$ from the joint cumulative distribution function $\boldsymbol{F}(\boldsymbol{v}^1, \ldots, \boldsymbol{v}^n)$ where $\boldsymbol{V}^i$ equals $(V_1^i, V_2^i)^\top$ in this case. Under the symmetric IPVP assumption,

$$\boldsymbol{F}(\boldsymbol{v}^1, \ldots, \boldsymbol{v}^n) = H(\boldsymbol{v}^1) \times \ldots \times H(\boldsymbol{v}^n)$$
$$\boldsymbol{f}(\boldsymbol{v}^1, \ldots, \boldsymbol{v}^n) = h(\boldsymbol{v}^1) \times \ldots \times h(\boldsymbol{v}^n).$$

Consider now the joint density of the component-wise, ordered pairs; *viz.*,

$$V_1^{(n:n)} < V_1^{(n-1:n)} < \ldots < V_1^{(2:n)} < V_1^{(1:n)}$$
$$\text{and}$$
$$V_2^{(n:n)} < V_2^{(n-1:n)} < \ldots < V_2^{(2:n)} < V_2^{(1:n)}.$$

Note the transformation from $[V_1^1, V_1^2, \ldots, V_1^{n-1}, V_1^n, V_2^1, V_2^2, \ldots, V_2^{n-1}, V_2^n]$ to $[V_1^{(1:n)}, V_1^{(2:n)}, \ldots, V_1^{(n-1:n)}, V_1^{(n:n)}, V_2^{(1:n)}, V_2^{(2:n)}, \ldots, V_2^{(n-1:n)}, V_2^{(n:n)}]$ is not one-to-one. In fact, a total of $(n!)^2$ possible combinations exist. Thus, $(n!)^2$ inverses exist to this transformation. For example, one of the permutations is

$$v_1^n < v_1^1 < \ldots < v_1^{n-1} < v_1^2 \quad \text{and} \quad v_2^{n-1} < v_2^2 < \ldots < v_2^n < v_2^1.$$

For this case, the corresponding inverse is then

$$v_1^n = v_1^{(n:n)}$$
$$v_1^1 = v_1^{(n-1:n)}$$
$$\vdots$$

$$
\begin{aligned}
v_1^{n-1} &= v_1^{(2:n)} \\
v_1^2 &= v_1^{(1:n)} \\
v_2^{n-1} &= v_2^{(n:n)} \\
v_2^2 &= v_2^{(n-1:n)} \\
&\vdots \\
v_2^n &= v_2^{(2:n)} \\
v_2^1 &= v_2^{(1:n)}.
\end{aligned}
$$

Notice that the Jacobian matrix $\mathbf{J}$ of this transformation is an $(2n \times 2n)$ identity matrix $\mathbf{I}_{2n}$ with its rows rearranged, so $|\mathbf{J}|$ is one. Therefore, the joint density of the order-statistics is simply

$$
\begin{aligned}
&\boldsymbol{g}[v_1^{(n-1:n)}, v_1^{(1:n)}, \ldots, v_1^{(2:n)}, v_1^{(n:n)}, v_2^{(1:n)}, v_2^{(n-1:n)}, \ldots, v_2^{(n:n)}, v_2^{(2:n)}] = \\
&\quad h[v_1^{(n-1:n)}, v_2^{(1:n)}] \times h[v_1^{(1:n)}, v_2^{(n-1:n)}] \times \ldots \times h[v_1^{(2:n)}, v_2^{(n:n)}] \times h[v_1^{(n:n)}, v_2^{(2:n)}].
\end{aligned}
$$

It then follows from Remark 4.4.2 in Rohatgi (1976) that

$$
\begin{aligned}
\boldsymbol{g}[v_1^{(1:n)}, v_1^{(1:n)}, \ldots, v_1^{(n-1:n)}, v_1^{(n:n)}, v_2^{(1:n)}, v_2^{(2:n)}, \ldots, v_2^{(n-1:n)}, v_2^{(n:n)}] = \\
\sum_{\text{all } (n!)^2 \text{ inverses}} \prod h(\cdot,\cdot).
\end{aligned}
$$

This can be written compactly as

$$
\begin{aligned}
&\boldsymbol{g}[v_1^{(1:n)}, v_1^{(1:n)}, \ldots, v_1^{(n-1:n)}, v_1^{(n:n)}, v_2^{(1:n)}, v_2^{(2:n)}, \ldots, v_2^{(n-1:n)}, v_2^{(n:n)}] = \\
&\qquad n!\,\mathrm{Perm}\begin{pmatrix}
h[v_1^{(1:n)}, v_2^{(1:n)}] & \ldots & h[v_1^{(n:n)}, v_2^{(1:n)}] \\
h[v_1^{(1:n)}, v_2^{(2:n)}] & \ldots & h[v_1^{(n:n)}, v_2^{(2:n)}] \\
\vdots & \ddots & \vdots \\
h[v_1^{(1:n)}, v_2^{(n:n)}] & \ldots & h[v_1^{(n:n)}, v_2^{(n:n)}]
\end{pmatrix}
\end{aligned}
$$

where "Perm" is the permanent operator which, for a (3×3) matrix, is defined as follows:

$$
\mathrm{Perm}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a(ei + hf) + b(di + gf) + c(dh + ge).
$$

The permanent formula has natural extensions to dimensions greater than three. Now, when n is two, for example, the following $(2!)^2$, four, inverses exist:

$$
v_1^1 = v_1^{(1:2)} \quad v_2^1 = v_2^{(1:2)}
$$

$$v_1^2 = v_1^{(2:2)} \quad v_2^2 = v_2^{(2:2)}$$

$$\begin{aligned} v_1^1 &= v_1^{(2:2)} \quad v_2^1 = v_2^{(2:2)} \\ v_1^2 &= v_1^{(1:2)} \quad v_2^2 = v_2^{(1:2)} \end{aligned}$$

$$\begin{aligned} v_1^1 &= v_1^{(1:2)} \quad v_2^1 = v_2^{(2:2)} \\ v_1^2 &= v_1^{(2:2)} \quad v_2^2 = v_2^{(1:2)} \end{aligned}$$

$$\begin{aligned} v_1^1 &= v_1^{(2:2)} \quad v_2^1 = v_2^{(1:2)} \\ v_1^2 &= v_1^{(1:2)} \quad v_2^2 = v_2^{(2:2)}, \end{aligned}$$

so

$$\begin{aligned} \boldsymbol{g}[v_1^{(1:2)}, v_2^{(1:2)}, v_1^{(2:2)}, v_2^{(2:2)}] &= h[v_1^{(1:2)}, v_2^{(1:2)}]h[v_1^{(2:2)}, v_2^{(2:2)}]+ \\ &\quad h[v_1^{(1:2)}, v_2^{(2:2)}]h[v_1^{(2:2)}, v_2^{(1:2)}]+ \\ &\quad h[v_1^{(2:2)}, v_2^{(1:2)}]h[v_1^{(1:2)}, v_2^{(2:2)}]+ \\ &\quad h[v_1^{(2:2)}, v_2^{(2:2)}]h[v_1^{(1:2)}, v_2^{(1:2)}]+ \\ \\ &= 2\Big\{ h[v_1^{(1:2)}, v_2^{(1:2)}]h[v_1^{(2:2)}, v_2^{(2:2)}]+ \\ &\qquad h[v_1^{(1:2)}, v_2^{(2:2)}]h[v_1^{(2:2)}, v_2^{(1:2)}] \Big\} \\ &= 2!\,\mathrm{Perm}\begin{pmatrix} h[v_1^{(1:2)}, v_2^{(1:2)}] & h[v_1^{(2:2)}, v_2^{(1:2)}] \\ \\ h[v_1^{(1:2)}, v_2^{(2:2)}] & h[v_1^{(2:2)}, v_2^{(2:2)}] \end{pmatrix}. \end{aligned}$$

To obtain the joint density of the winning bids Y_1 and Y_2 which, under the Milgrom–Weber clock model are $V_1^{(2:n)}$ and $V_2^{(2:n)}$, we must integrate out all the other order-statistics, so

$$\begin{aligned} g_{12}(y_1, y_2) = \int\int \ldots \int\int\int \ldots \int \boldsymbol{g}(x_1, z_1, y_1, y_2, x_3, z_3, \ldots, x_n, z_n) \\ dx_1, dx_3, \ldots, dx_n\ dz_1, dz_3, \ldots, dz_n. \end{aligned}$$

Continuing our two-bidder, two-object example

$$\begin{aligned} g_{12}(y_1, y_2) &= \int_{y_1}^{\infty}\int_{y_2}^{\infty} \boldsymbol{g}(x_1, z_1, y_1, y_2)\ dz_1 dx_1 \\ &= \int_{y_1}^{\infty}\int_{y_2}^{\infty} 2[h(x_1, z_1)h(y_1, y_2) + h(x_1, y_2)h(y_1, z_1)]\ dz_1 dx_1 \\ &= 2h(y_1, y_2)\int_{y_1}^{\infty}\int_{y_2}^{\infty} h(x_1, z_1)\ dz_1 dx_1 + \\ &\qquad 2\int_{y_1}^{\infty}\int_{y_2}^{\infty} h(x_1, y_2)h(y_1, z_1)\ dz_1 dx_1. \end{aligned}$$

Under the assumption of independence, for example, $h(v_1, v_2)$ is the product of the marginal probability density functions $f_1(v_1)f_2(v_2)$, so

$$\begin{aligned}
g_{12}(y_1, y_2) = {} & 2f_1(y_1)f_2(y_2) \int_{y_1}^{\infty} \int_{y_2}^{\infty} f_1(x_1)f_2(z_1)\ dz_1 dx_1 + \\
& 2 \int_{y_1}^{\infty} \int_{y_2}^{\infty} f_1(x_1)f_2(y_2)f_1(y_1)f_2(z_1)\ dz_1 dx_1 \\
= {} & 2f_1(y_1)f_2(y_2)[1 - F_1(y_1)][1 - F_2(y_2)] + \\
& 2f_1(y_1)f_2(y_2)[1 - F_1(y_1)][1 - F_2(y_2)] \\
= {} & 4f_1(y_1)f_2(y_2)[1 - F_1(y_1)][1 - F_2(y_2)].
\end{aligned}$$

It is then easy to show that the marginal probability density function of Y_1, the second-highest order-statistic of valuations for the first object for sale, for example, is

$$\begin{aligned}
g_1(y_1) &= 2f_1(y_1)[1 - F_1(y_1)] \int_0^{\infty} 2f_2(y_2)[1 - F_2(y_2)]\ dy_2 \\
&= 2f_1(y_1)[1 - F_1(y_1)].
\end{aligned}$$

But what about when V_1 and V_2 are dependent? Can one identify $h^0(v_1, v_2)$, within the family of Archimedean copulas, from just information on the joint density of winning bids $g^0_{12}(y_1, y_2)$? Yes.

Consider again our two-bidder, two-object example where we suppress the 0 superscript for notational parsimony. Now, the joint density can be written as

$$\begin{aligned}
g_{12}(y_1, y_2) = {} & 2h(y_1, y_2) \int_{y_1}^{\infty} \int_{y_2}^{\infty} h(x_1, z_1)\ dz_1 dx_1 + \\
& 2 \int_{y_1}^{\infty} \int_{y_2}^{\infty} h(x_1, y_2)h(y_1, z_1)\ dz_1 dx_1 \\
= {} & 2h(y_1, y_2)[1 - F_1(y_1) - F_2(y_2) + H(y_1, y_2)] + \\
& 2\left[\int_{y_2}^{\infty} h(y_1, z_1)\ dz_1 \int_{y_1}^{\infty} h(x_1, y_2)\ dx_1\right] \\
= {} & 2h(y_1, y_2)[1 - F_1(y_1) - F_2(y_2) + H(y_1, y_2)] + \\
& 2\left\{f_1(y_1) - \frac{\zeta'[F_1(y_1)]}{\zeta'(H)} f_1(y_1)\right\}\left\{f_2(y_2) - \frac{\zeta'[F_2(y_2)]}{\zeta'(H)} f_2(y_2)\right\}
\end{aligned}$$

where the last two lines derive from

$$\int_{y_2}^{\infty} h(y_1, z_1)\ dz_1 = f_1(y_1) - \frac{\partial H(y_1, y_2)}{\partial y_1}$$

and

$$\begin{aligned}
\frac{\partial H(y_1, y_2)}{\partial y_1} &= \frac{\partial C[F_1(y_1), F_2(y_2)]}{\partial F_1(y_1)} \frac{dF_1(y_1)}{dy_1} \\
&= \frac{\partial C[F_1(y_1), F_2(y_2)]}{\partial F_1(y_1)} f_1(y_1) \\
&= \frac{\zeta'[F_1(y_1)]}{\zeta'(H)} f_1(y_1).
\end{aligned}$$

We prove that the model is identified under the family of Archimedean copulas using the following *reductio ad absurdum* argument: Consider two different generating functions $\zeta_0(\cdot)$ and $\zeta_1(\cdot)$. Our model is unidentified if these two generating functions yield observationally equivalent joint density functions of the winning bids for all y_1 and y_2. When this is true,

$$
\begin{aligned}
&2h^0(y_1, y_2)[1 - F_1^0(y_1) - F_2^0(y_2) + H^0(y_1, y_2)]+\\
&2\left\{f_1^0(y_1) - \frac{\zeta_0'[F_1^0(y_1)]}{\zeta_0'[H^0(y_1, y_2)]} f_1^0(y_1)\right\}\left\{f_2^0(y_2) - \frac{\zeta_0'[F_2^0(y_2)]}{\zeta_0'[H^0(y_1, y_2)]} f_2^0(y_2)\right\} =\\
&2h^1(y_1, y_2)[1 - F_1^1(y_1) - F_2^1(y_2) + H^1(y_1, y_2)]+\\
&2\left\{f_1^1(y_1) - \frac{\zeta_1'[F_1^1(y_1)]}{\zeta_1'[H^1(y_1, y_2)]} f_1^1(y_1)\right\}\left\{f_2^1(y_2) - \frac{\zeta_1'[F_2^1(y_2)]}{\zeta_1'[H^1(y_1, y_2)]} f_2^1(y_2)\right\}
\end{aligned}
$$

for all y_1 and y_2. In particular, this must be true at y_2 equal zero, which implies that

$$h^0(y_1, 0) = h^1(y_1, 0)$$

for all y_1. This condition then implies that

$$\frac{\zeta_0''(0)\zeta_0'(0)\zeta_0'(u_1)}{[\zeta_0'(0)]^3} = \frac{\zeta_1''(0)\zeta_1'(0)\zeta_1'(u_1)}{[\zeta_1'(0)]^3}$$

for all u_1. Hence, the functions can only differ up to an affine transformation, so

$$\zeta_1(u) = a + b\zeta_0(u).$$

Now, $\zeta_0(1)$ and $\zeta_1(1)$ must both be zero, so this then implies that they can only differ by a multiplicative constant b. We know, however, from Genest and MacKay (1986) that this yields the same copula. Hence, we have established identification in this case. Although the algebra in the general case of n bidders is straightforward, it is cumbersome, so we omit it here as it lends little to our proof.

Note that in the literature concerning single-object auctions, authors have written on correlation (*e.g.*, within the affiliated private-values paradigm as well as the common-value paradigm); see, for example, Athey and Haile (2002). In that literature, the maximum (or second-highest) bid is drawn from a correlated distribution. In the multi-object case discussed in this paper, the second-highest bids are individually drawn from an independent and identically-distributed law, but they are correlated. The correlation of the observed prices is a nonlinear transformation of the correlation of the underlying value distribution. While these two problems have different structures, they are related.

3.3. Estimation within the Archimedean Family of Copulas

One flexible way of estimating the true generating function $\zeta^0(\cdot)$ would involve introducing a family of shape-preserving functions. For example, consider Figure 1 in which the logarithm of the true generating function is depicted as the solid curve, and the logarithms of successively-better approximations to that function are depicted as dashed and dotted curves. Examples of such approximating functions are shape-preserving polynomials. One would then estimate the coefficients of these polynomials. The estimation strategy would involve allowing the number of terms in the polynomials to increase as the sample size increases, but at a rate which is slower than that of the sample size. Sieve estimation with shape-preserved polynomials is an example of such a method; see, for example, Chen and Shen (1998). Numerical complexity simply prevents us from entertaining such a strategy. Instead, below, we pursue a "parametric" approach which also has certain semiparametric features.

In econometrics, the adjective *semiparametric*, when applied to an estimator, often means that the estimator depends on a first-stage nonparametric procedure that has slower than $\sqrt{T}$ parametric rate of convergence. In the second-stage, however, the semiparametric estimator still achieves the $\sqrt{T}$ parametric rate of convergence. In this paper, the first-stage estimator is also estimated at the $\sqrt{T}$ parametric rate. Hence, we refer to this estimator as semiparametric with some trepidation.

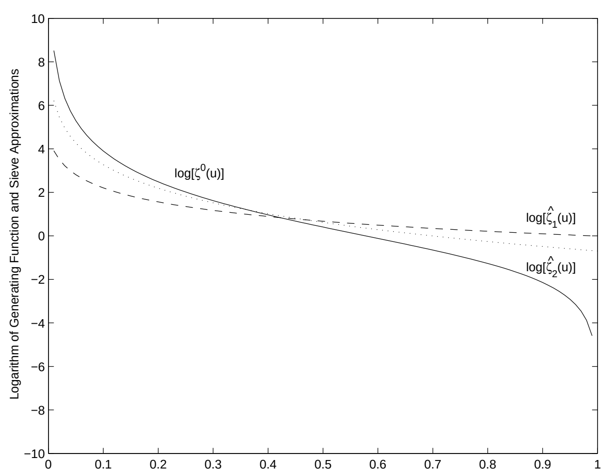

Fig. 1. Sieve Approximations of Copula Generating Function ζ^0.

As mentioned above, a number of members of the Archimedean family have simple parametric representations. Under Assumptions 1 and 2 as well as suitable regularity conditions, when the true copula $C^0(\cdot)$ belongs to a parametric family

$$\mathcal{C} = \{C_{\boldsymbol{\theta}},\ \boldsymbol{\theta} \in \boldsymbol{\Theta}\}$$

defined by the vector $\boldsymbol{\theta}$, then a parameter-consistent and asymptotically-normal estimator of $\boldsymbol{\theta}^0$, the true value, can be obtained by applying the method of maximum likelihood.

In applying the method of maximum likelihood, two approaches exist — first, a fully parametric one or, second, a semiparametric one. Under the first approach, the marginal cumulative distribution functions are assumed to belong to specific parametric families. Each parametric marginal distribution is then substituted into the parametric copula and the corresponding likelihood function is then maximized with respect to all parameters.

Under the second approach, the marginal empirical cumulative distribution functions are substituted into the parametric copula and the corresponding "likelihood function" is then maximized with respect to the parameters of the copula. We employ this latter approach because we do not want to rely on potentially restrictive as well as unnecessary assumptions concerning the marginal cumulative distribution functions. In order to do this, however, we must extend the work of Shih and Louis (1995) as well as Genest, Ghoudi, and Rivest (1995) because, typically, at English auctions only the winning bids, the second-highest order-statistics under the Milgrom–Weber clock model, are observed, rather than the valuations of all the bidders.

From Theorem 2.6 of Embrechts, Lindskog, and McNeil (2001), we know that, because the transformation $\varphi(\cdot)$ is strictly increasing, it is unnecessary to carry it around in the analysis; we can just work with $\hat{G}_1(y_1)$ and $\hat{G}_2(y_2)$. Note, too, that we can view U_i equal to $\hat{G}_i(Y_i)$ as a draw from a uniformly-distributed random variable without changing the copula function or, thereby, the dependence parameters. Hence, in what follows, we shall work with $U_{i,t}$ equal $\hat{G}_i(Y_t)$, $i = 1, 2$.

Having derived the joint density function of the winning bids, we can then apply the work of Shih and Louis (1995) as well as Genest *et al.* (1995) to estimate the copula parameter vector $\boldsymbol{\theta}$ by $\hat{\boldsymbol{\theta}}$ which solves

$$\sum_{t=1}^{T} \frac{\partial \log g_{12}(U_{1,t}, U_{2,t}; \hat{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}} = \sum_{t=1}^{T} \boldsymbol{s}(U_{1,t}, U_{2,t}; \hat{\boldsymbol{\theta}}) = \mathbf{0}_p \tag{3.1}$$

where $g_{12}(\cdot, \cdot; \boldsymbol{\theta})$ is the joint density of (U_1, U_2) under the copula $C(\cdot; \boldsymbol{\theta})$ with p denoting the dimension of $\boldsymbol{\theta}$.

Although we use the empirical distribution functions $\hat{G}_1(\cdot)$ and $\hat{G}_2(\cdot)$, instead of the true cumulative distribution functions $G_1^0(\cdot)$ and $G_2^0(\cdot)$, Genest *et al.* (1995) have demonstrated that, at least under independence, the solution to the first-order conditions defined by equation (3.1) yields an efficient estimator of $\boldsymbol{\theta}^0$, the true population parameter vector of the copula. Independence is, however, a special case. Because we use the empirical distribution functions, instead of the true cumulative distribution functions themselves, some might call this approach the method of *pseudo maximum-likelihood*.

3.3.1. Parameter Consistency and Asymptotic Distribution

We now characterize the asymptotic distribution of $\hat{\theta}$, the pseudo maximum-likelihood estimator, under certain regularity conditions. To describe these conditions, we first introduce the function

$$\rho(u) = u(1 - u)$$

and a positive constant δ as well as the positive constants p and q where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Assumption 3. The function $s(u_1, u_2; \theta)$ has continuous derivatives in u_1 and u_2 where

$$\frac{\partial s(u_1, u_2; \theta)}{\partial u_i} \quad i = 1, 2$$

is dominated by $M\rho(u_1)^{d_1}\rho(u_2)^{d_2}$ with M being a positive constant, while

$$d_1 = \frac{\delta - \frac{1}{2}}{p} \quad \textit{and} \quad d_2 = \frac{\delta - \frac{1}{2}}{q}.$$

Under the above assumptions, we have

Theorem. $\hat{\theta}$, the pseudo maximimum-likelihood estimator of the true copula parameter θ^0, has asymptotic distribution

$$\sqrt{T}(\hat{\theta} - \theta^0) \xrightarrow{\mathrm{d}} \mathcal{N}\left(0, \frac{\sigma^2}{\beta^2}\right)$$

where

$$\beta = \mathcal{E}_{12}\left\{s\left[G_1^0(V_1), G_2^0(V_2); \theta^0\right]^2\right\}$$

and

$$\sigma^2 = \mathcal{V}\left\{s\left[G_1^0(V_1), G_2^0(V_2); \theta^0\right] + W_1^0(V_1) + W_2^0(V_2)\right\}$$

with

$$W_i^0(V_i) = \int \mathbf{1}\left[G_i(V_i) \leq u_i\right] \frac{\partial s\left(u_1, u_2; \theta^0\right)}{\partial u_i}\, dC(u_1, u_2) \quad i = 1, 2.$$

Proof: Assumption 3 implies that we can invoke Proposition A1 in Genest *et al.* (1995), whereby the result follows. QED

Of course, because several of the quantities above depend on unknown parameters or distributions, implementation is a problem. One strategy is to use the empirical analogues of the true population values when estimating the variance. One estimator of the asymptotic variance of $\hat{\theta}$ would be $(\hat{\sigma}^2/\hat{\beta}^2)$ where the sample analogues are used as estimates of β and σ^2. But what sample analogues?

Consider the random variables

$$\Xi^0(V_1, V_2) = s[G_1^0(V_1), G_2^0(V_2); \theta^0]$$
$$\Omega^0(V_1, V_2) = \Xi^0(V_1, V_2) + W_1^0(V_1) + W_2^0(V_2).$$

We should like to investigate the sampling variation in these quantities. To this end, we introduce pseudo-values. In particular, we define the rescaled empirical copula by

$$\hat{C}(u_1, u_2) = \frac{1}{T}\sum_{t=1}^{T} \mathbf{1}\big[\hat{G}_1(Y_{1,t}) \leq u_1\big]\mathbf{1}\big[\hat{G}_2(Y_{2,t}) \leq u_2\big]$$

and introduce

$$\hat{\Xi}_t = s\big[\hat{G}_1(Y_{1,t}), \hat{G}_2(Y_{2,t}); \hat{\theta}\big]$$

as well as

$$\hat{\Omega}_t = \hat{\Xi}_t + \hat{W}_{1,t} + \hat{W}_{2,t}$$

where

$$\hat{W}_i(V_{i,t}) = \int \mathbf{1}\big[\hat{G}_i(V_{i,t}) \leq u_i\big] \; \frac{\partial s(u_1, u_2; \hat{\theta})}{\partial u_i} \; d\hat{C}(u_1, u_2) \quad i = 1, 2.$$

We then use the sample variances of these pseudo-values as estimates of β and σ^2.

The reader should note that not all copulas satisfy the conditions of Assumption 3. Moreover, under the null hypothesis of independence, the association parameter θ is often on the boundary of the feasible parameter set; *e.g.*, in the case of Clayton and Gumbel copulas. Thus, we cannot always rely on first-order asymptotic methods to provide a useful characterization of our estimator's distribution.

3.3.2. Monte Carlo Experiment

To decide whether first-order asymptotic distribution theory provides approximations that are valid for the small samples typically encountered by empirical researchers investigating field data from auctions, we conducted a series of Monte Carlo experiments. Our experiments had the following structure.

Because we did not want to introduce confounding factors related to boundary problems, we considered the Frank copula, so

$$C_\zeta(u_1, u_2) = \log_\theta \left[1 + \frac{(\theta^{u_1} - 1)(\theta^{u_2} - 1)}{(\theta - 1)}\right].$$

Under independence, θ^0 is one, while under dependence θ^0 can differ from one. Note that the Frank copula satisfies assumption 3. In addition, simulating from the Frank copulas is relatively easy as we illustrate below, so conducting a Monte Carlo study is relatively straightforward.

To generate pairs of dependent random variables, we adopted the methods described by Genest (1987). Specifically, when θ^0 is different from one (*i.e.*, in the absence of independence), F_1 and F_2 are drawn independently from the uniform $(0, 1)$. The random variable

$$W = \theta^{F_1} + (\theta - \theta^{F_1})F_2$$

is then created. Finally,

$$V_1 = F_1$$

$$V_2 = \frac{\log\left\{\frac{W}{[W+(1-\theta)F_2]}\right\}}{\log\theta}.$$

We assumed that the number of bidders n is three. Thus, for each simulation, indexed by $\ell = 1, \ldots, L$, from $(V_{1,\ell}^1, V_{1,\ell}^2, V_{1,\ell}^3)$ and $(V_{2,\ell}^1, V_{2,\ell}^2, V_{2,\ell}^3)$, we found the second-highest order-statistic for each of the two random variables; *i.e.*, $Y_{1,\ell}$ equals $V_{1,\ell}^{(2:3)}$ and $Y_{2,\ell}$ equals $V_{2,\ell}^{(2:3)}$. We chose a sample size T of 250 observations, and estimated θ^0 using the method described above. We replicated this 100 times, so L was 100.

In Table 2, we present the descriptive statistics for a subset of our experiments. In the first row of this table, a θ^0 of one, where V_1 and V_2 are independent so $H(v_1, v_2)$ equals $F_1(v_1)F_2(v_2)$, we report the mean, median, minimum, maximum and standard deviation. A bias of about seven percent exists. The distribution is skewed to the right as Figure 2 confirms.

Table 2.
Monte Carlo Descriptive Statistics.
Sample Size $T = 250$; Number of Replications $L = 100$.

θ^0	Mean	Median	Minimum	Maximum	St.Dev.
1	1.0632	1.0185	0.5157	2.1788	0.2979
5	5.2875	4.9769	2.3041	11.8825	1.8791
10	11.2433	10.3128	5.4279	30.3115	4.0539
15	16.3272	14.5879	6.3534	38.1251	5.9473
20	20.9187	18.2559	9.5473	59.5250	7.7470
25	26.6134	24.7846	10.8459	77.3694	11.0263

In Figure 3, we present the histogram of $\hat{\theta}$ when θ^0 is five; *i.e.*, when V_1 and V_2 are dependent. The mean of $\hat{\theta}$ is 5.2875, which implies a bias of about six percent, suggesting that the method does reasonably well under dependence too.

The remaining rows of Table 2 document the performance of our estimator for larger values of θ^0, those for 10 to 25 in increments of 5. In all of these cases, the bias remains between four and thirteen percent. The standard deviation of the estimator increases as θ^0 rises. To perform additional Monte Carlo experiments, both for larger sample sizes and for larger numbers of bidders, is, in principle, straightforward. However, while admitting larger sample sizes simply requires additional computer time, admitting additional bidders requires simulation because we must integrate out $2(n-1)$ dimensions of unobservables when calculating $g_{12}(\cdot,\cdot;\theta)$.

4. An Application: The *Grenaa Fiskeauktion*

We now present an empirical analysis obtained by applying our methods to data from a particular auction, the *Grenaa Fiskeauktion*, which is an English auction

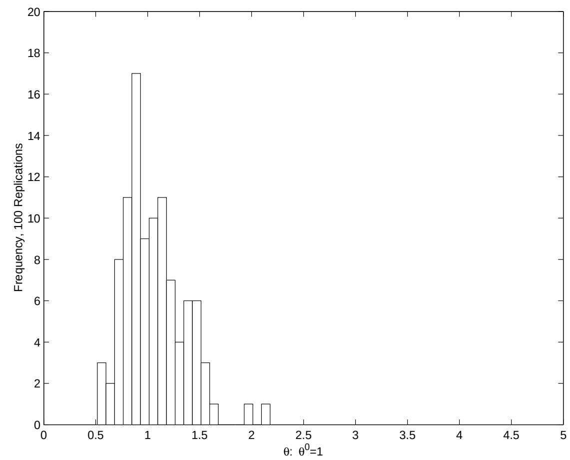

Fig. 2. Histogram of $\hat{\theta}$ under Independence, $\theta^0 = 1$.

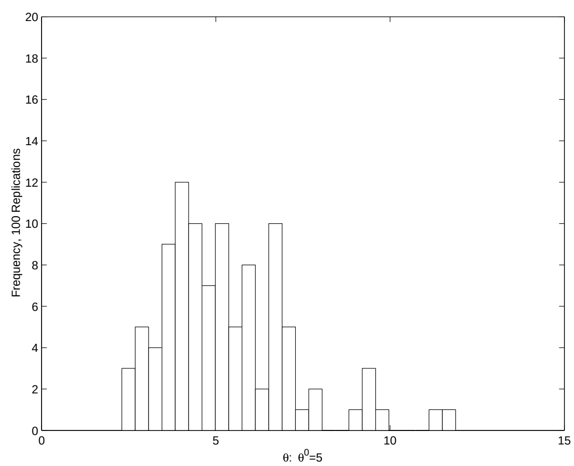

Fig. 3. Histogram of $\hat{\theta}$ under Dependence, $\theta^0 = 5$.

held each weekday morning at 5:00 a.m. in Grenå, Denmark, which is located on the east coast of Jutland, to the northwest of Copenhagen.

The English-auction format is frequently used to sell fish because it is fast and thus well-suited to selling such perishable commodities. However, one goal of the empirical work presented below is to provide estimates of the primitives (*i.e.*, H) necessary to make statements concerning bundling, for example.

By international standards, the *Grenaa Fiskeauktion* is very small. The sellers are the local fishermen who ply the Kattegat and beyond. They have banded together to create the auction house.

The bidders at the *Grenaa Fiskeauktion* are mostly resale trade firms who can be considered agents of retail sellers who have placed orders at pre-specified prices. We think of these retail sellers as living in spacially-separated markets where, because of location, some market power exists. In these markets, the retail sellers have individual-specific marginal revenue curves. We imagine that these are the source of the variation in valuations of bidders. In short, we believe that the IPVP is a reasonable model of the market for fish in and around Grenå.

While other species are sold, often irregularly, the three main species on sale in Grenå are Atlantic cod (*Gadus morhua*), Greenland halibut (*Reinhardtius hippoglossoides*), and plaice (*Pleuronectes platessa*). Plaice is called *rødspætte* in Danish and often referred to as *right-eyed flounder* in North America because both of its eyes are on the right-hand side of its head. Another fish, frequently sold, but often in small volumes, is the wolf-fish or sea catfish (*Anarhichas lupus*), which is called *havkat* in Danish.

For each species of fish, a reserve price exists; this is set by the Danish government in accordance with regulations determined by the European Union. The local auction is allowed to deviate from this reserve price by up to ten percent. During our sample period, the reserve price did not vary.

The particular two products we chose to study were plaice and wolf-fish because each was sold steadily throughout the three-year period we chose to examine, 2 January 2000 to 31 December 2002. The fish supplied at the auction are packed into thirty-five kilogram units which we define to be an object.

After consulting with the auctioneer in Grenå and after examining the raw data, we found a total of seven potential bidders. Each of these bidders attended virtually every auction so, despite the presence of a reserve price which typically induces endogenous participation, we believe that issues of endogenous participation can be safely ignored in this case. Moreover, in our sample of data, the winning bids were always above the posted reserve prices; none of the fish ever went unsold.

In the archives at the *Grenaa Fiskeauktion*, for each auction indexed by t, information concerning the following variables was available:

1) winning bids of the plaice and wolf-fish, $y_{1,t}$ and $y_{2,t}$, respectively;
2) number of potential bidders n, seven.

Table 3.
Sample Descriptive Statistics.
Number of Observations $T = 239$.

Species	Mean	Minimum	Maximum	St.Dev.
Plaice	19.08	7.00	37.00	7.47
Wolf-Fish	24.34	10.00	45.00	6.34

In total, we were able to gather data concerning 239 two-object auctions of plaice and wolf-fish. In Table 3, we present some univariate sample descriptive statistics that may be of interest to the reader, while in in Figure 4, we present a scatterplot of the winning prices of each product. The sample correlation matrix of winning bids for plaice and wolf-fish,

$$\begin{pmatrix} 1 & 0.0668 \\ 0.0668 & 1 \end{pmatrix},$$

is of particular interest as it may provide some hint concerning the dependence in winning prices. For these data, the sample correlation between the winning price for plaice Y_1 and the winning price for wolf-fish Y_2 is quite small, 0.0668; Kendall's tau-statistic was 0.0473.

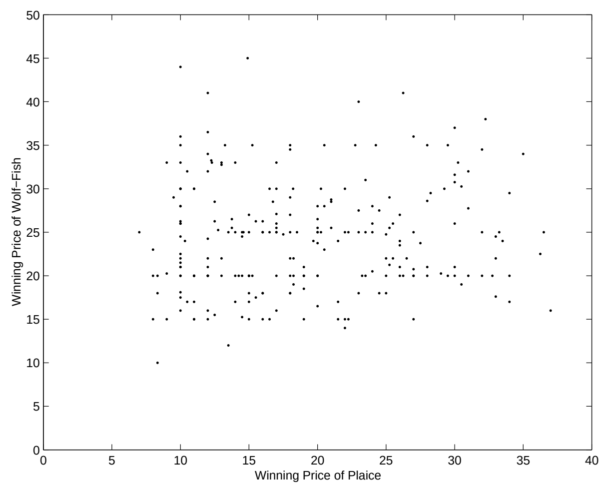

Fig 4. Scatterplot of Winning Prices.

Subsequently, we applied our methods to calculate an estimate of θ^0. We assumed the Frank copula described in section 3. To form $g_{12}(\cdot, \cdot; \theta)$ in the logarithm of the likelihood function, we needed to integrate out twelve dimensions of unobservables. This proved impossible to do either in closed-form or with quadrature, so we resorted to simulation methods. Thus, one might call our method *simulated, pseudo maximum-likelihood* estimation. We used $50,000$ draws to calculate the simulated, pseudo-MLE. Our estimate of θ^0 is 1.48, with a standard error of 0.44. Assuming first-order asymptotic theory applies, the p-value for the standard-normal test-statistic is 0.138, suggesting that V_1 and V_2 are independent. In Figure 5, we present the level sets of the estimated copula.

In Figure 6, we present estimates of the level sets of $h^0(v_1, v_2)$, the joint density of V_1 and V_2. In constructing this figure, we used kernel-smoothed estimates of the cumulative distribution functions of valuations.

Using these point estimates, we have been able to decide whether bundling is, in fact, optimal for the auctioneer. We estimated Chakraborty's τ_0 to be about 0.743, while in Figure 7, we present an estimate of Chakraborty's $Q(\tau)$ function. Based on these estimates we can conclude that bundling is optimal when n is less than five, but selling the objects individually is optimal when n is five or greater. At the *Grenaa Fiskeauktion*, typically seven bidders participate at the auction.

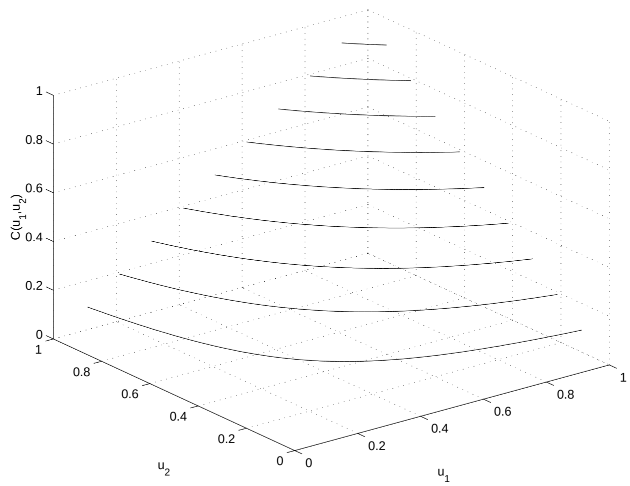

Fig 5. Level Sets of Frank Copula: $\hat{\theta} = 1.48$.

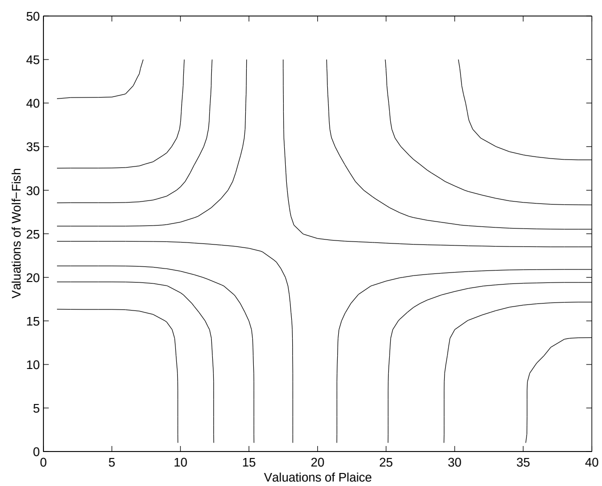

Fig 6. Level Sets for Estimate of Joint Density $h(v_1, v_2)$.

5. Summary and Conclusions

We have undertaken a preliminary investigation of equilibrium behaviour at auctions when several different objects are sold sequentially under the English format. Among other things, in models of multi-object auctions, empirical researchers are often interested in whether to bundle objects to sell as one. The decision to bundle often hinges on the properties of the joint density function of valuations. We have developed an empirical private-values model of sequential, two-object, English auctions within which dependence in valuations for a particular bidder is admitted. We have derived the data-generating process of the winning bids for two

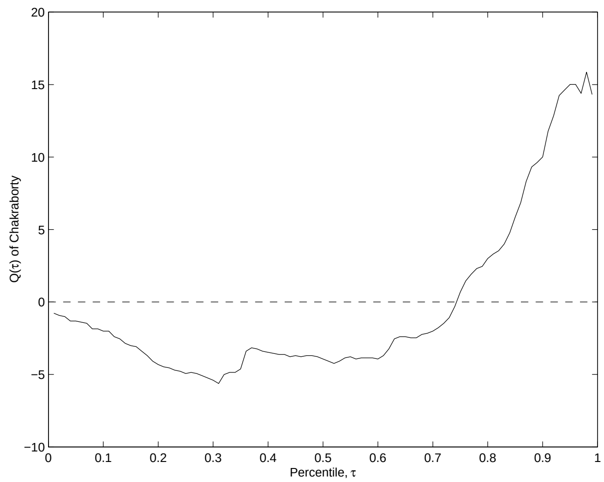

Fig 7. Chakraborty's $Q(\tau)$ Function.

objects sold sequentially and then demonstrated identification under the family of Archimedean copulas. Using only data concerning the observed winning bids, under the Archimedean family, we have proposed a semiparametric structural-econometric strategy to identify and to estimate the joint distribution of latent valuations. Numerical complexity prevented us from implementing the most flexible of empirical methods, so we resorted to parametric specifications which had some semiparametric features. Subsequently, we have demonstrated that our proposed estimation strategy provides parameter-consistent estimates of population quantities and we have described its asymptotic distribution. Because samples of data concerning auctions are often small, we conducted several Monte Carlo experiments to gauge the small-sample behaviour of our estimator. The results of these experiments suggest that our methods are useful to analyze data from the small samples typically encountered by empirical researchers investigating auctions. We have implemented these methods using daily data concerning plaice and wolf-fish from a fish auction held in Grenå, Denmark and the Frank member of the Archimedean family of copulas. Our methods allow us to estimate whether bundling would be profitable. In this case, we found that it is most profitable for the auctioneer to sell each object individually.

A. Appendix

In this appendix, we describe the creation of the data set used. Our data set was derived from information contained in the archives at the *Grenaa Fiskeauktion.* The administrators of the auction graciously gave us access to files from 2 January 2000 to 31 December 2002. This period was chosen because it is recent, so we hope that our empirical work will be relevant. Also, during this period, data at the *Grenaa Fiskeauktion* were recorded in real time on a laptop computer, so an electronic record of each transaction was available. These records were given to us on magnetic media. From them we selected two species of fish, plaice and wolf-fish. The choice of species was made exclusively because they were sold frequently; we wanted a large sample of two commonly-consumed products. Of the 718 potential auction days during our sample period, plaice and wolf-fish, were both sold on 239 days. These two species of fish are consumed quite regularly by Danes. Having received the electronic files on which data were recorded, we selectively retrieved information relevant to our empirical work. We then organized these into ASCII files which were the inputs to the analysis described in the text of the paper.

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