FOUNDATIONS OF BAYESIAN THEORY

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January 9, 2003

Abstract

This paper presents a new axiomatic subjective expected utility model of Bayesian decision making under uncertainty with state-dependent preferences and moral hazard. The theory provides choice-theoretic foundations for the existance of prior probabilities representing decision makers' beliefs about the likely realization of events and for the updating of these probabilities according to Bayes' rule.

1 Introduction

Subjective expected utility theory is nowadays the standard economic model of individual decision making under uncertainty and the choice-theoretic foundation of the Bayesian statistics. Because of its fundamental importance, the model was subjected, over the years, to careful scrutiny as a result of which three unsatisfactory features were identified:

- The theory ascribes to decision makers probabilities that do not necessary represent their beliefs.
- The theory does not imply the updating of subjective probabilities by Bayes' rule.
- The theory requires that preferences be state independent and the probabilities be independent of the action taken which significantly limits the scope of its applications.

*I am grateful to Eddie Dekel, Jacques Dreze, Peter Fishburn, Simon Grant, Faruk Gul, George Mailath, Zvi Safra, David Schmeidler, and Peter Wakker and three anonymous referees for their valuable comments.

In this paper I present an alternative axiomatic subjective expected utility model that does not suffer from any of these drawbacks. In this theory the probabilities ascribed to decision makers represent their beliefs, the probabilities are necessarily updated using Bayes' rule, and the theory applies to decisions making situations in which preferences are state-dependent and the action taken affects the probabilities.

1.1 On the representation of beliefs by probabilities

The search for a choice-theoretic definition of subjective probabilities began with the pioneering work of Ramsey (1931) and de Finetti (1937) and attained its definitive formulation in the work of Savage (1954). Ultimately, however, this quest failed to achieve its goal: the representation, by a probability measure, of decision makers' beliefs regarding the likely realization of events. The definitions of subjective probabilities in these and later works invoke a convention that is neither part of nor implied by the underlying axioms, namely, that the utility functions are state independent. Whereas state-independent preferences are implied by the axiomatic structure, state-independent utility functions are not the only utility functions consistent with the axioms. In fact, there are infinitely many combinations of state-dependent utilities and arbitrary probability measures that are consistent with the axioms. Consequently, the curvature of the utility functions (and the ranking of "objective" lotteries, if such lotteries exist, as, for example, in Anscombe and Aumann [1963]) must be independent of the underlying states but the utility functions themselves may be positive linear transformations of each other.[1] In many situations this convention is untenable. For instance, it requires that the value attributed to possessing a fur coat be independent of the temperature. Moreover, if a decision maker's valuations of outcomes are not independent of the underlying states, the imposition of state-independent utility functions means that, even when the decision maker's beliefs (that is, a binary relation on the set of events that have the interpretation "more likely of being realized than,") are consistent with a representation by a probability measure, they may be inconsistent with the subjective probabilities ascribed to the decision maker by the theory.

Choices among alternative courses of action, or acts, reveal the decision maker's marginal rates of substitution of outcomes across states. These trade-offs confound subjective probabilities and marginal utilities and are too coarse to allow a meaningful separation of the two. Misconstrued separa-

[1] See the discussion in Dreze (1987); Schervish, Seidenfeldt, and Kadane (1990); Karni (1993), (1996), (2001); and Karni and Schmeidler (1993).

tion of probabilities and utilities may result in inconsistencies between verbal expression of preferences and observed choice behavior.[2] More importantly, however, as the following example shows, in the context of principal-agent problems if the principal ascribes to the agent incorrect subjective probabilities and utilities, he may fail to induce the agent to choose the optimal action.

EXAMPLE**:** A gambler (the principal) bets that a certain boxer will win a boxing match and then pays off the other boxer (the agent) to throw the match.[3] To render this scenario concrete, assume that the winner's take is a title and a prize of $\$2,000$ and the loser's take is \$1000. The two fighters, Abe and Ben, are equally matched and the two possible results of the match are, A, Abe wins and B, Ben wins. Assume that both the gambler and Abe are expected utility-maximizing Bayesian decision makers whose preferences over income are state independent (that is, they display the same attitude toward risk regardless of who wins the fight), that each of them believes that A and B are equally likely events, and that these beliefs are private information. The beliefs of both the gambler and Abe are represented by the uniform probability distribution $\pi(A) = \pi(B) = 1/2$, but the gambler does not know this and must infer Abe's probabilities from his observed choice behavior. Assume that the gambler is risk neutral and his utility function is state independent and that Abe is risk averse and that he cares about winning the title. Specifically let Abe's valuations of the payoff, w, be depicted by state-dependent utility functions $u_A(w) = 2\sqrt{w}$, and $u_B(w) = \sqrt{w}$. That is, winning the fight makes the payoff more worthwhile.

The gambler ascribes to Abe utilities and (prior) subjective probabilities implied by the subjective expected utility model. In other words, if the two fighters indeed "give all they've got," as far as the gambler can infer from Abe's choice behavior, the boxer's preferences are represented by:

$$p(A)\sqrt{w_A} + p(B)\sqrt{w_B}, \tag{1}$$

where $p(A) = 2\pi(A)/(2\pi(A) + \pi(B)) = 2/3$, $p(B) = \pi(B)/(2\pi(A) + \pi(B)) = 1/3$.[4]

[2] See example and discussion in Karni (1996).

[3] I use an example of a boxing match to make the story more "realistic". A reader concerned that a fight is a game involving two players rather than a game against nature, is welcome to substitute an archery contest (Dreze (1987)) for the fight. The main protagonists are the archer whose next shot will determines whether he wins or lose, and a gambler staking money on the outcome. Aiming amiss is equivalent to throwing the fight.

[4] In the present scenario if the payoff in the losing state is sufficiently large Abe will throw the fight. This is a violation of Savage' Sure Thing Principle. However, if the mon-

The outcome of the fight depends on the "effort" of the boxers. While Abe cannot be sure of winning the fight even if he tries, he can ensure that he loses. He may take one of two courses of action: "fight to win," f, with the possible results given by the event $\{A, B\}$ or "throw the fight," t. with the possible result described by the event $\{B\}$. Conditional on these actions, Abe thinks that the probabilities of his winning are $\pi(A \mid f) = 1/2$ and $\pi(A \mid t) = 0$. The gambler's perception of Abe's conditional probabilities of winning are $p(A \mid f) = 2/3$ and $p(A \mid t)) = 0$.

Suppose that the odds of each fighter winning the bout are even and that the gambler bets \$$x$ on Ben. Suppose that the gambler offers Abe a bribe, \$$b$, to throw the fight. The gambler's problem is then to choose the smallest payment $b \geq 0$ that satisfies the incentive compatibility constraint

$$p(B \mid t)\sqrt{b} \geq p(A \mid f)\sqrt{2000} + p(B \mid f)\sqrt{1000 + b}. \tag{2}$$

It is easy to verify that the gambler thinks that any amount of money $b > 1000$ is enough to persuade Abe to throw the fight and will offer him the smallest possible sum over \$1000. It is also clear that Abe is willing to take the money since, by taking it, he increases his expected utility whether or not he actually throws the fight (that is, the participation constraint is satisfied). Consider next what Abe does once he accepts the bribe. Abe's beliefs and valuations imply that throwing the fight yields $\sqrt{1000 + b}$ with certainty, whereas fighting for real entails an expected utility of

$$\pi(A \mid f)\, 2\sqrt{2000} + \pi(B \mid f)\sqrt{1000 + b}.$$

But $\pi(A \mid f) = \pi(B \mid f)$, hence, for all $b < 7000$,

$$\pi(A \mid f)\, 2\sqrt{2000} + \pi(B \mid f)\sqrt{1000 + b} > \sqrt{1000 + b}.$$

Abe will take the money and fight for real. *Because he ascribed to the agent subjective probabilities that do not represent the agent's beliefs, the principal designed a contract that induced the agent to chose an action that was not in the principal's best interest.* This problem is endemic and raises serious doubts about the validity and even the meaning of the common prior assumption that is often invoked in the analysis of principal-agent problems.

etary payoffs in the range $[1000, 4000)$ all the axioms underlying the subjective expected utility model are satisfied. In other words, the subjective expected utility model applies piecewise event though Abe is a subjective expected utility maximizer.

1.2 On Bayesian updating, state-dependent preferences, and moral hazard

From the point of view of Bayesian statistics, to which subjective expected utility theory is supposed to provide a choice-theoretic foundation of prior probabilities, the failure to obtain a correct representation of beliefs by probabilities is catastrophic. Moreover, whereas subjective expected utility theory is consistent with the updating of the subjective probabilities according to Bayes' rule it does not imply it. In other words subjective probabilities are not required to be reasonable in any sense except of internal consistency, and subjective expected utility theory does not entail any conclusion about the relation between decision makers' beliefs and empirical distributions representing the relative frequencies produced by repeated trials.[5] The notion that reasonable decision makers must agree on the probabilities of outcomes that, in repeated trials, produces stable, long-run frequency distribution is based on a leap of faith and is not implied by subjective expected utility theory.[6] In view of these observations it is natural to define subjective expected utility maximizing decision makers as *Bayesian* if, in addition to being subjective expected utility theory maximizers they also update of their prior beliefs according to Bayes' rule.

A last, well known, point criticism of subjective expected utility theory is its requirement that the preferences be state-independent and the event that obtain be independent of the action taken. This imposes sever limitations on its possible applications. For example, the theory is inappropriate for the analysis of the demand for health or life insurance as well as principal-agent relationships in the presence of moral-hazard problem. The last point is particularly disturbing since, as the boxing match example illustrates, a decision maker may be Bayesian and yet does not abide by the Sure Thing Principle.

[5] Ghirardato (2002) is one exception. A more detailed discussion of this point is provided in Kyburg (1968).

[6] The notion of conditional preferences on acts is well defined in subjective expected utility theory. These conditional preferences are sometimes interpreted as the updated preferences. However, this interpretation, appealing as it may sound, is not implied by the axioms. In other words, the axioms do not imply that if a decision maker receives information that makes him believe that a certain event is null, he must update his prior probabilities for the subevents in complemetary event equiproportionally.

1.3 Preferences on conditional acts

The failure of the choice-theoretic models to quantify decision makers' beliefs by a probability measure is due to the restrictive nature of preference relations defined solely on acts (that is, on the set of functions from the set of states of nature to the set of consequences). The extension of the choice set to include conditional acts allows the expression of preferences that makes it possible to separate utilities from probabilities in a more satisfactory manner. The idea of extending the choice set to include conditional acts is not new. (Preferences on conditional acts were studied in Pflanzagl [1968], Luce and Krantz [1971], Fishburn [1973], and in Drèze and Rustichini [1999]. I review these contributions in Section 5.2 and contrast them with the approach advanced here.) However, a few words on the meaning of preference relations on conditional acts are in order.

Conditional acts represent alternative courses of action when the decision maker knows that a particular event obtains. One way of interpreting conditional acts is to regard them as hypothetical entities and to treat preferences on them as thought experiments that may be deliberately invoked by decision makers when trying to clarify to themselves, or articulate to others, the reasoning underlying their actual choice behavior. Savage (1954) uses this interpretation to justify his celebrated sure thing principle. To motivate this principle, he gives the following example (italics are mine):

> A businessman contemplates buying a certain piece of property. He considers the outcome of the next presidential election relevant to the attractiveness of the purchase. So, *to clarify the matter to himself*, he asks whether he would buy it if he *knew* that the Republican candidate were going to win, and decides that he would do so. Similarly, he considers whether he would buy if he *knew* that the Democratic candidate would win, and again finds that he would do so. Seeing that he would buy in either event, he decides that he should buy. (Savage [1972] p. 21)

The businessman compares the act of buying conditional on the event that the Republican candidate wins and the act of not buying conditional on the same event. He then proceeds to compare the same two acts conditional on the complementary event (a slightly different interpretation is given in Grant, Kajii and Polak [2000]). According to this description, an act conditional on an event corresponds to a *subset of unconditional acts that agree on that event.* The comparison between two acts conditional on any given event does not require any restriction on the values the two acts assume outside of the conditioning event. If preferences among conditional acts are to be expressed in terms of unconditional acts, it requires the comparison of *subsets of unconditional acts* that agree with the conditional acts on the conditioning event. Yet, when Savage formalized this concept, he took a different approach which he expressed as follows: (italics and expressions in parentheses are mine)

> What technical interpretation can be attached to the idea that (the act) **f** would be preferred to (the act) **g**, if (the event) B were known to obtain? *Under any reasonable interpretation, the matter would seem not to depend on the values* **f** *and* **g** *assume at states outside of* B. There is, then no loss of generality in supposing that **f** and **g agree** with each other except in B. (Savage [1972] p. 22)

In what follows I take conditional acts to correspond to Savage's original description. Interpreting preferences among conditional acts as a thought experiment renders the axiomatic model presented in Sections 2 and 3 a normative theory.[7] Alternatively, conditional acts may be interpreted as conditional state-contingent payoffs implemented by actions of decision makers that restrict the set of states that might obtain. The control over events envisioned here is the theoretical counterpart of what in reality is one essential ingredient of the moral hazard problem (the other ingredient being the unobservability of the action taken by the decision maker). According to this interpretation, the axiomatic structure articulated in Sections 2 and 3 may be regarded as a positive theory of decision making under uncertainty with moral hazard and costless actions, whereas the model in Section 4 constitutes a positive theory of decision making under uncertainty with moral hazard and costly actions. Moreover, because it involves choice among alternative action-act pairs and, thus, entails "hypotheses about empirical data which could conceivably be refuted, if only under ideal conditions" (Samuelson [1947] p. 4) this approach renders the definition of probabilities compatible with the revealed preference methodology and hence constitutes a behavioral foundations of Bayesian theory.

Unlike some theories of state-dependent preferences with or without moral hazard, the Bayesian decision theory advanced in this paper does not invoke objective probabilities for its formulation. As in Savage's (1954)

[7] Further comments on the methodolgical issue raised by the use of conditional acts appear in section 5.?

theory, probabilities do not enter as undefined (primitive) ingredients of the model, appearing instead as a derived concept. Moreover, this theory does not rely on the use of or require the availability of all constant acts, thereby avoiding a problematic aspect of Savage's model.

A different issue concerning the uniqueness of subjective probabilities in the choice-theoretic approach has to do with the valuation of outcomes. Generally speaking, if a decision maker believes that some state may obtain but all the outcomes in this state are equally preferred his choice behavior is no different than if he believes it is virtually impossible that this state obtains. The choice-theoretic model assigns such a state probability zero. Hence the interpretation of the probabilities as representation of beliefs is based on the implicit and *unverifiable assumption* that in every state some outcomes are better than others. If this assumption is not warranted, the procedure may result in misrepresentation of beliefs. The theory advanced here circumvents this problem by permitting decision makers to express their preferences among acts conditional on singleton events, thereby allowing direct verification of whether or not they are indifferent among all outcomes in a give state (see Karni, Schmeidler and Vind [1983].)

In the next section I describe the analytical framework and derive some preliminary results. The main results are presented in Section 3. The implications for the theory of moral hazard are examined in Section 4. Further discussion and review of the relevant literature appears in Section 5. Proofs are provided in the Appendix.

2 Subjective Expected Utility Theory

2.1 The analytical framework

Let $S = \{1, ..., n\}$, $3 \leq n < \infty$, be a set of *states of nature* one and only one of which is the *true* state. Nonempty subsets of S are *events*. Let $\mathcal{E}'$ denote the set of all events. When the true state belongs to the event E, we say that E *obtains*. Uncertainty is the lack of knowledge regarding which state is the true state. For each $s \in S$, let X_s be a connected separable topological space whose elements are *outcomes* that are feasible in s. *Unconditional acts* are an $n-$tuples $\mathbf{x} = (x_1, ..., x_n)$, where $x_s \in X_s$, representing possible courses of action. The set of all unconditional acts is the product set $\mathbf{X} := X_1 \times ... \times X_n$. Note that the feasible sets of outcomes do not have to be the same across states. This is a significant departure from Savage's (1954) model in which the set of acts includes all the constant acts and, consequently, requires that the feasible outcomes be the same in every state. Let $\succsim$ be a preference relation on $\mathbf{X}$ and denote by $\approx$ the symmetric part of $\gtrapprox$.

For each s, denote by $(\mathbf{x}^{-s}, y)$ the act obtained from $\mathbf{x}$ by replacing its $s-th$ coordinate with $y \in X_s$. Given a preference relation $\succcurlyeq$ on $\mathbf{X}$, a state s is *null* if $(\mathbf{x}^{-s}, y) \approx (\mathbf{x}^{-s}, z)$ for all y and z in X_s; otherwise it is *nonnull*. Denote by $\mathbf{x}_E\mathbf{y}$ the act that coincides with $\mathbf{x}$ on E and with $\mathbf{y}$ on $S-E$ (that is, $(\mathbf{x}_E\mathbf{z})_s = x_s$ if $s \in E$ and $(\mathbf{x}_E\mathbf{z})_s = z_s$ if $s \in S-E$.) Then an event $E \in \mathcal{E}'$ is null if $\mathbf{x}_E\mathbf{y} \approx \mathbf{x}_E\mathbf{z}$ for all $\mathbf{y}, \mathbf{z} \in \mathbf{X}$. Denote by $\mathcal{E}$ the subset of $\mathcal{E}'$ that consists of all the nonnull events. Henceforth I assume that S contains at least three nonnull states.

Given $E \in \mathcal{E}$ a *conditional act,* $\mathbf{x}_E$, is the element of $\mathbf{X}_E := \prod_{s\in E} X_s$. For each $E \in \mathcal{E}$ assume that $\mathbf{X}_E$ is endowed with the product topology. Let $\mathbb{X} = \cup_{E\in\mathcal{E}}\mathbf{X}_E$ denote the set of all conditional acts and assume that it is endowed with the topology whose open sets are the unions of the open sets in the product spaces $\mathbf{X}_E$, $E \in \mathcal{E}$.

Decision makers are characterized by a preference relation $\succcurlyeq$ on $\mathbb{X}$ where $\succcurlyeq = \gtrapprox$ on $\mathbf{X}$ I assume throughout that $\succcurlyeq$ is a continuous weak order. Formally, $\succcurlyeq$ is a complete and transitive binary relation on $\mathbb{X}$ such that the sets $\{\mathbf{x}_E \in \mathbb{X} \mid \mathbf{x}_E \succcurlyeq \mathbf{y}_{E'}\}$ and $\{\mathbf{x}_E \in \mathbb{X} \mid \mathbf{y}_{E'} \succcurlyeq \mathbf{x}_E\}$ are closed for all $\mathbf{y}_{E'} \in \mathbb{X}$. The interpretation of the statement $\mathbf{x}_B \succcurlyeq \mathbf{y}_A$ requires some explanation. Taken literally it means that if the decision maker could choose between the course of action represented by the act $\mathbf{x}$ and the force the event B and the course of action represented by the act $\mathbf{y}$ and force the event A, he would either choose the first or be indifferent between the two. In view of the prevalence of moral hazard problems in economics, it is conceivable that the decision maker could take actions as a result of which certain nonnull events would obtain. In general these actions are costly. To introduce the main ideas in a way that will make them more transparent I assume provisionally that the actions are costless and postpone the development of a full fledge model of choice among actions-acts pairs in which distinct actions entail different costs to Section 4. Thus, for the moment, I suppress the actions and assume that the preference relation $\succcurlyeq$ represents choice behavior among conditional acts in the aforementioned sense. The strict preference relation $\succ$ and the indifference relation $\sim$ are defined as usual and have the usual interpretation.

An act $\mathbf{x}^* \in \mathbf{X}$ is an unconditional *constant-valuation act* if $\mathbf{x}^*_{\{s\}} \sim \mathbf{x}^*_{\{t\}}$ for all nonnull $s, t \in S$.[8] (Similarly, $\mathbf{x}^*_E \in \mathbf{X}_E$ is a conditional constant valu-

[8] The idea of constant valuation acts is similar to Drèze's (1987) notion of "omnipotent" acts. In its present form it was used Karni (1993a). A similar concept appears in Skiadas (1997).

ation act if $\mathbf{x}^*_{\{s\}} \sim \mathbf{x}^*_{\{t\}}$ for all $s, t \in E$.) Note that the relation $\mathbf{x}^*_{\{s\}} \sim \mathbf{x}^*_{\{t\}}$ means that, faced with the choice between the outcome-state pairs (x^*_s, s) and (x^*_t, t) the decision maker is indifferent between them. Constant valuation acts are analogous to constant acts in Savage (1954). However, unlike Savage, who assumes implicitly that constant acts are constant-valuation acts, the present analysis recognizes that the same outcome may be assigned distinct values in different states. Moreover, whereas Savage requires the existence of all constant acts, I require only the existence of some constant-valuation acts. I assume that there exist constant valuation acts $\overline{\mathbf{x}}$ and $\underline{\mathbf{x}}$ such $\overline{\mathbf{x}} \succ \underline{\mathbf{x}}$.

A decision maker's *beliefs* are represented by a binary relation, $\trianglerighteq$, on $\mathcal{E}$ that has the following interpretation: For all $T, Q \subset S$, $T \trianglerighteq Q$ means that the decision maker considers the event T as at least as likely to obtain as the event Q. Following Ramsey (1931), it is now commonplace to infer a decision maker's beliefs from his willingness to bet on different events. However, considering the fact that the outcome valuations may be state dependent, care must taken in defining bets. Let $\trianglerighteq$ be defined as follows: For all constant valuation acts, $\mathbf{x}^{**}, \mathbf{x}^* \in \mathbf{X}$, satisfying $\mathbf{x}^{**} \succ \mathbf{x}^*$, and for all $T, Q \in \mathcal{E}$, $T \trianglerighteq Q$ if $\mathbf{x}^{**}_T \mathbf{x}^* \succcurlyeq \mathbf{x}^{**}_Q \mathbf{x}^*$.

2.2 Preferences on conditional acts and their representation

For any given event, E, assume that the preference between any two acts conditional on E is independent of outcomes in states to which the two acts assign the same outcomes. This assumption is analogous to Savage's (1954) Sure Thing Principle (see Wakker [1989] for more details). Formally,

(A.1) Conditional Coordinate Independence - *For all* $\mathbf{x}, \mathbf{y} \in \mathbf{X}$*,* $E \in \mathcal{E}$*,* $s \in S$ *and* $w, z \in X_s$*,* $(\mathbf{x}^{-s}, z)_E \succcurlyeq (\mathbf{y}^{-s}, z)_E$ *if and only if* $(\mathbf{x}^{-s}, w)_E \succcurlyeq (\mathbf{y}^{-s}, w)_E$.

The second axiom links the preferences on distinct conditional acts. It asserts that, for any conditional constant valuation act $\mathbf{x}^*_E$, all the conditional valuation acts $\mathbf{x}^*_G$ such that $G \subset E$ are equally preferred. In particular, $\mathbf{x}^*_E \sim \mathbf{x}^*$ for all $E \in \mathcal{E}$. Formally,

(A.2) Consequentialism - *If* $\mathbf{x}^*_E$ *is a conditional constant valuation act then* $\mathbf{x}^*_E \sim \mathbf{x}^*_G$ *for all nonnull* $G \subset E$.

The logic underlying (A.2) is that ultimately the outcome of a decision is a state-outcome pair (s, x^*_s). With constant valuation acts the ultimate outcome is of the same valued regardless which event obtains. Indifference among all conditional constant valuation acts is form of consequentialism. It means that the decision maker is solely concerned with the ultimate outcome.

A function $F : \mathbb{X} \to \mathbb{R}$ is said to be *additive valued* if there exist functions $f_E(\cdot;t) : X_t \to \mathbb{R}$, for all $E \in \mathcal{E}$ and $t \in E$, such that $F(\mathbf{x}_E) = \sum_{s \in E} f_E(x_s; s)$. The functions $f_E(\cdot; s)$ are called additive-valued functions. The following result gives necessary and sufficient conditions for the existence of an additive-valued representation of $\succcurlyeq$ and establishes its uniqueness.

Theorem 1 *Suppose that there are at least three nonnull states. Then the following conditions are equivalent:*

(i) *The relation $\succcurlyeq$ is a continuous weak-order on $\mathbb{X}$ satisfying (A.1) and (A.2).*

(ii) *There exist continuous functions $\{w_E(\cdot; s) : X_s \to \mathbb{R} \mid E \in \mathcal{E}, s \in E\}$ such that for all $\mathbf{x}_E, \mathbf{y}_A \in \mathbb{X}$,*

$$\mathbf{x}_E \succcurlyeq \mathbf{y}_A \Leftrightarrow \sum_{s \in E} w_E(x_s; s) \geq \sum_{s \in A} w_A(y_s; s).$$

and, for every conditional constant valuation act $x_E^* \in \mathbf{X}$ and every nonnull event $G \subset E$,

$$\sum_{s \in G} w_G(x_s^*; s) = \sum_{s \in E} w_E(x_s^*; s).$$

Moreover, if $\{\hat{w}_E(\cdot; s) \mid E \in \mathcal{E},\ s \in E\}$ is another set of continuous functions that represent $\succcurlyeq$ in the sense of (ii), then, for all $E \in \mathcal{E}$ and $s \in E$, $\hat{w}_E(\cdot; s) = \beta w_E(\cdot; s) + \gamma_E(s)$, where $\beta > 0$ and $\sum_{s \in E} \gamma_E(s) = C$. These functions are constant if and only if s is null.

The proof of Theorem 1 is given in the Appendix.

Remark 1: If there are only two nonnull states, Theorem 1 holds if $\succcurlyeq$ satisfies the following hexagon condition (see Wakker [1989, Ch. III]):

Hexagon condition - *Let s and r be the only two nonnull states. Then, for all x_s, y_s, z_s in X_s, x_r, y_r, z_r in X_r,* and $\mathbf{x} \in \mathbf{X}$, *if $((\mathbf{x}^{-s}, x_s)^{-r}, y_r) \sim ((\mathbf{x}^{-s}, y_s)^{-r}, x_r)$ and $((\mathbf{x}^{-s}, z_s)^{-r}, x_r) \sim ((\mathbf{x}^{-s}, y_s)^{-r}, y_r) \sim ((\mathbf{x}^{-s}, x_s)^{-r}, z_r)$ then $((\mathbf{x}^{-s}, y_s)^{-r}, z_r) \sim ((\mathbf{x}^{-s}, z_s)^{-r}, y_r)$.*

3 Subjective Expected Utility Theory

3.1 Coherence

Bayesian decision makers update their beliefs regarding the likely realization of events independently of their valuation of outcomes in different states. To capture this idea, I assume that the preference relation $\succcurlyeq$ reflects the same valuation of state-outcome pairs regardless of the conditioning event. Formally, let $\mathcal{E}_1$ the collection of all nonsingleton events in $\mathcal{E}$, then

(A.3) Coherence - *For all* $\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$*, nonnull* $E \in \mathcal{E}_1$*,* $s \in E$*, and* $a_s, b_s, c_s, d_s \in X_s$*, if* $(\mathbf{x}^{-s}, a_s) \succcurlyeq (\mathbf{y}^{-s}, b_s)$*,* $(\mathbf{y}^{-s}, c_s) \succcurlyeq (\mathbf{x}^{-s}, d_s)$*, and* $(\mathbf{z}^{-s}, b_s)_E \succcurlyeq (\mathbf{w}^{-s}, a_s)_E$ *then* $(\mathbf{z}^{-s}, c_s)_E \succcurlyeq (\mathbf{w}^{-s}, d_s)_E$.

Axiom (A.3) is an adaptation of Wakker's (1987) cardinal consistency axiom. (Wakker [1989] discusses the earlier literature on the idea underlying cardinal consistency.) To grasp the meaning of this axiom, think of the preferences $(\mathbf{x}^{-s}, a_s) \succcurlyeq (\mathbf{y}^{-s}, b_s)$ and $(\mathbf{y}^{-s}, c_s) \succcurlyeq (\mathbf{x}^{-s}, d_s)$ as indicating that the "intensity" of the prior preference for c_s over d_s is sufficiently greater than that of a_s over b_s as to reverse the order of preference between the other coordinates of $\mathbf{x}$ and $\mathbf{y}$. Coherence requires that these intensities are not contradicted by the conditioning of the acts on nonnull events.

If axiom (A.3) is added to part (i) of Theorem 1 then the functions $\{w_E(\cdot; s) \mid s \in E, E \in \mathcal{E}\}$ are positive affine or constant transformations of one another (see Lemma 8 in the Appendix). Consequently, if a state is null it must remain so when included in the conditioning events (that is, for every $E \in \mathcal{E}$, if s is a null state and $s \in E$ then $(\mathbf{x}^{-s}, y)_E \sim (\mathbf{x}^{-s}, z)_E$ for all $y, z \in X_s$.) The next theorem shows that the subjective prior probability of null states is zero. Hence the implication is that the conditional probability of such states cannot become positive in view of new information.

3.2 Subjective expected utility representation of state-dependent preferences

The next theorem establishes the main result: that there exists a unique subjective probability distribution on the set of states representing the decision maker's prior beliefs, unique posterior probabilities obtained from the given prior by Bayes' rule, state-dependent real-valued utility functions on the respective sets of outcomes representing the decision maker's valuations, and subjective expected utility representations of his conditional and unconditional preferences. Implicit in this result is the notion that the decision maker may choose among act-event pairs, but once the event is chosen, the probabilities assigned to states belonging to the chosen event must increase proportionally.

Theorem 2 *Suppose that there are at least three nonnull states. Then:*

a. *The following two conditions are equivalent:*

(i) *The relation $\succcurlyeq$ is a continuous weak-order on $\mathbb{X}$ satisfying (A.1), (A.2) and (A.3).*

(ii) *There exists a probability measure π on S and an array of continuous functions $\{u_s : X_s \to \mathbb{R}\}_{s\in S}$ such that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$,*

$$\mathbf{x} \succcurlyeq \mathbf{y} \quad \Leftrightarrow \quad \sum_{s\in S} \pi(s)\, u_s(x_s) \;\; \geq \sum_{s\in S} \pi(s)\, u_s(y_s),$$

for every conditional constant valuation act, $\mathbf{x}_E^*$, $u_s(x_s^*) = u_t(x_t^*)$ for all $s, t \in E$, and, for all $\mathbf{x}_E, \mathbf{y}_A \in \mathbb{X}$,

$$\mathbf{x}_E \succcurlyeq \mathbf{y}_A \Leftrightarrow \sum_{s\in E} \pi(s \mid E)\, u_s(x_s) \geq \sum_{s\in A} \pi(s \mid A)\, u_s(y_s),$$

where, for all $B \in \mathcal{E}$, $\pi(s \mid B) = \pi(s) / \sum_{t\in B} \pi(t)$ is the probability of state s conditional on the event B.

b. *The utility functions $\{u_s\}_{s\in S}$ are cardinally measurable fully-comparable. (That is, if $\{\hat{u}_s\}_{s\in S}$ is another array of functions representing $\succcurlyeq$ in the sense of (ii) then $\hat{u}_s = \beta u_s + \alpha$, $\beta > 0$, for all $s \in S$.)*

c. *π is unique and $\pi(s) = 0$ if and only if s is null.*

The proof of Theorem 2 is given in the Appendix.

3.3 Subjective expected utility representation of state-independent preferences

State-independent preferences are a special case of the theory of the preceding sections. To study this case, assume, without essential loss of generality, that the same outcomes are feasible in all states (i.e., $X_1 = ... = X_n = X$). To help keep this in mind I denote $\mathbf{X}$ by X^n. Intuitively speaking, state-independent preferences requires that the "intensity" of the preferences be the same across states. To formalize this idea I invoke the condition of cardinal coordinate independence of Wakker (1989, Ch. IV).

(A.4) Cardinal Coordinate Independence - *For all* $\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w} \in X^n$, *nonnull* $s,t \in S$, *and* $a,b,c,d \in X$, *if* $(\mathbf{y}^{-s},b) \succcurlyeq (\mathbf{x}^{-s},a)$, $(\mathbf{x}^{-s},c) \succcurlyeq (\mathbf{y}^{-s},d)$ *and* $(\mathbf{z}^{-t},a) \succcurlyeq (\mathbf{w}^{-t},b)$ *then* $(\mathbf{z}^{-t},c) \succcurlyeq (\mathbf{w}^{-t},d)$.

The interpretation of cardinal coordinate independence is analogous to that of coherence. The relations $(\mathbf{y}^{-s},b) \succcurlyeq (\mathbf{x}^{-s},a)$ and $(\mathbf{x}^{-s},c) \succcurlyeq (\mathbf{y}^{-s},d)$ indicate that the "intensity" of the preference for c over d in state s is sufficiently greater than that of b over a as to reverse the order of preference between the other coordinates of $\mathbf{x}$ and $\mathbf{y}$. State independence requires that these intensities are not contradicted by the preferences between the same outcomes in any other state t.

The next lemma gives necessary and sufficient conditions for the state-dependent utility functions to be affine transformations of one another.

Lemma 3 *Let* $\succcurlyeq$ *be a continuous weak order on* X^n. *Then the following conditions are equivalent:*

(i) $\succcurlyeq$ *satisfies (A.4).*

(ii) *There exist* $u : X \to \mathbb{R}$ *and positive affine or constant functions* $\varphi_s : u(X) \to \mathbb{R}$ *for all* $s \in S$ *such that, for all* $\mathbf{x},\mathbf{y} \in X^{\cdot n}$,

$$\mathbf{x} \succcurlyeq \mathbf{y} \quad \Leftrightarrow \quad \sum_{s=1}^{n} \varphi_s \circ u(x_s) \geq \sum_{s=1}^{n} \varphi_s \circ u(y_s).$$

The proof of Lemma 3 follows immediately from Wakker's (1989) Theorem IV.2.7 and is omitted.[9]

In general, even if the preference relation has an expected utility representation, *state-independence preferences does not imply state-independent utility functions.* However, if the utility functions are not the same across states then, by Lemma 3, they must be positive affine transformations of one another (i.e., for all $s \in S$ and $x \in X_s$, $u_s(x) := \sigma_s u(x) + \xi_s$, where $\sigma_s > 0$). In other words, the dependence of the evaluation of an outcome on the underlying states is quantifiable by the multiplicative coefficients σ_s and the additive constants ξ_s. Note that if φ_s is a constant function, then s is null. The next theorem captures this fact and is analogous to Theorem 2.

[9] If the assumption $X_s = X_t$ does not hold, then the utility functions of nonnull states are positive affine transformations of one another over the outcomes that are in the intersection of the sets of feasible outcomes.

Theorem 4 *Suppose that there are at least three nonnull states. Then:*

a. *The following conditions are equivalent:*

(i) $\succcurlyeq$ *is a continuous weak-order on* $\mathbb{X}$ *satisfying (A.2), (A.3), and (A.4).*

(ii) There exists a probability measure π *on* S*, a continuous nonconstant function* $u : X \to \mathbb{R}$*, and for all* $s \in S$ *, there are numbers* $\sigma_s > 0$ *and* ξ_s *such that, for all* $\mathbf{x}, \mathbf{y} \in X^n$*,*

$$\mathbf{x} \succcurlyeq \mathbf{y} \quad \Leftrightarrow \quad \sum_{s \in S} \pi(s) \, \sigma_s u(x_s) \geq \sum_{s \in S} \pi(s) \, \sigma_s u(y_s),$$

for every conditional constant valuation act, $\mathbf{x}_E^*$*,* $\sigma_s u(x_s^*) + \xi_s = \sigma_t u(x_t^*) + \xi_t$*, for all* $s, t \in E$*, and, for every* $\mathbf{x}_E, \mathbf{y}_A \in \mathbb{X}$*,*

$$\mathbf{x}_E \succcurlyeq \mathbf{y}_A \Leftrightarrow \sum_{s \in E} \pi(s \mid E) \left[\sigma_s u(x_s) + \xi_s\right] \geq \sum_{s \in A} \pi(s \mid A) \left[\sigma_s u(y_s) + \xi_s\right],$$

where, for all $B \in \mathcal{E}$*,* $\pi(s \mid B) = \pi(s) / \sum_{t \in B} \pi(t)$ *is the probability of the state* s *conditional on the event* B*.*

b. *The triplet* (u, σ_s, ξ_s) *is unique. (That is, if* (v, ζ_s, τ_s) *represent the preference relation as in (ii) then* $v = \beta u + \alpha$ *and, for all* $s \in S$*,* $\zeta_s = \sigma_s / \beta$*, and* $\tau_s = \xi_s - \alpha \zeta_s$*.)*

c. π *is unique and* $\pi(s) = 0$ *if and only if* s *is null.*

The proof of Theorem 4 is similar to that of Theorem 2 and is outlined in the Appendix.

The definitions of subjective probabilities in Theorems 2 and 4 represent the decision makers' prior beliefs. Letting the probability of an event E be given by $\pi(E) = \sum_{s \in E} \pi(s)$, these definitions imply that for all $T, Q \subset S$,

$$T \trianglerighteq Q \quad \Leftrightarrow \quad \pi(T) \geq \pi(Q).$$

Moreover, for every given event $E \in \mathcal{E}$ the posterior beliefs, $\trianglerighteq_E$, are represented by the conditional probabilities $\pi(\cdot \mid E)$. These are the only representations of the prior and posterior beliefs of Bayesian decision makers by probabilities.

4 Subjective Expected Utility Theory with Moral Hazard

4.1 The analytical framework

The analysis of schemes designed to mitigate the welfare loss associated with moral hazard have been a focal issue in economic theory in the past 30 years. Less attention has been devoted to examining the analytical underpinning of the theory, which still lacks satisfactory foundations. In this section, I build on the model of the preceding sections to address this issue. In so doing I also develop a choice-theoretic model that encompasses the preceding analysis and leads to a choice-theoretic definition of subjective probabilities that represent decision makers' beliefs. The approach taken here is based on the idea that the moral hazard problem arises when decision makers can to affect, by unobservable actions, the event that obtains.

Let $\mathbf{A}$ be a topological space of feasible *actions* and let F be a mapping of $\mathbf{A}$ onto $\mathcal{E}$. (In general the set of actions is abstract. In specific situations it may be more structured. For example, if actions correspond to levels of effort then $\mathbf{A}$ may be a compact interval in the real line and the topology the metric topology. The same interpretation applies if actions correspond to monetary expenditure.) Define an induced mapping, $\hat{F} : \mathbf{A} \times \mathbf{X} \to \mathbf{A} \times \mathbb{X}$ by $\hat{F}(a; \mathbf{x}) = \left(a; \mathbf{x}_{F(a)}\right)$. A *default action* is an action, a^0, such that $F\left(a^0\right) = S$ and, consequently, $\hat{F}(a^0; \mathbf{x}) = \left(a^0; \mathbf{x}\right)$. Let $\mathbf{A}^0$ be a nonempty set of default actions and assumed that $\mathbf{A}^0 \subset \mathbf{A}$. Using the default actions define null states the set $\mathbb{X}$ as in Section 2.1. Assume further that $\mathbf{A} \times \mathbb{X}$ is endowed with the product topology.

Decision makers are characterized by preference relations, $\hat{\succcurlyeq}$, on $\mathbb{A} := \left\{\left(a; \mathbf{x}_{F(a)}\right) \mid a \in \mathbf{A}, \mathbf{x}_{F(a)} \in \mathbb{X}\right\}$. The interpretation of $\left(a; \mathbf{x}_{F(a)}\right) \hat{\succcurlyeq} \left(b; \mathbf{y}_{F(b)}\right)$ is that the decision maker is better off with the alternative $\left(a; \mathbf{x}_{F(a)}\right)$ that involves taking the action a when facing the payoff depicted by the act $\mathbf{x}$ than with the alternative $\left(b; \mathbf{y}_{F(b)}\right)$ involving taking the action b when the payoff is depicted by the act $\mathbf{y}$. I assume throughout that $\hat{\succcurlyeq}$ is a continuous weak order and denote by $\hat{\succ}$ and $\hat{\sim}$ the asymmetric and the symmetric parts of $\hat{\succcurlyeq}$, respectively.

If, at a given $\mathbf{x}$, imposing a sufficiently severe penalty on the event $S - F(a)$ induces a decision maker to take an action, a, to make sure that $F(a)$ obtains the action a is implementable. Formally, an action a is *implementable at* $\left(a^0, \mathbf{x}\right) \in \mathbf{A}^0 \times \mathbf{X}$ if there exist $\mathbf{z}\left(a; \left(a^0, \mathbf{x}\right)\right) \in \mathbf{X}$ such that $\left(a^0, \mathbf{x}_{F(a)} \mathbf{z}\left(a; \left(a^0, \mathbf{x}\right)\right)\right) \hat{\sim} \left(a, \mathbf{x}_{F(a)}\right)$.

4.2 Axioms and preliminary results

The preference relation $\hat{\succcurlyeq}$ is assumed to satisfy conditional coordinate independence, consequentialism, and coherence properly modified to accommodate the extended framework. The first axiom is analogous to conditional coordinate independence and, like it, requires that the preference between any two acts conditional on the decision maker taking the action a is independent of outcomes in states to which the two acts assign the same outcomes. In addition, it requires that the ranking of all the acts conditional on a given event be independent of the action that induces that event.

(A.1') *For all* $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $a \in \mathbf{A}$, $s \in S$, *and* $w, z \in X_s$, $\left(a; (\mathbf{x}^{-s}, z)_{F(a)}\right) \hat{\succcurlyeq} \left(a; (\mathbf{y}^{-s}, z)_{F(a)}\right)$ *if and only if* $\left(a; (\mathbf{x}^{-s}, w)_{F(a)}\right) \hat{\succcurlyeq} \left(a; (\mathbf{y}^{-s}, w)_{F(a)}\right)$, *and for all* $E \in \mathcal{E}$, $a, b \in F^{-1}(E)$, $(a, \mathbf{x}_E) \hat{\succcurlyeq} (a, \mathbf{y}_E)$ *if and only if* $(b, \mathbf{x}_E) \hat{\succcurlyeq} (b, \mathbf{y}_E)$.

The fact that actions must be taken in order to induce a conditioning event implies that direct comparisons of state-outcome pairs is no longer possible and, consequently, constant valuation acts may no longer be observable. Constant valuation acts served a dual purpose in the preceding analysis: they provided a link among preferences on conditional acts and the mean to identify the decision maker's utility function. To link together the preference on conditional acts I assume that there exist a default action $a^0 \in \mathbf{A}^0$ and an act $\hat{\mathbf{x}} \in \mathbf{X}$ such that every action in $\mathbf{A}$ is implementable at $(a^0, \hat{\mathbf{x}})$.

The following result is analogous to Theorem 1.

Theorem 5 *If there are at least three nonnull states then the following conditions are equivalent:*

(i) $\hat{\succcurlyeq}$ *is a continuous weak-order on* $\mathbb{A}$ *satisfying (A.1') and there exist* $(a^0, \hat{\mathbf{x}}) \in \mathbf{A}^0 \times \mathbf{X}$ *such that all actions are implementable at* $(a^0, \hat{\mathbf{x}})$.

(ii) There exist continuous functions $\{\hat{w}_{F(a)}(\cdot; s) : X_s \to \mathbb{R} \mid a \in \mathbf{A}, s \in F(a)\}$ *and a function* $v : \mathbf{A} \to \mathbb{R}$ *such that, for all* $\left(a, \mathbf{x}_{F(a)}\right), \left(b, \mathbf{y}_{F(b)}\right) \in \mathbb{A}$, $\left(a, \mathbf{x}_{F(a)}\right) \hat{\succcurlyeq} \left(b, \mathbf{y}_{F(b)}\right)$ *if and only if*

$$\sum_{s \in F(a)} \hat{w}_{F(a)}(x_s; s) + v(a) \geq \sum_{s \in F(b)} \hat{w}_{F(b)}(y_s; s) + v(b) \tag{3}$$

and, for all $a \in \mathbf{A}$,

$$\hat{w}_{F(a)}(\hat{x}_s; s) = 0, \;\; v(a) = -\sum_{s \in S - F(a)} \hat{w}_S\left(z\left(a^0, \hat{\mathbf{x}}\right)_s; s\right). \tag{4}$$

Moreover, if $\{\hat{w}'_{F(a)}(\cdot;s) : X_s \to \mathbb{R} \mid a \in \mathbf{A}, s \in F(a)\}$ *and* $\hat{v}$ *is another set of continuous functions that represent* $\hat{\succcurlyeq}$ *in the sense of equation (3) then, for all* $E \in \mathcal{E}$ *and* $s \in E$, $\hat{w}'_{F(a)}(\cdot;s) = \beta\hat{w}_{F(a)}(\cdot;s) + \gamma_{F(a)}(s)$, *and* $\hat{v} = \beta v + \gamma$, *where* $\beta > 0$ *and* $\sum_{s\in F(a)} \gamma_{F(a)}(s) = C$. *These functions are constant if and only if* s *is null.*

The proof is given in the Appendix.

Because constant valuation acts may no longer be used to define the utilities, I introduce the concept of compensating-variations payoff profiles. Formally, denote by a_s an action whose image under F is $\{s\}$. Fix $\mathbf{a} = (a_1, ..., a_n)$ than an act $\mathbf{x}^*$ is a *compensating-variations payoff profile* (CVPP) if $\left(a_s, \mathbf{x}^*_{\{s\}}\right) \hat{\sim} \left(a_t, \mathbf{x}^*_{\{t\}}\right)$. Compensating-variations payoff profiles have the following interpretation: The decision maker's valuation of the difference in the impact on the state-outcome pairs $(s, \mathbf{x}^*_s)$ and $(t, \mathbf{x}^*_t)$ on his well-being is equal to the difference in the direct effect on his well-being of the corresponding actions a_s and a_t that yield them. Given $\mathbf{a}$ as above, let $\mathbf{X}(\mathbf{a}) = \{\mathbf{x} \in \mathbf{X} \mid (a_s, \mathbf{x}_{\{s\}}) \hat{\sim} (a_t, \mathbf{x}_{\{t\}})\}$ be the set of CVPPs generated by $\mathbf{a}$. Assume that the set of actions is sufficiently rich so that given $\mathbf{x}^* \in \mathbf{X}(\mathbf{a})$ and any $E, E' \in \mathcal{E}$ there exist $a \in F^{-1}(E)$ and $b \in F^{-1}(E')$ such that $\left(a, \mathbf{x}^*_{F(a)}\right) \hat{\sim} \left(b, \mathbf{x}^*_{F(b)}\right)$. Analogous to (A.2) I assume:

(A.2') *For every given* $\mathbf{a} = (a_1, ..., a_n) \in \mathbf{A}^n$, *for all compensating-variations payoff profiles* $\mathbf{x}^*, \mathbf{x}^{**} \in \mathbf{X}(\mathbf{a})$ *and actions* $a, b \in \mathbf{A}$, $\left(a, \mathbf{x}^*_{F(a)}\right) \hat{\sim} \left(b, \mathbf{x}^*_{F(b)}\right)$ *if and only if* $\left(a, \mathbf{x}^{**}_{F(a)}\right) \hat{\sim} \left(b, \mathbf{x}^{**}_{F(b)}\right)$.

The next axiom is analogous to coherence (A.3) and has a similar interpretation.

(A.3') *For all* $\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$, $a \in \mathbf{A}$, $s \in F(a)$, *and* $a_s, b_s, c_s, d_s \in X_s$, *if* $\left(a^0, (\mathbf{x}^{-s}, a_s)\right) \hat{\succcurlyeq} \left(a^0, (\mathbf{y}^{-s}, b_s)\right)$, $\left(a^0, (\mathbf{y}^{-s}, c_s)\right) \hat{\succcurlyeq} \left(a^0, (\mathbf{x}^{-s}, d_s)\right)$, *and* $\left(a, (\mathbf{z}^{-s}, b_s)_{F(a)}\right) \hat{\succcurlyeq} \left(a, (\mathbf{w}^{-s}, a_s)_{F(a)}\right)$ *then* $\left(a, (\mathbf{z}^{-s}, c_s)_{F(a)}\right) \hat{\succcurlyeq} \left(a, (\mathbf{w}^{-s}, d_s)_{F(a)}\right)$.

The meaning of (A.3') is easy to grasp if the preferences $\left(a^0, (\mathbf{x}^{-s}, a_s)\right) \hat{\succcurlyeq} \left(a^0, (\mathbf{y}^{-s}, b_s)\right)$ and $\left(a^0, (\mathbf{y}^{-s}, c_s)\right) \hat{\succcurlyeq} \left(a^0, (\mathbf{x}^{-s}, d_s)\right)$ are taken to indicate that, given that act of omission, a^0, the "intensity" of the preference for c_s over d_s is sufficiently greater than that of a_s over b_s as to reverse the order of preference between the other coordinates of $\mathbf{x}$ and $\mathbf{y}$. Axiom (A.3') requires that these intensities not be contradicted when some other action is taken that restricts the event that obtains. The implication of this assumption is that intensity of preferences between pairs of outcomes in any given state is unaffected by the action taken provided that the state may still be true following that action.

4.3 Subjective expected utility with moral hazard and state-dependent preferences

The next theorem extends the main result, Theorem 2, to the case of decision making under uncertainty with moral hazard and state-dependent preferences.

Theorem 6 *Suppose that there are at least three nonnull states. Then:*

a. *The following two conditions are equivalent:*

(i) The relation $\hat{\succcurlyeq}$ *is a continuous weak-order on* $\mathbb{A}$ *satisfying (A.1'), (A.2'), and (A.3') and there exist* $\left(a^0, \hat{\mathbf{x}}\right) \in \mathbf{A}^0 \times \mathbf{X}$ *such that all actions are implementable at* $\left(a^0, \hat{\mathbf{x}}\right)$.

(ii) There exist probability measure π *on* S, *continuous functions* $u_s : X_s \to \mathbb{R}$, $s \in S$, *and a function* $v : \mathbf{A} \to \mathbb{R}$ *such that, for all* $\mathbf{x}, \mathbf{y} \in \mathbf{X}$,

$$\left(a^0, \mathbf{x}\right) \hat{\succcurlyeq} \left(a^0, \mathbf{y}\right) \quad \Leftrightarrow \quad \sum_{s \in S} \pi(s)\, u_s(x_s) \geq \sum_{s \in S} \pi(s)\, u_s(y_s).$$

For every $\left(a, \mathbf{x}_{F(a)}\right), \left(b, \mathbf{y}_{F(b)}\right) \in \mathbb{A}$, $\left(a, \mathbf{x}_{F(a)}\right) \hat{\succcurlyeq} \left(b, \mathbf{y}_{F(b)}\right)$ *if and only if*

$$\sum_{s \in F(a)} \pi(s \mid F(a))\, u_s(x_s) + v(a) \geq \sum_{s \in F(b)} \pi(s \mid F(b))\, u_s(y_s) + v(b),$$

where $\pi(s \mid F(a)) = \pi(s) / \sum_{t \in F(a)} \pi(t)$ *is the probability of state* s *conditional on the event* $F(a)$.

b. *The utility functions* $\{u_s\}_{s \in S}$ *and* v *are cardinally measurable fully-comparable.*

c. π *is unique and* $\pi(s) = 0$ *if and only if* s *is null.*

An outline of the proof of Theorem 6 is given in the Appendix.

4.4 Subjective expected utility with moral hazard and state-independent preferences

The case of state-independent preferences may be treated similarly. Specifically, the axiom of cardinal coordinate independence may be restated as follows:

(A.4') *For all* $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in X^n$*, nonnull* $s, t \in S$*, and* $\alpha, \beta, \gamma, \delta \in X$*, if* $\left(a^0, (\mathbf{y}^{-s}, \beta)\right) \hat{\succsim} \left(a^0, (\mathbf{x}^{-s}, \alpha)\right)$, $\left(a^0, (\mathbf{x}^{-s}, \gamma)\right) \hat{\succsim} \left(a^0, (\mathbf{y}^{-s}, \delta)\right)$ *and* $\left(a^0, (\mathbf{z}^{-t}, \alpha)\right) \hat{\succsim} \left(a^0, (\mathbf{w}^{-t}, \beta)\right)$ *then* $\left(a^0, (\mathbf{z}^{-t}, \gamma)\right) \hat{\succsim} \left(a^0, (\mathbf{w}^{-t}, \delta)\right)$.

Analogous to Theorem 4, the next theorem provides necessary and sufficient conditions for the existence of subjective expected utility representation of state-independent preferences with moral hazard:

Theorem 7 *Suppose that there are at least three nonnull states and that all actions are implementable. Then:*

a. *The following conditions are equivalent:*

(i) $\hat{\succsim}$ *is a continuous weak-order on* $\{(a, \mathbf{x}) \mid a \in A, \mathbf{x} \in X^n\}$ *satisfying (A.2'), (A.3') and (A.4').*

(ii) There exists a probability measure π *on* S*, a continuous, nonconstant, functions* $u : X \to \mathbb{R}$ *and* $v : \mathbf{A} \to \mathbb{R}$ *and numbers* $\sigma_s > 0$, $\xi_s, s \in S$ *such that , for all* $\mathbf{x}, \mathbf{y} \in X^n$,

$$\left(a^0, \mathbf{x}\right) \hat{\succsim} \left(a^0, \mathbf{y}\right) \quad \Leftrightarrow \quad \sum_{s \in S} \pi(s)\, \sigma_s u(x_s) \geq \sum_{s \in S} \pi(s)\, \sigma_s u(y_s).$$

For all $\left(a, \mathbf{x}_{F(a)}\right), \left(b, \mathbf{y}_{F(b)}\right) \in \mathbb{A}$, $\left(a, \mathbf{x}_{F(a)}\right) \hat{\succsim} \left(b, \mathbf{y}_{F(b)}\right)$ *if and only if*

$$\sum_{s \in F(a)} \pi(s \mid F(a)) \left[\sigma_s u(x_s) + \xi_s\right] + v(a) \geq \sum_{s \in F(b)} \pi(s \mid F(b)) \left[\sigma_s u(y_s) + \xi_s\right] + v(b),$$

where $\pi(s \mid F(a)) = \pi(s) / \sum_{t \in F(a)} \pi(t)$ *is the probability of state* s *conditional on the event* $F(a)$.

b. *The triplet* (u, σ_s, ξ_s) *is unique. (That is, if* (u', ζ_s, τ_s) *represent the preference relation as in (ii) then* $u' = \beta u + \alpha$ *and, for all* $s \in S$, $\zeta_s = \sigma_s / \beta$*, and* $\tau_s = \xi_s - \alpha \zeta_s$*.).*

c. π *is unique and* $\pi(s) = 0$ *if and only if* s *is null.*

The proof of Theorem 7 follows immediately from the proofs of Theorems 4 and 6 and is omitted.

4.5 The moral hazard problem

To relate the axiomatic models of the preceding sections to the literature on optimal contracts in the presence of moral hazard let $\mathbf{X} = \mathbb{R}^n$ and $\mathbf{x}$ represents state-contingent output levels. An *incentive contract* is a vector $\mathbf{w} \in \mathbb{R}^n$ satisfying $w_s = w_t$ if $x_s = x_t$, for all $s, t \in S$. Denote by $\succcurlyeq^P$ the principal's preference relation on $\mathbb{X}$ and suppose that it satisfies axioms (A.1), (A.2), and (A.3). Denote by $\hat{\succcurlyeq}^A$ the agent's preference relation on $\mathbf{A} \times \mathbb{X}$, and suppose that it satisfies axioms *(A.1'),* (A.2'), (A*.3')*. Let $\mathbf{B} \subset \mathbf{A}$ be a subset of *feasible actions*. Then the principal's problem may be stated as follows: Given $\mathbf{x} \in \mathbb{R}^n$ design a contract $\mathbf{w}^*$ and choose an action $a^* \in \mathbf{B}$ such that $(\mathbf{x} - \mathbf{w}^*)_{F(a^*)} \succcurlyeq^P (\mathbf{x} - \mathbf{w})_{F(a)}$ for all $(a, \mathbf{w}) \in \mathbf{B} \times \mathbb{R}^n$ subject to the incentive compatibility constraints:

$$\left(a^*, \mathbf{w}^*_{F(a^*)}\right) \hat{\succcurlyeq}^A \left(a, \mathbf{w}^*_{F(a)}\right), \text{ for all } a \in \mathbf{B},$$

and the participation constraint

$$\left(a^*, \mathbf{w}^*_{F(a^*)}\right) \hat{\succcurlyeq}^A \left(c, \mathbf{z}_{F(c)}\right),$$

where $\left(c, \mathbf{z}_{F(c)}\right)$ represents the agent's best alternative course of action if he refuses the contract.

By Theorems 2 and 6, this problem may be restated as follows: Choose $(a^*, \mathbf{w}^*) \in \mathbf{B} \times \mathbb{R}^n$ so as to maximize the principal's objective function

$$\sum_{s \in S} \pi^P (s \mid F(a^*))\, u^P_s (x_s - w^*_s)$$

subject to the participation constraint:

$$\sum_{s \in S} \pi^A (s \mid F(a^*))\, u^A_s (w^*_s) + v(a^*) \geq v_0$$

and the incentive compatibility constraints: for all $a \in \mathbf{B}$

$$\sum_{s \in S} \pi^A (s \mid F(a^*))\, u^A_s (w^*_s) + v(a^*) \geq \sum_{s \in S} \pi^A (s \mid F(a))\, u^A_s (w^*_s) + v(a),$$

where $v_0 = \sum_{s \in S} \pi^A (s \mid F(c))\, u^A_s (z_s) + v(c)$ and the superscripts P and A denote the variables corresponding to the principal and agent, respectively.

Note that in the usual formulation of the moral hazard problem (e.g., Shavell [1979], Holmstorm [1979]) it is assumed that the principal and the agent agree on the probabilities (i.e., $\pi^A = \pi^P$) and that the utility functions are state independent, so that $u_s^P = u^P$ and $u_s^A = u^A$ for all $s \in S$. Neither of these assumptions is necessary or compelling. Notice also that the formulation above is one way of stating the moral hazard problem. An alternative approach is to assume that the subjective probability on the state-space is unaffected by the agent's action and that there is a "production function" that selects a random variable, that is, an act in $\mathbf{X}$, as a function of the action. Consequently, the choice of action affect the probability of the outcomes, through the selection of the random variable without affecting the probabilities of the underlying states.

5 Discussion

5.1 Beliefs and probabilities

The theory developed here yields a unique subjective probability distribution regardless of whether the preference relation is state independent. The underlying premise is that the mental processes at work in the assessment of the likelihood of events and the valuation of outcomes are the same whether or not the preferences are state independent. It is reassuring, therefore, that both cases are addressed using the same approach and that the case of state-independent preferences is merely a special instance of the more general model.

A crucial aspect of the definition of subjective probabilities in this paper is the fact that they quantify the decision makers' prior and posterior beliefs *correctly*. The following example illustrates this assertion. Consider a subjective expected utility - maximizing Bayesian decision maker whose preferences are state independent. Let there be three states of nature, $S = \{1, 2, 3\}$. Suppose that the decision maker's valuations of outcomes are depicted by state-dependent utility functions satisfying $u_1 = u_2 = u$, and $u_3 = 2u$. Assume further that the decision maker believes the three states to be equally likely. This belief is represented by the uniform probability distribution $p_1 = p_2 = p_3$. However, based on the observations of his choice among acts, the choice-theoretic models ascribe to the decision maker utilities and subjective probabilities so that his preferences are represented by:

$$q_1 u(x) + q_2 u(y) + q_3 u(z), \tag{5}$$

where $q_1 = p_1/(p_1 + p_2 + 2p_3) = 1/4$, $q_2 = p_2/(p_1 + p_2 + 2p_3) = 1/4$, and $q_3 = 2p_3/(p_1 + p_2 + 2p_3) = 1/2$.

Consider next the decision maker's choices among acts conditional on the events $\{1,2\}$ and $\{2,3\}$. According to the decision maker's beliefs the probabilities of these events are: $p(\{1,2\}) = p(\{2,3\}) = p(\{1,3\}) = 2/3$ and according to the ascribed probabilities they are: $q(\{1,2\}) = 1/2$, $q(\{2,3\}) = q(\{1,3\}) = 3/4$.

Let $\mathbf{x}_{\{2,3\}} \sim \mathbf{y}_{\{1,2\}}$ and suppose that $x_2 = y_2$ then, by Theorem 4,

$$\frac{3}{2}\left[\frac{1}{3}u(x_2) + \frac{2}{3}u(x_3)\right] = \frac{3}{2}\left[\frac{1}{3}u(y_1) + \frac{1}{3}u(y_2)\right]. \tag{6}$$

According to the ascribed probabilities, the same preference is represented by:

$$\frac{4}{3}\left[\frac{1}{4}u(x_2) + \frac{1}{2}u(x_3)\right] = 2\left[\frac{1}{4}u(y_1) + \frac{1}{4}u(y_2)\right]. \tag{7}$$

But equations (6) and (7) cannot both be true. In fact, the choice-theoretic subjective expected utility model fails to predict the behavior of an expected utility - maximizing Bayesian decision maker even though his preferences satisfy the underlying axiomatic structure. This failure is endemic.

In Karni (1996) I argued that a correct representation of beliefs is useful since it is consistent with both decision-makers' choice behavior and the verbal expressions used to exchange information. The boxing example in the introduction and subsequent work by Grant and Karni (2002) illustrate the kind of economic problems that may arise if agent's beliefs and utilities are misconstrued. As I argue next, correct representation of beliefs is also important for normative economic analyses.

Aggregation of Beliefs and the Pareto Principle: Harsanyi's (1955) aggregation theorem shows that if individuals and social preference relations satisfy the axioms of expected utility theory of von Neumann and Morgenstern (1944) and a Pareto indifference condition, then the social preferences may be represented as a linear combination of individual utilities. Harsanyi's theorem takes the probabilities of the social state-lotteries as given. If the probabilities are subjective, then Harsanyi's approach suggests that individual utilities and probabilities be aggregated separately into social utilities and probabilities and then combined to obtain an expected utility representation of social preferences. Unfortunately, as noted by Hylland and Zeckhauser (1979) and Mongin (1995), such an aggregation is inconsistent with Pareto indifference.

Gilboa, Samet, and Schmeidler (2001) claim that this inconsistency does not pose an ethical problem. They argue, convincingly, that the Pareto condition is an expression of individual preferences that combines beliefs (subjective probabilities) and tastes (utilities). Hence the Pareto condition that requires that when all members of a society are indifferent between two alternatives the social preferences must also be indifferent is compelling only when the individual members do not hold contradictory beliefs. In other words, without some qualification, the Pareto condition cannot be used as an argument to justify social preference over alternatives about which individual members hold conflicting beliefs. On the positive side, Gilboa et al. show that if the Pareto indifference condition is imposed only when there is agreement among individuals' beliefs, the inconsistency disappears. Hence if the individual and social preferences satisfy the axioms of subjective expected utility theory, then imposing Pareto indifference implies that the social preferences are represented by a subjective expected utility functional with probabilities that are an affine combination of the individual subjective probabilities, and a social utility function that is a linear combination of the individual utilities.

Gilboa et al. do not distinguish between probabilities and beliefs. In fact, following the traditional practice in decision theory, they tacitly define beliefs by probabilities and use these probabilities in the formulation of their main axiom, namely, the restricted Pareto condition. This approach opens a gap between the verbal argument, which is stated, quite compellingly, using the language of beliefs, and the formal argument ,which is presented in terms of Savage-type ascribed probabilities. What happens if beliefs are not represented by the ascribed probabilities? Not surprisingly, this may lead to two types of errors. Errors of the first type occur when the restricted Pareto condition is not used to justify social preferences when in fact it should be. Errors of the second type occur when the restricted Pareto condition is used to justify social preferences when it should not be.

To understand the problem, consider the following example. Let there be two individuals, a and b, and two states of nature, 1 and 2. Suppose that individual tastes are captured by state-dependent utility functions defined on the level of wealth as follows:

$\frac{State}{Individual}$	1	2
a	w^{α}	$2w^{\alpha}$
b	w^{β}	w^{β}

Consider next the beliefs of the individuals.

Case 1: Both individuals believe that state 1 is twice is likely to obtain as state 2. Being subjective expected utility maximizers, their subjective probabilities are $\pi(1) = 2/3$ and $\pi(2) = 1/3$. However, in the context of traditional subjective expected utility theory, the representation of the individual preferences are:

$$U^{a}\left(\left(w_{1},w_{2}\right)\right)=\frac{1}{2}w_{1}^{\alpha}+\frac{1}{2}w_{2}^{\alpha}$$

$$U^{b}\left(\left(w_{1},w_{2}\right)\right)=\frac{2}{3}w_{1}^{\beta}+\frac{1}{3}w_{2}^{\beta}$$

(These are the probabilities and representation that would figure in Gilboa et al. [2001].) Thus the two individuals appear to disagree on the probabilities and therefore, the restricted Pareto condition of Gilboa et. al. does not apply, even though, by the normative argument, it should.

Case 2: Individual a believes that state 1 is four times more likely to obtain than state 2 (i.e., $\pi^{a}(1)=4/5$ and $\pi^{a}(2)=1/5$), while individual b believes, as before, that state 1 is twice is likely to obtain as state 2. The preference of the two individuals are represented by:

$$U^{a}\left(\left(w_{1},w_{2}\right)\right)=\frac{2}{3}w_{1}^{\alpha}+\frac{1}{3}w_{2}^{\alpha},$$

and

$$U^{b}\left(\left(w_{1},w_{2}\right)\right)=\frac{2}{3}w_{1}^{\beta}+\frac{1}{3}w_{2}^{\beta}.$$

In this case, the model of Gilboa, Samet, and Schmeidler implies that restricted Pareto indifference should apply, even though, in fact, the individuals hold conflicting beliefs. In other words, Gilboa et al. would use the restricted Pareto condition to justify social preferences even though, by their own argument, the situation does not warrant it.

Conclusion: *To avoid making errors in using the restricted Pareto condition to justify social preferences it is necessary to use the corrected probability representations of individual beliefs.*

5.2 Related literature

Luce and Krantz (1971) maintain that, in many circumstances, decisions delimit which events may obtain and that in such circumstances the application of Savage's theory is cumbersome and unintuitive. They propose instead a theory based on choice among conditional acts that, they believe, is simpler and more natural. The critical view of the adequacy of Savage's theory is shared by Fishburn, according to whom "although the Luce-Krantz theory might seem a bit more intricate than Savage's, it surely comes closer to making contact with the structure of actual decision situations" (Fishburn [1973] p. 5). The analytical framework of Luce and Krantz includes a set of states (finite or infinite), an algebra of events, and an arbitrary set of consequences that is the same across events (as opposed to the structured sets of state-dependent outcomes in this paper).

Both the Luce and Krantz model and the model presented here require that the preference relations be continuous weak orders satisfying a version of the "sure thing principle." Moreover, both models include an axiom that calibrates the intensity of preference (or preferential "differences"). The difference between the two models is the formalization of this idea. Loosely speaking, the coherence axiom (A.2) requires that the intensity of preferences between pairs of outcomes in a given state be independent of the event in which this state occurs. Axiom 5 of Luce and Krantz requires that $f_A^{(i)} \sim g_B^{(i)}, i = 1, 2, 3, 4$ (a conditional act f_A is a function from the subset of states A to the set of consequences) implies that if the preferential difference between pairs of conditional acts $f_A^{(3)}$ and $f_A^{(4)}$ exceeds that of $f_A^{(1)}$ and $f_A^{(2)}$, then the preferential difference between $g_B^{(3)}$ and $g_B^{(4)}$ must exceed that between $g_B^{(1)}$ and $g_B^{(2)}$ independently of the (disjoint) conditional act to which they may be attached. This distinction has significant implications for the representation. In particular, the axioms of Luce and Krantz do not imply a subjective expected utility representation in which the utility function is defined on the set of consequences. To obtain such an event-independent representation they require, in addition, that there exist constant acts, namely, that there exist consequences whose valuations are event independent. This is in contrast to the present model in which the utility function, defined on the state-dependent sets of outcomes, may be either state dependent or state independent and its existence does not require the availability of constant acts.

Fishburn (1973) proposed an alternative axiomatization of subjective expected utility that combines elements of the Luce-Krantz model with elements of the model of Anscombe and Aumann (1963). Fishburn's analytical framework includes extraneous probabilities, and his choice set consists of "objective" probability mixtures of conditional acts. Fishburn defines probability mixtures on acts conditional on the same event and introduces a version of the independence axiom requiring that if two mixture-acts conditional on one event are each indifferent to a corresponding act on a second event, then the 50-50 mixture of the first pair of conditional acts is indifferent to the same mixture of the second pair. He also requires that if a mixture-act conditional on one even is weakly preferred over the same mixture-act conditional on another, disjoint, event, then the same mixture-act conditional on the union of the two events is (weakly) less desirable than the former and (weakly) more desirable than the latter. With these and some additional innocuous conditions, Fishburn shows that there exist a subjective conditional expected utility representation of the preference relation on conditional acts with an event-dependent utility function that is unique up to positive linear transformations and the unique subjective conditional probabilities.

The main difference between Fishburn's approach and the approach taken here and by Luce and Krantz concerns the role of probabilities. The former relies on extraneous probabilities in the statement of the axioms, while the latter works do not invoke the notion of probabilities at the primitive level.

Unlike the present work, neither Luce and Krantz (1971) nor Fishburn (1973) attempted to apply their theories to the formulation of the moral-hazard problem. A decision theory with moral hazard and state-dependent preferences was proposed by Drèze (1961, 1987). Building on the model of Anscombe and Aumann (1963), Drèze (1987) and Drèze and Rustichini (1999) replace the formers' assumption of reversal of order with preference for early resolution of the outcome of a random device used to choose among acts ("games" in their terminology). This preference for information reflects the decision makers confidence in their ability to influence the likely realization of alternative events by taking appropriate actions. To exploit their power decision makers need to know in advance what game they are engaged in (the payoff associated with every state). They obtain a utility representation of choice among games that is the maximum of subjective expected utility over a closed convex set of probability distributions over a finite state space. Moreover, under additional assumptions there is a unique minimal set of such probability distributions.

Drèze (1987) is critical of the approach that uses preferences on conditional acts as primitive, even though implicit in his preference for information is the assumption that decision makers are capable of foreseeing and evaluating the merits of alternative games conditional on their "preemptive" actions. Drèze and Rustichini (1999) use the preference relation on conditional acts as a central ingredient of their model. Their work spells out the link between the model of Drèze (1987) which uses the (derived) conditional preferences and conditional expected utility theory (e.g., of Luce and Kranz [1971]). In fact, the main novelty of their analysis is the imposition of consistency between the (primitive) preference relations on conditional acts and derived conditional preferences over unconditional acts. Thus the difference between the approaches taken by Drèze (1987) and Drèze and Rustichini (1999) and the one pursued here has less to do with the use of conditional acts and more with the other assumptions of the model. In particular, it is noteworthy that, in general, in these works the sets of probabilities over which decision makers exercise choice are not the conditional probabilities obtained from a given prior. In fact, in the model of Dreze (1987) conditionally on a specific action, Bayes's rule holds; but across actions it need not hold since different actions may imply different relative probabilities across states in the conditioning event.[10] Second, even though their formulations depend critically on the presence of moral hazard, no (dis)utility is assigned to actions intended to change the probability distribution on the states space (Drèze and Rustichini [1999] allude to this point). Third, by avoiding using probabilities as a primitive concept, the model presented here is more appealing as a foundation of subjective probabilities.

An axiomatic model of subjective expected utility and Bayesian updating using derived conditional preferences over unconditional acts was studied by Ghirardato (2002). The framework is similar to that of Savage (1954) with the sure thing principle replaced by dynamic consistency. The latter assumption connects the unconditional and conditional preferences. In addition, the model imposes consistency between unconditional and conditional preferences over constant acts, which together with Savage's P4 imply state-independent preferences. This implies the existence of a unique prior and event-dependent posterior probability distributions connected through Bayes' rule. Ghirardato's model is an important extension of Savage's work. It is different from the results presented here in some significant ways. First, unlike the present results, the prior and posterior probabilities obtained by Ghirardato do not necessarily represent the decision makers' beliefs. Second, like Savage, Ghirardato's representation does not admit state-dependent preferences. Third, by not including the events as an ingredient of the choice set, Ghirardato's model is not (and was not meant to be) a framework that can accommodate decision making in the presence of moral hazard. These divergent results are manifestations of the fundamental difference between Ghirardato's approach and the approach pursued here, namely, the former approach events signify information while in the latter approach events constitute an ingredient of the choice variable.

Skiadas (1997, 1997a) axiomatized subjective probabilities representing decision makers' beliefs in a model that accommodates state-dependent preferences and admits non-separability (across sates) of the evaluation of acts.

[10] I am grateful to Jacques Drèze for clarifying this point.

In Skiadas' model acts and states are primitive concepts and the set of consequences is a derived concept intended to capture the subjective nature of its elements. Preferences are defined on acts for any given event expressing the decision maker's desire for the overall consequences of an act on an event not knowing whether the event occurred. In fact, to express disappointment aversion, given an event, one act may be conditionally preferred over another with the preferences between the same two acts if the same event were known to occurred is reversed. Thus conditional preferences is an expression of anticipated feeling and do not have clear choice-theoretic interpretation. It is also worth mentioning that Skiadas' model was not intended to nor does it imply Bayesian updating. In fact, disappointment aversion means that learning that an even was realized affects the decision maker's beliefs at the same time that it affects his valuations of different acts and it is not clear what restrictions need to be imposed if beliefs are to be updated using Bayes rule.

A different branch of literature is related to the coherence axiom. This axiom serves the purpose of linking the preference relations on acts conditional on distinct events. The idea of axiomatically linking different preference relations was originally used in Karni and Schmeidler (1980) and in Karni, Schmeidler, and Vind (1983) to connect the preference relations on actual and hypothetical acts in the framework of Anscombe and Aumann (1963) and, thereby, model subjective expected utility with state-dependent preferences. Subsequently Wakker (1987) extended the work of Karni, Schmeidler, and Vind by replacing the roulette lotteries of Anscombe and Aumann with topologically connected consequence spaces. The different structure of the consequence spaces requires the use of a linkage axiom different from the one used by Karni, Schmeidler, and Vind and much closer in spirit to the coherence axiom of this paper. In both cases, however, the linkage imposes state-by-state consistency of conditional preferences.

Karni and Mongin (2000) observed that only the model of Karni and Schmeidler (1980) leads to a definition of subjective probabilities that faithfully represents the decision maker's beliefs. Other models, including Karni, Schmeidler, and Vind (1983) and Wakker (1987), involve a choice of hypothetical probabilities over the states that renders the resulting subjective probabilities arbitrary. This observation lends weight to the work of Grant and Karni (2000) extending the work of Karni and Schmeidler (1980) to nonexpected utility preferences and to Karni (2001) extending it to the framework of Wakker (1987).

A common feature of all these contributions is the reliance on expressed preferences among hypothetical lotteries or objective probability distributions on the states. The present model is different in that it relies on the use of preferences on conditional acts. It thus circumvents the need to use probabilities as primitive concepts.

APPENDIX

A. **Proof of Theorem 1 -** $(i) \Rightarrow (ii)$. Null states do not affect the preferences among acts. Thus, without loss of generality, when writing $\mathbf{x}_E$ it is assumed that all the states in E are nonnull. If s is a null state set $w_S(\cdot; s) = 0$. By Theorem III.4.1 of Wakker (1989) there exist additive value function $w_S : \mathbf{X} \to \mathbb{R}$ that represent $\succcurlyeq$ on $\mathbf{X}$ with jointly cardinal continuous functions $\{w_S(\cdot; s) : X_s \to \mathbb{R}\}_{s \in S}$. (I.e., for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}, \mathbf{x} \succcurlyeq \mathbf{y}$ if and only if $w_S(\mathbf{x}) = \sum_{s \in S} w_S(x_s; s) \geq \sum_{s \in S} w_S(y_s; s) = w_S(\mathbf{y})$.) Let $\bar{\mathbf{x}}$ and $\underline{\mathbf{x}}$ be constant valuation acts such that $\bar{\mathbf{x}} \succ \underline{\mathbf{x}}$. Using the uniqueness property of the jointly cardinal representation normalize $\{w_S(\cdot; s)\}_{s \in S}$ as follows: Set $w_S(\underline{x}_s; s) = 0$ for all $s \in S$ and let

$$w_S(\bar{\mathbf{x}}) = \sum_{s \in S} w_S(\bar{x}_s; s) = 1. \tag{8}$$

To construct functions $w_E(\cdot; s)$ on X_s note that, by (A.2), $\underline{\mathbf{x}} \sim \underline{\mathbf{x}}_E$. Set $w_E(\underline{x}_s; s) = 0$ for all $s \in E$. For each $s \in E$ and $x_s \in X_s$ let $x'_s(x_s, E) \subset X_s$ be defined by: $(\underline{\mathbf{x}}^{-s}, x_s) \sim (\underline{\mathbf{x}}^{-s}, x'_s(x_s, E))_E$. Set $w_E(y_s; s) = w_S(x_s, s)$ for all $y_s \in x'_s(x_s, E)$. If the $\cup_{x_s} x'_s(x_s; E)$ is a proper subset of X_s extend it as follows: For $z_s \notin \cup_{x_s} x'_s(X_s; E)$ take $\mathbf{x} \in \mathbf{X}$ such that $\mathbf{x} \sim (\underline{\mathbf{x}}^{-s}, z_s)_E$ and define $w_E(z_s; s) = \sum_{t \in S} w_S(x_t, t)$.

Define a function w_E on $\mathbf{X}_E$ by $w_E(\mathbf{y}_E) = w_S(\mathbf{x}_E \underline{\mathbf{x}})$ whenever $\mathbf{y}_E \sim \mathbf{x}_E \underline{\mathbf{x}}$. But, by (A.1), $\mathbf{y}_E \sim \mathbf{x}_E \underline{\mathbf{x}}$ implies that $y_s \in x'_s(x_s; E)$. Thus $w_S(\mathbf{x}_E \underline{\mathbf{x}}) = \sum_{s \in E} w_S(x_s; s) = \sum_{s \in E} w_E(y_s; s)$. Hence $w_E(\mathbf{y}_E) = \sum_{s \in E} w_E(y_s; s)$ for all $\mathbf{y}_E \in \mathbf{X}_E$. On the set of conditional constant valuation acts (A.2) implies that $w_E(\mathbf{x}^*_E) = w_G(\mathbf{x}^*_G)$ for all $E \in \mathcal{E}$ and $G \subset E$.

Next we show that the functions $\{w_E(\cdot, s) \mid s \in S, E \in \mathcal{E}\}$ constitute an additive-valued representation of $\succcurlyeq$ on $\mathbb{X}$. Let $\hat{\mathbf{x}}_E, \hat{\mathbf{y}}_A \in \mathbb{X}$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that $\hat{x}_s \in x'_s(x_s; E)$ for all $s \in E$ and $\hat{y}_s \in x'_s(y_s; A)$ for all $s \in A$. Then $\hat{\mathbf{x}}_E \sim \mathbf{x}_E \underline{\mathbf{x}}$ and $\hat{\mathbf{y}}_A \sim \mathbf{y}_A \underline{\mathbf{x}}$. By transitivity and the presentation, $\hat{\mathbf{x}}_E \succcurlyeq \hat{\mathbf{y}}_A$ if and only if $\mathbf{x}_E \underline{\mathbf{x}} \succcurlyeq \mathbf{y}_A \underline{\mathbf{x}}$ if and only if $\sum_{t \in E} w_S(x_t; t) \geq \sum_{t \in S-A} w_S(y_t; t)$. But, by definition, $w_S(x_s; s) = w_E(\hat{x}_s; s)$ for all $s \in E$ and $w_S(y_s; s) = w_A(\hat{y}_s; s)$ for all $s \in A$. Hence

$$\sum_{t \in E} w_S(x_t; t) \geq \sum_{t \in A} w_S(y_t; t) \Leftrightarrow \sum_{t \in E} w_E(\hat{x}_t; t) \geq \sum_{t \in A} w_A(\hat{y}_t; t), \tag{9}$$

for all $E, A \in \mathcal{E}$. Thus

$$\hat{\mathbf{x}}_E \succcurlyeq \hat{\mathbf{y}}_A \;\Leftrightarrow\; \sum_{t\in E} w_E\left(\hat{x}_t; t\right) \geq \sum_{t\in A} w_A\left(\hat{y}_t; t\right). \tag{10}$$

This completes the proof that $(i) \rightarrow (ii)$. The proof that (ii) implies (ii) is immediate.

To prove the uniqueness of $\{w_E(\cdot; s) \mid s \in S, E \in \mathcal{E}\}$ note that, by Wakker (1989) Observation III.6.6', $\{w_E(\cdot; s) \mid s \in S\}$ is unique up to unit-comparable transformations. (That is, $\{\hat{w}_E(\cdot; s) \mid s \in S\}$ represent $\succcurlyeq$ on $\mathbf{X}_E$ in the sense of (ii) if and only if $\hat{w}_E(\cdot; s) = \beta w_E(\cdot; s) + \gamma_E(s)$, $\beta > 0$ for all $s \in E$.) But, by (ii), for all $E \in \mathcal{E}$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x}_S \sim \mathbf{y}_E$ if and only if

$$\sum_{s\in S} w_S\left(x_s; s\right) = \sum_{s\in E} w_E\left(y_s; s\right),$$

which is equivalent to

$$\sum_{s\in S} \left[\beta w_S\left(x_s; s\right) + \gamma_S(s)\right] = \sum_{s\in E} \left[\beta w_E\left(y_s; s\right) + \gamma_E(s)\right].$$

But this implies that $C = \sum_{s\in S} \gamma_S(s) = \sum_{s\in E} \gamma_E(s)$ for all E. □

B. Proof of the Main Result.

The following Lemma is needed for the proof of the main result.

Lemma 8 *Assume that there are at least three nonnull states and that the relation $\succcurlyeq$ is a continuous weak-order on $\mathbb{X}$. Then the following conditions are equivalent:*

(i) *$\succcurlyeq$ satisfies (A.1), (A.2) and (A.3).*

(ii) *For every $E \in \mathcal{E}$ there exist positive affine or constant function $\phi_E : \cup_{s\in E} w_S(X_s; s) \rightarrow \mathbb{R}$ such that, for all and $s \in E$, $w_E(\cdot; s) = \phi_E \circ w_S(\cdot; s)$, where $\{w_E(\cdot; s) \mid E \in \mathcal{E},\, s \in E\}$ are as in Theorem 1.*

Proof of Lemma 8. The proof of Lemma 8 involves two stages. The first stage is an adaptation of the proof of Wakker's (1987) Proposition 4.5 and proves the results for $E \in \mathcal{E}_1$. The second stage is new and extends the result to nonnull singleton events.

Stage 1 - $(i) \Rightarrow (ii)$. Suppose that $\succcurlyeq$ is a continuous weak order satisfying (A.1), (A.2) and (A.3). Fix $E \in \mathcal{E}_1$ then, by Theorem 1, for every $t \in E$ there exist $\mathbf{w}, \mathbf{z} \in \mathbf{X}$ such that

$$\sum_{r \in S - \{t\}} \left[w_S \left(w_r; r \right) - w_S \left(z_r; r \right) \right] = \zeta > 0, \tag{11}$$

and $\mathbf{x}_E, \mathbf{y}_E \in \mathbf{X}_E$ satisfying

$$\sum_{r \in E - \{t\}} \left[w_E \left(x_r; r \right) - w_E \left(y_r; r \right) \right] = \varepsilon > 0. \tag{12}$$

By continuity of the additive valued functions $w_E(\cdot; s)$ and the connectedness of the sets X_s, for every $\hat{\zeta} \in [-\zeta, \zeta]$, $\hat{\varepsilon} \in [-\varepsilon, \varepsilon]$, and $t \in E$ there exist $\bar{\mathbf{w}}, \bar{\mathbf{z}} \in \mathbf{X}$ and $\bar{\mathbf{x}}_E, \bar{\mathbf{y}}_E \in \mathbf{X}_E$ such that

$$\sum_{r \in S - \{t\}} \left[w_S \left(\bar{w}_r; r \right) - w_S \left(\bar{z}_r; r \right) \right] = \hat{\zeta} \tag{13}$$

and

$$\sum_{r \in E - \{t\}} \left[w_E \left(\bar{x}_r; r \right) - w_E \left(\bar{y}_r; r \right) \right] = \hat{\varepsilon}. \tag{14}$$

Define ϕ_E by $w_E(\cdot; \cdot) = \phi_E \circ w_S(\cdot; \cdot)$. Then ϕ_E is continuous. To show that ϕ_E is positive affine or constant function fix $t \in E$ and let $W_t = w_S(X_t; t)$. Then, by the connectedness of X_t and the continuity of $w_S(\cdot; t)$, W_t is an interval in $\mathbb{R}$. Take $\alpha, \beta, \gamma, \delta \in W_t$ such that $-\zeta \leq \alpha - \beta = \gamma - \delta \leq \zeta$ and $-\varepsilon \leq \phi_E(\alpha) - \phi_E(\beta) \leq \varepsilon$. Let $a_t, b_t, c_t, d_t \in X_t$ satisfy $w_S(a_t; t) = \alpha$, $w_S(b_t; t) = \beta$, $w_S(c_t; t) = \gamma$ and $w_S(d_t; t) = \delta$. Take $\hat{\mathbf{w}}, \hat{\mathbf{z}} \in \mathbf{X}$ such that $w_S(\hat{w}_r; r) - w_S(\hat{z}_r; r) = (\alpha - \beta) / (|S| - 1)$ for all $r \in S$. Then

$$\sum_{r \in S - \{t\}} \left[w_S \left(\hat{w}_r; r \right) - w_S \left(\hat{z}_r; r \right) \right] = \alpha - \beta. \tag{15}$$

By Theorem 1 $\left(\hat{\mathbf{w}}^{-t}; a_t \right) \sim \left(\hat{\mathbf{z}}^{-t}; b_t \right)$ and $\left(\hat{\mathbf{w}}^{-t}; c_t \right) \sim \left(\hat{\mathbf{z}}^{-t}; d_t \right)$.

Take $\hat{\mathbf{x}}_E, \hat{\mathbf{y}}_E \in \mathbf{X}_E$ such that $w_E(\hat{x}_r; r) - w_E(\hat{y}_r; r) = (\phi_E(\alpha) - \phi_E(\beta)) / (|E| - 1)$ for all $r \in E$. Then

$$\sum_{r \in E - \{t\}} \left[w_E \left(\hat{x}_r; r \right) - w_E \left(\hat{y}_r; r \right) \right] = \phi_E(\alpha) - \phi_E(\beta). \tag{16}$$

Since $w_E(\cdot;t) = \phi_E \circ w_S(\cdot;t)$ this implies $\left(\hat{\mathbf{x}}^{-t};a_t\right)_E \sim \left(\hat{\mathbf{y}}^{-t};b_t\right)_E$. Applying (A.3) twice yields $\left(\hat{\mathbf{x}}^{-t};c_t\right)_E \sim \left(\hat{\mathbf{y}}^{-t};d_t\right)_E$. Thus

$$\phi_E(\gamma) - \phi_E(\delta) = \sum_{r\in E-\{t\}} \left[w_E(\hat{x}_r;r) - w_E(\hat{y}_r;r)\right] = \phi_E(\alpha) - \phi_E(\beta). \tag{17}$$

By Wakker (1987) Lemma 4.4 this implies that ϕ_E is affine. But ϕ_E is nondecreasing. (To see this, let $\left(\mathbf{x}^{-t},a_t\right) \succcurlyeq \left(\mathbf{x}^{-t},b_t\right)$. But $\left(\mathbf{x}^{-t},a_t\right) \succcurlyeq \left(\mathbf{x}^{-t},a_t\right)$, $\left(\mathbf{x}^{-t},a_t\right) \succcurlyeq \left(\mathbf{x}^{-t},b_t\right)$, and $\left(\mathbf{x}^{-t},a_t\right)_E \succcurlyeq \left(\mathbf{x}^{-t},a_t\right)_E$. Thus, by (A.3) $\left(\mathbf{x}^{-t},a_t\right)_E \succcurlyeq \left(\mathbf{x}^{-t},b_t\right)_E$. The conclusion is implied by the representation of $\succcurlyeq$.) Hence ϕ_E is constant or positive.

$(ii) \Rightarrow (i)$. Assume that there exist positive affine or constant transformations ϕ_E such that $w_E(\cdot;\cdot) = \phi_E \circ w_S(\cdot;\cdot)$. Suppose that $(\mathbf{x}^{-t},a_t) \succcurlyeq (\mathbf{y}^{-t},b_t)$, $(\mathbf{y}^{-t},c_t) \succcurlyeq (\mathbf{x}^{-t},d_t)$ and $\left(\mathbf{z}^{-t},b_t\right)_E \succcurlyeq \left(\mathbf{w}^{-t},a_t\right)_E$. By Theorem 1, $(\mathbf{x}^{-t},a_t) \succcurlyeq (\mathbf{y}^{-t},b_t)$ if and only if

$$w_S(a_t;t) + \sum_{s\in S-\{t\}} w_S(x_s;s) \geq w_S(b_t;t) + \sum_{s\in S-\{t\}} w_S(y_s;s) \tag{18}$$

and $(\mathbf{y}^{-t},c_t) \succcurlyeq (\mathbf{x}^{-t},d_t)$ if and only if

$$w_S(d_t;t) + \sum_{s\in S-\{t\}} w_S(x_s;s) \leq w_S(c_t;t) + \sum_{s\in S-\{t\}} w_S(y_s;s). \tag{19}$$

Hence

$$w_S(b_t;t) - w_S(a_t;t) \leq \sum_{s\in S-\{t\}} \left[w_S(x_s;s) - w_S(y_s;s)\right] \leq w_S(c_t;t) - w_S(d_t;t). \tag{20}$$

By positive affineness or constancy of ϕ_E these inequalities imply

$$w_E(b_t;t) - w_E(a_t;t) \leq w_E(c_t;t) - w_E(d_t;t). \tag{21}$$

By Theorem 1 $\left(\mathbf{z}^{-t},b_t\right)_E \succcurlyeq \left(\mathbf{w}^{-t},a_t\right)_E$ if and only if

$$\sum_{s\in E-\{t\}} w_E(z_s;s) + w_E(b_t;t) \geq \sum_{s\in E-\{t\}} w_E(w_s;s) + w_E(a_t;t). \tag{22}$$

Thus

$$w_E(b_t;t) - w_E(a_t;t) \geq \sum_{s\in E-\{t\}} \left[w_E(w_s;s) - w_E(z_s;s)\right]. \tag{23}$$

But $w_E(b_t;t) - w_E(a_t;t) \le w_E(c_t;t) - w_E(d_t;t)$ implies

$$\sum_{s\in E-\{t\}} w_E(z_s;s) + w_E(c_t;t) \ge \sum_{s\in E-\{t\}} w_E(w_s;s) + w_E(d_t;t). \qquad (24)$$

Hence, by Theorem 1, $\left(\mathbf{z}^{-t}, c_t\right)_E \succcurlyeq \left(\mathbf{w}^{-t}, d_t\right)_E$. This completes the proof of stage 1.

Stage 2 - $(ii) \to (i)$. For the nonnull singleton events $\{s\}$, $s \in S$ suppose that

$$w_{\{s\}}(x_s;s) = \lambda_s^{-1} w_S(x_s,s) + \kappa_s,\ \forall x_s \in X \qquad (25)$$

where $\lambda_s > 0$. By Theorem 1, for every constant valuation act, $\mathbf{x}^*$, and all nonnull $s \in S$,

$$w_{\{s\}}(x_s^*;s) = \sum_{t\in S} \lambda_t w_{\{t\}}(x_t^*;t) = \sum_{t\in S} w_S(x_t^*,t). \qquad (26)$$

(Notice that, by the normalization of $w_S(\cdot,s)$, $w_{\{s\}}(\bar{x}_s;s) = \sum_{t\in S} w_S(\bar{x}_t,t) = 1$. Hence, for all s, $\lambda_s = w_S(\bar{x}_s,s)$, $\kappa_s = 0$, and $\sum_{s\in S}\lambda_s = 1$.) But equations (26) implies (A.2) for all nonnull singleton events. Axiom (A.1) and (A.3) are implied trivially. Thus $(ii) \to (i)$.

Next we show that (A.1)-(A.3) imply the definition in equations (25). Without loss of generality assume that $S = \{1,2,3\}$ and that all three states are nonnull. Suppose also, without loss of generality that, for some constant valuation act $\mathbf{x}^*$,

$$w_{\{1\}}(x_1^*;1) = \theta_1^{-1} w_S(x_1^*,1),\ w_{\{2\}}(x_2^*;2) = \theta_2^{-2} w_S(x_2^*,2),\ w_{\{3\}}(x_3^*;3) = \lambda_3^{-1} w_S(x_3^*,3). \qquad (27)$$

where $\theta_1 < \lambda_1$ and $\theta_2 > \lambda_2$. Let $E = \{2,3\}$, then, since E is nonnull, the proof of stage 1 implies that there exist $b_E > 0$ and a_E such that, for all $x \in X_t$ and $t \in E$,

$$w_E(x;t) = b_E w_S(x;t) + a_E. \qquad (28)$$

Moreover, by (A.2),

$$w_E(\bar{x}_2;2) + w_E(\bar{x}_3;3) = b_E\left[w_S(\bar{x}_2;2) + w_S(\bar{x}_3;3)\right] + 2a_E = 1. \qquad (29)$$

Hence $b_E = \left[w_S(\bar{x}_2;2) + w_S(\bar{x}_3;3)\right]^{-1}$ and $a_E = 0$. By (A.2), for every constant valuation act, $\mathbf{x}^*$,

$$b_E\left[w_S(x_2^*;2) + w_S(x_3^*;3)\right] = w_{\{2\}}(x_2^*;2) = w_{\{3\}}(x_3^*;3). \qquad (30)$$

But

$$w_S(x_3^*;3) = \lambda_3 w_{\{3\}}(x_3^*;3) = w_{\{3\}}(x_3^*;3)\, w_S(\bar{x}_3;3) \tag{31}$$

and

$$w_S(x_2^*;2) = \theta_2 w_{\{2\}}(x_2^*;2) > w_{\{2\}}(x_2^*;2)\, w_S(\bar{x}_2;2) = \lambda_2 w_{\{2\}}(x_2^*;2). \tag{32}$$

Hence, using the fact that $w_{\{2\}}(x_2^*;2) = w_{\{3\}}(x_3^*;3)$, $\lambda_s = w_S(\bar{x}_s;s)$ for all s,

$$b_E\left[w_S(x_2^*;2) + w_S(x_3^*;3)\right] > b_E(\lambda_2+\lambda_3)\, w_{\{2\}}(x_2^*;2) = w_{\{2\}}(x_2^*;2). \tag{33}$$

This contradicts equation (30). Hence, for all s and $x_s \in X_s$, $w_{\{s\}}(x_s;s) = \lambda_s^{-1} w_S(x_s,s)$. □

Proof of Theorem 2 - (a) $(i) \Rightarrow (ii)$. Suppose that $\succcurlyeq$ satisfies (A.1), (A.2), and (A.3). Then, by Theorem 1, for all $E \in \mathcal{E}$, $t \in E$, $\mathbf{x}_E \in \mathbf{X}_E$ and $x, x' \in X_t$,

$$(\mathbf{x}^{-t}, x)_E \succcurlyeq (\mathbf{x}^{-t}, x')_E \quad \Leftrightarrow \quad w_E(x;t) \ \geq \ w_E(x';t). \tag{34}$$

By Lemma 8 there exist $b_E \geq 0$ and a_E such that, for all $x \in X_t$ and $t \in E$,

$$w_E(x;t) = b_E w_S(x;t) + a_E \tag{35}$$

In particular,by the proof of stage 2 of Lemma 8, for all nonull $t \in S$, $w_{\{t\}}(x;t) = b_{\{t\}} w_S(x;t)$, where $b_{\{t\}} = w_S(\bar{x}_t;t)^{-1}$.

For each $s \in S$ let $u_s(\cdot) = w_{\{s\}}(\cdot;s)$ and define $\pi(s) = b_{\{s\}}^{-1} = w_S(\bar{x}_s;s)$, if $b_{\{s\}} > 0$ (i.e., if s is nonnull) and $\pi(s) = 0$ otherwise. Then $w_S(x;s) = \pi(s)\, u_s(x)$ and, by Theorem 1, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$,

$$\mathbf{x} \succcurlyeq \mathbf{y} \quad \Leftrightarrow \quad \sum_{s\in S} \pi(s)\, u_s(x_s) \geq \sum_{s\in S} \pi(s)\, u_s(y_s). \tag{36}$$

Moreover, by Theorem 1 and equations (35), for all $\mathbf{x}_E, \mathbf{y}_A \in \mathbb{X}$, $\mathbf{x}_E \succcurlyeq \mathbf{y}_A$ if and only if

$$b_E \sum_{s\in E} \pi(s)\, u_s(x) + \nu(E) \geq b_A \sum_{s\in A} \pi(s)\, u_s(x) + \nu(A), \tag{37}$$

where $\nu(B) = \mid B \mid a_B$, for all $B \in \mathcal{E}$.

Next use the constant valuation acts to determine the probabilities. Let $\mathbf{x}^*$ and $\mathbf{x}^{**}$ be constant valuation acts and suppose that $\mathbf{x}^{**} \succ \mathbf{x}^*$. Denote by $\bar{u}$ and $\underline{u}$ the common value $u_s(x_s^{**})$ and $u_s(x_s^*)$, respectively. Then, by (A.3), $\mathbf{x}_E^{**} \sim \mathbf{x}_A^{**}$ and $\mathbf{x}_E^* \sim \mathbf{x}_A^*$. Thus equations (37) imply

$$\bar{u}[b_E \sum_{s\in E} \pi(s) - b_A \sum_{s\in A} \pi(s)] + [\nu(E) - \nu(A)] = 0. \tag{38}$$

and

$$\underline{u}[b_E \sum_{s\in E} \pi(s) - b_A \sum_{s\in A} \pi(s)] + [\nu(E) - \nu(A)] = 0. \tag{39}$$

But $\bar{u} > \underline{u}$ hence equations (38) and (39) imply that $b_E \sum_{s\in E} \pi(s) = b_A \sum_{s\in A} \pi(s)$ and $\nu(E) = \nu(A)$. Let $A = S$ then, by definition, $b_S = 1$ and $\nu(S) = 0$. Since $\sum_{s\in S} \pi(s) = 1$ equation (39) implies that $b_E \sum_{s\in E} \pi(s) = 1$ and $\nu(E) = 0$ for all $E \in \mathcal{E}$. Letting $\pi(t \mid E) = b_E \pi(t) = \pi(t) / \sum_{s\in E} \pi(s)$ and invoking equations (37) we conclude that, for all $\mathbf{x}_E, \mathbf{y}_A \in \mathbb{X}$,

$$\mathbf{x}_E \succcurlyeq \mathbf{y}_A \Leftrightarrow \sum_{s\in E} \pi(s \mid E) u_s(x_s) \geq \sum_{s\in A} \pi(s \mid A) u_s(y_s).$$

This completes the proof that $(i) \Rightarrow (ii)$.

$(ii) \Rightarrow (i)$. The fact that (ii) implies (A.1) is an immediate implication of Theorem 1. The fact that it implies (A.3) is straightforward. To show that (ii) implies (A.2) take $E \in \mathcal{E}_1$, $\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$, $a_j, b_j, c_j, d_j \in X_j$ and suppose that $(\mathbf{x}^{-j}, a_j) \succcurlyeq (\mathbf{y}^{-j}, b_j)$, $(\mathbf{y}^{-j}, c_j) \succcurlyeq (\mathbf{x}^{-j}, d_j)$, $\left(\mathbf{z}^{-j}, b_j\right)_E \succcurlyeq \left(\mathbf{w}^{-j}, a_j\right)_E$. By (ii) $(\mathbf{x}^{-j}, a_j) \succcurlyeq (\mathbf{y}^{-j}, b_j)$ implies

$$\sum_{s\in S-\{j\}} \pi(s) u_s(x_s) + \pi(j) u_j(a_j) \geq \sum_{s\in S-\{j\}} \pi(s) u_s(y_s) + \pi(j) u_j(b_j), \tag{40}$$

and $(\mathbf{y}^{-j}, c_j) \succcurlyeq (\mathbf{x}^{-j}, d_j)$ implies

$$\sum_{s\in S-\{j\}} \pi(s) u_s(x_s) + \pi(j) u_j(d_j) \leq \sum_{s\in S-\{j\}} \pi(s) u_s(y_s) + \pi(j) u_j(c_j). \tag{41}$$

and $\left(\mathbf{z}^{-j}, b_j\right)_E \succcurlyeq \left(\mathbf{w}^{-j}, a_j\right)_E$ implies

$$\sum_{s\in S-\{j\}} \pi(s \mid E) u_s(z_s) + \pi(j \mid E) u_j(b_j) \geq \sum_{s\in S-\{j\}} \pi(s \mid E) u_s(w_s) + \pi(j \mid E) u_j(a_j). \tag{42}$$

But equations (40) and (41) imply that

$$u_j(c_j) - u_j(d_j) \geq u_j(b_j) - u_j(a_j). \tag{43}$$

Hence equations (42) and (43) imply

$$\sum_{s \in S - \{j\}} \pi(s \mid E) u_s(z_s) + \pi(j \mid E) u_j(c_j) \geq \sum_{r \in S - \{j\}} \pi(s \mid E) u_s(w_s) + \pi(j \mid E) u_j(d_j). \tag{44}$$

Equation (44) and (ii) imply $(\mathbf{z}^{-j}, c_j)_E \not\succ (\mathbf{w}^{-j}, d_j)_E$. Hence (ii) implies (A.2).

(b) The uniqueness of $\{u_s\}_{s \in S}$ follows directly from the uniqueness of $\{w_E(\cdot, s) \mid s \in S, E \in \mathcal{E}\}_{s \in S}$ of Theorem 1. In particular, $\alpha = C$.

(c) Let π and $\{u_s\}_{s \in S}$ satisfy part (a) of Theorem 2. If $t \in S$ is nonnull then, by the maintained assumption (A.0), for some $\bar{x}_t, \underline{x}_t \in X_t$, and $\mathbf{x} \in \mathbf{X}$, $(\mathbf{x}^{-t}, \bar{x}_t)_{\{t\}} \succ (\mathbf{x}^{-t}, \underline{x}_t)_{\{t\}}$. Hence $u_t(\bar{x}_t) - u_t(\underline{x}_t) > 0$. Moreover, $(\mathbf{x}^{-t}, \bar{x}_t) \succ (\mathbf{x}^{-t}, \underline{x}_t)$ implies $\pi(t)[u_t(\bar{x}_t) - u_t(\underline{x}_t)] > 0$. Thus $\pi(t) > 0$. If t is null then $(\mathbf{x}^{-t}, \bar{x}_t) \sim (\mathbf{x}^{-t}, \underline{x}_t)$ implying $\pi(t)[u_t(\bar{x}_t) - u_t(\underline{x}_t)] = 0$. Hence $\pi(t) = 0$.

To prove the uniqueness of π suppose, by way of negation, that there exists a probability measure, μ, on S and utility functions $\{\hat{u}_s\}_{s \in S}$ that satisfy the representation in (a.ii), but $\mu \neq \pi$. Then there are states $s, t \in S$ such that $\mu(s) > \pi(s)$ and $\pi(t) > \mu(t)$. Note that $\mu(s) > \pi(s)$ and $\pi(t) > \mu(t)$ imply that s and t are nonnull. Moreover, let $u_s(\cdot) = c_s \hat{u}_s(\cdot)$, for some $c_s > 0$. Then the representation requires that $\mu(s) = c_s \pi(s) / C$ for all $s \in S$, where $C = \sum_{t \in S} c_t \pi(t)$. Moreover, $\pi(s) u_s(\cdot) = C \mu(s) \hat{u}_s(\cdot)$ for all $s \in S$. Let $r \in S - \{s, t\}$ be a nonnull state and consider the events $E = \{s, r\}$ and $A = \{t, r\}$. Suppose that $\mathbf{x}_E \sim \mathbf{y}_A$. Then

$$\frac{\pi(s) u_s(x_s) + \pi(r) u_r(x_r)}{\pi(s) + \pi(r)} = \frac{\pi(t) u_t(y_t) + \pi(r) u_r(y_r)}{\pi(t) + \pi(r)}.$$

But $\mu(s) + \mu(r) > \pi(s) + \pi(r)$ and $\mu(t) + \mu(r) < \pi(t) + \pi(r)$ which, together with $\pi(s) u_s(\cdot) = C \mu(s) \hat{u}_s(\cdot)$ for all $s \in S$, imply

$$\frac{\mu(s) \hat{u}_s(\cdot) + \mu(r) \hat{u}_r(x_r)}{\mu(s) + \mu(r)} < \frac{\mu(t) \hat{u}_t(y_t) + \mu(r) \hat{u}_r(y_r)}{\mu(t) + \mu(r)}.$$

This contradicts $\mathbf{x}_E \sim \mathbf{y}_A$. Hence $\mu = \pi$. □

C. Proof of Theorem 4 - Theorem 4 follows from Theorem 2 with the following additional specifications: To show that, in part (a), $(i) \Rightarrow (ii)$ note that (A.4) makes (A.1) superfluous. In particular, the existence of additive representation is implied by Wakker (1989) Theorem IV.2.7. Without loss of generality let 1 be a nonnull state and set $u(x) = u_1(x)$, $x \in X$. Then, by Lemma 3 and Theorem 2, for every nonnull $s \in S$, $u_s(x) = \sigma_s u(x) + \xi_s$, where $\sigma_s > 0$. Hence, by Theorem 2, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$,

$$\mathbf{x} \succcurlyeq \mathbf{y} \quad \Leftrightarrow \quad \sum_{s\in S} \pi(s)\, \sigma_s u(x_s) \geq \sum_{s\in S} \pi(s)\, \sigma_s u(y_s). \tag{45}$$

and, for every $\mathbf{x}_E, \mathbf{y}_A \in \mathbb{X}$,

$$\mathbf{x}_E \succcurlyeq \mathbf{y}_A \Leftrightarrow \sum_{s\in E} \pi(s \mid E)\left[\sigma_s u(x_s) + \xi_s\right] \geq \sum_{s\in A} \pi(s \mid A)\left[\sigma_s u(y_s) + \xi_s\right]. \tag{46}$$

The proof that (ii) implies (i) is follows immediately from the representation.

The proofs of the part (b) is straight forward. The proof of part (c) follow from the proof of part (c) in Theorem 2. In particular, $\left(\mathbf{x}^{-1}, \bar{x}\right)_{\{1\}} \succ \left(\mathbf{x}^{-1}, \underline{x}\right)_{\{1\}}$. Hence $u(\bar{x}) - u(\underline{x}) > 0$ and, by (A.4), for all $s \in S$, $(\mathbf{x}^{-s}, \bar{x})_{\{s\}} \succ (\mathbf{x}^{-s}, \underline{x})_{\{s\}}$. Hence $u_s(\bar{x}) - u_s(\underline{x}) = \sigma_s\left[u(\bar{x}) - u(\underline{x})\right] > 0$. Thus, $\sigma_s > 0$ for all $s \in S$. □

D. Proof of Theorem 5 - $(i) \Rightarrow (ii)$. Suppose that $\hat{\succcurlyeq}$ is a continuous weak order satisfying (A.1') and that every action in $\mathbf{A}$ is implementable at $\left(a^0, \hat{\mathbf{x}}\right)$. By Theorem III.4.1 of Wakker (1989), for every $a^0 \in \mathbf{A}^0$ there exist jointly cardinal continuous additive value functions $\{\hat{w}_S\left(\cdot; s, a^0\right)\}_{s\in S}$ that represent $\hat{\succcurlyeq}$ on $\{a^0\} \times \mathbf{X}$. Furthermore, (A.1') implies that, for all $s \in S$ and $a^0 \in \mathbf{A}^0$, $\hat{w}_S\left(\cdot; s, a^0\right) = \hat{w}_S(\cdot; s) + v\left(a^0\right)$. Using the uniqueness of the jointly cardinal representation normalize $\{\hat{w}_S(\cdot; s) + v\left(a^0\right)\}_{s\in S}$ as follows: Set $\hat{w}_S(\hat{x}_s; s) = 0$ for all $s \in S$, $v\left(a^0\right) = 0$.

Fix $E \in \mathcal{E}$ and $a \in F^{-1}(E)$ and let $\left(a^0, \hat{\mathbf{x}}_E \mathbf{z}(a)\right) \hat{\sim} (a, \hat{\mathbf{x}}_E)$. (Such $\mathbf{z}(a) := \mathbf{z}\left(a; \left(a^0, \hat{\mathbf{x}}\right)\right)$ exist since all actions are implementable at $\left(a^0, \hat{\mathbf{x}}\right)$.) Define $v(a) = -\sum_{s\in S-E} \hat{w}_S(z_s(a); s)$ and set $\hat{w}_E(\hat{x}_s; s) = 0$ for all $s \in E$. Hence (i) implies (4).

For each $s \in E$ and $x_s \in X_s$ let $x'_s(x_s, a) \subset X_s$ be defined by:

$$\left(a^0, \left(\hat{\mathbf{x}}^{-s}, x_s\right)_E \mathbf{z}(a)\right) \hat{\sim} \left(a, \left(\hat{\mathbf{x}}^{-s}, x'_s(x_s, a)\right)_E\right).$$

Set $\hat{w}_E(y_s; s) = \hat{w}_S(x_s, s)$ for all $y_s \in x'_s(x_s; a)$, $s \in S$. (If the $\cup_{x\in X_s} x'_s(x; a) \not\supseteq X_s$ extend the correspondence $x'_s(\cdot; a)$ it as follows: For $q_s \in X_s - \cup_{x\in X_s} x'_s(x; a)$

take $\mathbf{x} \in \mathbf{X}$ such that $\left(a^0, \mathbf{x}_E \mathbf{z}(a)\right) \hat{\sim} \left(a, \left(\hat{\mathbf{x}}^{-s}, q_s\right)_E\right)$ and define $\hat{w}_E(q_s; s) = \sum_{s \in E} \hat{w}_S(x_s; s)$.)

Let

$$\hat{W}_{F(a)}(\mathbf{x};a) = \sum_{s \in F(a)} \hat{w}_{F(a)}(x_s; s) + v(a). \tag{47}$$

Next we show that the functions $\{\hat{w}_{F(a)}(\cdot; s) + v(a) \mid s \in S, a \in \mathbf{A}\}$ constitute an additive-valued representation of $\hat{\succcurlyeq}$ on $\mathbb{A}$. Let $\left(a, \tilde{\mathbf{x}}_{F(a)}\right), \left(b, \tilde{\mathbf{y}}_{F(b)}\right) \in \mathbb{A}$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that $\tilde{x}_s \in x'_s(x_s; a)$ for all $s \in F(a)$ and $\tilde{y}_s \in x'_s(y_s; b)$ for all $s \in F(b)$. Then, by transitivity of $\hat{\succcurlyeq}$ and (A.1'), $\left(a, \tilde{\mathbf{x}}_{F(a)}\right) \hat{\sim} \left(a^0, \left(\mathbf{x}_{F(a)} \mathbf{z}(a)\right)\right)$ and $\left(b, \tilde{\mathbf{y}}_{F(b)}\right) \hat{\sim} \left(a^0, \left(\mathbf{y}_{F(b)} \mathbf{z}(b)\right)\right)$. By transitivity and the presentation, $\left(a, \tilde{\mathbf{x}}_{F(a)}\right) \hat{\succcurlyeq} \left(b, \tilde{\mathbf{y}}_{F(b)}\right)$ if and only if $\left(a^0, \left(\mathbf{x}_{F(a)} \mathbf{z}(a)\right)\right) \hat{\succcurlyeq} \left(a^0, \left(\mathbf{y}_{F(b)} \mathbf{z}(b)\right)\right)$ if and only if $\sum_{t \in F(a)} \hat{w}_S(x_t; t) + v(a) \geq \sum_{t \in F(b)} \hat{w}_S(y_t; t) + v(b)$. But, by definition, $\hat{w}_S(x_s; s) = \hat{w}_{F(a)}(\tilde{x}_s; s)$ for all $s \in F(a)$ and $\hat{w}_S(y_s; s) = \hat{w}_{F(b)}(\tilde{y}_s; s)$ for all $s \in F(b)$. Hence

$$\left(a, \tilde{\mathbf{x}}_{F(a)}\right) \hat{\succcurlyeq} \left(b, \tilde{\mathbf{y}}_{F(b)}\right) \Leftrightarrow \sum_{t \in F(a)} \hat{w}_{F(a)}(\tilde{x}_t; t) + v(a) \geq \sum_{t \in F(b)} w_{F(b)}(\tilde{y}_t; t) + v(b). \tag{48}$$

This completes the proof that $(i) \rightarrow (ii)$. The proof that (ii) implies that $\hat{\succcurlyeq}$ on $\mathbb{A}$ is a continuous weak order satisfying (A.1') that is immediate. To show that (ii) implies that all actions are implementable at $\left(a^0, \hat{\mathbf{x}}\right)$ note that

$$\sum_{s \in F(a)} \hat{w}_{F(a)}(\hat{x}_s; s) + v(a) = \sum_{s \in F(a)} \hat{w}_{F(a)}(\hat{x}_s; s) - \sum_{s \in S - F(a)} \hat{w}_S(z_s(a); s) + v\left(a^0\right). \tag{49}$$

Hence $(a, \hat{\mathbf{x}}) \hat{\sim} \left(a^0, \hat{\mathbf{x}}_{F(a)} \mathbf{z}(a)\right)$ and a is implementable at $\left(a^0, \hat{\mathbf{x}}\right)$. The proof of uniqueness is similar to the proof of uniqueness in Theorem 1. □

E. Proof of Theorem 6

The following lemma is analogous to Lemma 8.

Lemma 9 *Assume that there are at least three nonnull states and that the relation $\hat{\succcurlyeq}$ is a continuous weak-order on $\mathbb{A}$. Then the following conditions are equivalent:*

(i) $\hat{\succcurlyeq}$ *is a continuous weak order satisfying (A.1') and (A.3') and there exist $\left(a^0, \hat{\mathbf{x}}\right) \in \mathbf{A}^0 \times \mathbf{X}$ such that all actions are implementable at $\left(a^0, \hat{\mathbf{x}}\right)$.*

(ii) *For every* $a \in \mathbf{A}$ *and* $s \in F(a)$, $\hat{w}_{F(a)}(\cdot; s) = \beta_{F(a)} \hat{w}_S(\cdot; s) + \alpha_{F(a)}$, $\beta_{F(a)} \geq 0$, *where the functions* $\{\hat{w}_{F(a)}(\cdot; s) \mid a \in \mathbf{A},\ s \in F(a)\}$ *are as in Theorem 5.*

Proof of Lemma **9 -** $(i) \Rightarrow (ii)$. Suppose that (i) holds the, by Theorem 5, for every $t \in S$ there exist $\mathbf{w}, \mathbf{z} \in \mathbf{X}$ such that

$$\sum_{r \in S - \{t\}} [\hat{w}_S(w_r; r) - \hat{w}_S(z_r; r)] = \zeta > 0,$$

and, for every $a \in \mathbf{A}$ there exist $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that

$$\sum_{r \in F(a) - \{t\}} \left[\hat{w}_{F(a)}(x_r; r) - \hat{w}_{F(a)}(y_r; r)\right] = \varepsilon > 0.$$

By continuity of the functions $\hat{w}_{F(a)}(x_s; s)$ and the connectedness of the sets X_s, for every $\hat{\zeta} \in [-\zeta, \zeta]$, $\hat{\varepsilon} \in [-\varepsilon, \varepsilon]$, and $t \in F(a)$ there exist $\bar{\mathbf{w}}, \bar{\mathbf{z}} \in \mathbf{X}$ and $\left(a, \bar{\mathbf{x}}_{F(a)}\right), \left(a, \bar{\mathbf{y}}_{F(a)}\right) \in \{a\} \times \mathbb{X}$, such that

$$\sum_{r \in S - \{t\}} [\hat{w}_S(\bar{w}_r; r) - \hat{w}_S(\bar{z}_r; r)] = \hat{\zeta}$$

and

$$\sum_{r \in F(a) - \{t\}} \left[\hat{w}_{F(a)}(\bar{x}_r; r) - \hat{w}_{F(a)}(\bar{y}_r; r)\right] = \hat{\varepsilon}.$$

Define $\hat{\phi}_{F(a)}$ by $\hat{w}_{F(a)}(\cdot; \cdot) = \hat{\phi}_{F(a)} \circ \hat{w}_S(\cdot; \cdot)$. To show that $\hat{\phi}_{F(a)}$ is positive affine or constant function, let $W_t = \hat{w}_S(X_t; t)$. Then, by the connectedness of X_t and the continuity of $\hat{w}_S(X_t; t)$, W_t is an interval in $\mathbb{R}$. Take $\alpha, \beta, \gamma, \delta \in W_t$ such that $-\zeta \leq \alpha - \beta = \gamma - \delta \leq \zeta$ and $-\varepsilon \leq \hat{\phi}_{F(a)}(\alpha) - \hat{\phi}_{F(a)}(\beta) \leq \varepsilon$. Let $a_t, b_t, c_t, d_t \in X_t$ satisfy $\hat{w}_S(a_t; t) = \alpha$, $\hat{w}_S(b_t; t) = \beta$, $\hat{w}_S(c_t; t) = \gamma$ and $\hat{w}_S(d_t; t) = \delta$. Take $\hat{\mathbf{w}}, \hat{\mathbf{z}} \in \mathbf{X}$ such that

$$\sum_{r \in S - \{t\}} [\hat{w}_S(\hat{w}_r; r) - \hat{w}_S(\hat{z}_r; r)] = \alpha - \beta.$$

Then, by Theorem 5, $\left(a^0, \left(\hat{\mathbf{w}}^{-t}; a_t\right)\right) \tilde{\sim} \left(a^0, \left(\hat{\mathbf{z}}^{-t}; b_t\right)\right)$ and $\left(a^0, \left(\hat{\mathbf{w}}^{-t}; c_t\right)\right) \tilde{\sim} \left(a^0, \left(\hat{\mathbf{z}}^{-t}; d_t\right)\right)$.

Take $\left(a, \hat{\mathbf{x}}_{F(a)}\right), \left(a, \hat{\mathbf{y}}_{F(a)}\right) \in \{a\} \times \mathbb{X}$, such that

$$\sum_{r \in F(a) - \{t\}} \left[\hat{w}_{F(a)}(\hat{x}_r; r) - \hat{w}_{F(a)}(\hat{y}_r; r)\right] = \hat{\phi}_{F(a)}(\alpha) - \hat{\phi}_{F(a)}(\beta).$$

Since $\hat{w}_{F(a)}(\cdot;t) = \hat{\phi}_{F(a)} \circ \hat{w}_S(\cdot;t)$ this implies $\left(a, \left(\hat{\mathbf{x}}^{-t}; a_t\right)_{F(a)}\right) \hat{\sim} \left(a, \left(\hat{\mathbf{y}}^{-t}; b_t\right)_{F(a)}\right)$. Applying (A.3') twice yields $\left(a, \left(\hat{\mathbf{x}}^{-t}; c_t\right)_{F(a)}\right) \hat{\sim} \left(a, \left(\hat{\mathbf{y}}^{-t}; d_t\right)_{F(a)}\right)$. Thus

$$\hat{\phi}_{F(a)}(\gamma) - \hat{\phi}_{F(a)}(\delta) = \sum_{r \in F(a) - \{t\}} \left[\hat{w}_{F(a)}(\hat{x}_r; r) - \hat{w}_{F(a)}(\hat{y}_r; r)\right] = \hat{\phi}_{F(a)}(\alpha) - \hat{\phi}_{F(a)}(\beta).$$

By Wakker (1987) Lemma 4.4 this implies that $\hat{\phi}_{F(a)}$ is affine. But $\hat{\phi}_{F(a)}$ is nondecreasing. Hence it is constant or positive.

Since F may be many-to-one it is possible that there are $a, b \in \mathbf{A}$ such that $F(a) = F(b) = E$. Let $(a, \mathbf{x}_E) \hat{\succcurlyeq} (b, \mathbf{x}_E)$ then

$$(\beta(a, E) - \beta(b, E)) \sum_{s \in E} \hat{w}_S(x_s; s) + \alpha_{F(a)} - \alpha_{F(b)} + v(a) - v(b) \geq 0.$$

By (3) in Theorem 5, for all $\mathbf{y} \in \mathbf{X}$,

$$(\beta(a, E) - \beta(b, E)) \sum_{s \in E} \hat{w}_S(y_s; s) + \alpha_{F(a)} - \alpha_{F(b)} + v(a) - v(b) \geq 0.$$

Let $\hat{\mathbf{x}} \gg \mathbf{y}$ then $\sum_{s \in E} \hat{w}_S(y_s; s) < \sum_{s \in E} \hat{w}_S(\hat{x}_s; s) = 0$, where the last equality follows from Theorem 5 and the definition of $\left(a^0, \hat{\mathbf{x}}\right)$. Hence for sufficiently unfavorable payoff profile $\mathbf{y}$ this inequalities implies that $\beta(a, E) = \beta(b, E) = \beta(E) = \beta_{F(a)}$. Hence (i) implies (ii).

$(ii) \Rightarrow (i)$. The proof that (ii) implies (i) follows the same argument as in the proofs of Lemma 8 and of Theorem 5. □

Proof of Theorem 6 - (a) $(i) \Rightarrow (ii)$. Suppose that $\hat{\succcurlyeq}$ satisfies the assumptions in (i). Then, by Theorem 5, for all $a \in \mathbf{A}$, $\mathbf{x} \in \mathbf{X}$, $t \in F(a)$ and $x, x' \in X_t$,

$$\left(a, (\mathbf{x}^{-t}, x)_{F(a)}\right) \hat{\succcurlyeq} \left(a, (\mathbf{x}^{-t}, x')_{F(a)}\right) \quad \Leftrightarrow \quad \hat{w}_{F(a)}(x; t) \;\geq\; \hat{w}_{F(a)}\left(x'; t\right). \tag{50}$$

By Lemma 9, $\hat{w}_{F(a)}(x; t) = \beta_{F(a)} \hat{w}_S(x; t) + \alpha_{F(a)}$, $\beta_{F(a)} \geq 0$. Hence, for all $a \in F^{-1}(\{s\})$ $\hat{w}_{\{s\}}(\cdot; s) = \beta_{\{s\}} \hat{w}_S(x; t) + \alpha_{\{s\}}$. For each $s \in S$ let $u_s(\cdot) = \hat{w}_{\{s\}}(\cdot; s)$.and define $\pi(s) = \beta_{\{s\}}^{-1} / \sum_{t \in S} \beta_{\{t\}}^{-1}$ if $\beta_{\{s\}} > 0$ and $\pi(s) = 0$ otherwise. Then $\hat{w}_S(x; t) = \pi(s) \left[u_s(x) - \alpha_{\{s\}}\right] \sum_{t \in S} \beta_{\{t\}}^{-1}$ and, by Theorem 5, for all $\left(a^0, \mathbf{x}\right), \left(a^0, \mathbf{y}\right) \in \{a^0\} \times \mathbf{X}$,

$$\left(a^0,\mathbf{x}\right) \hat{\succsim} \left(a^0,\mathbf{y}\right) \Leftrightarrow \sum_{s\in S} \pi\left(s\right) u_s\left(x_s\right) \geq \sum_{s\in S} \pi\left(s\right) u_s\left(y_s\right). \tag{51}$$

Moreover, by Theorem 5, for all $\left(a,\mathbf{x}_{F(a)}\right), \left(b,\mathbf{y}_{F(b)}\right) \in \mathbb{A}$, $\left(a,\mathbf{x}_{F(a)}\right) \hat{\succsim} \left(b,\mathbf{y}_{F(b)}\right)$ if and only if

$$\beta_{F(a)} \sum_{s\in F(a)} \pi\left(s\right) u_s\left(x_s\right) + \kappa\left(a\right) + v\left(a\right) \geq \beta_{F(b)} \sum_{s\in F(b)} \pi\left(s\right) u_s\left(y_s\right) + \kappa\left(b\right) + v\left(b\right), \tag{52}$$

where $\kappa\left(a\right) = -\sum_{s\in F(a)} \pi\left(s\right) \alpha_{\{s\}} \sum_{t\in S} \beta^{-1}_{\{t\}} + \mid F\left(a\right) \mid \alpha_{F(a)}$, $a \in \mathbf{A}$.

Let $\mathbf{x}^*, \mathbf{x}^{**} \in \mathbf{X}\left(\mathbf{a}\right)$ be CVPP. Then $u_s\left(x^{**}_s\right) - u_s\left(x^*_s\right) = u_t\left(x^{**}_t\right) - u_t\left(x^*_t\right)$ for all $s,t \in S$. By (A.3') $\left(a,\mathbf{x}^*_{F(a)}\right) \hat{\sim} \left(b,\mathbf{x}^*_{F(b)}\right)$ if and only if $\left(a,\mathbf{x}^{**}_{F(a)}\left(a\right)\right) \hat{\sim} \left(b,\mathbf{x}^{**}_{F(b)}\left(b\right)\right)$. Hence, by equation (52),

$$\beta_{F(a)} \sum_{s\in F(a)} \pi\left(s\right) \left[u_s\left(x^{**}_s\right) - u_s\left(x^*_s\right)\right] = \beta_{F(b)} \sum_{s\in F(b)} \pi\left(s\right) \left[u_s\left(x^{**}_s\right) - u_s\left(x^*_s\right)\right]. \tag{53}$$

Thus

$$\left[u_s\left(x^{**}_s\right) - u_s\left(x^*_s\right)\right] \left[\beta_{F(a)} \sum_{s\in F(a)} \pi\left(s\right) - \beta_{F(b)} \sum_{s\in F(b)} \pi\left(s\right)\right] = 0. \tag{54}$$

Without loss of generality assume that $\left(a,\mathbf{x}^{**}_{F(a)}\right) \hat{\succ} \left(a,\mathbf{x}^*_{F(a)}\right)$ then $\left[u_s\left(x^{**}_s\right) - u_s\left(x^*_s\right)\right] > 0$. Equations (53) and (54) imply that $\beta_{F(a)} \sum_{s\in F(a)} \pi\left(s\right) = \beta_{F(b)} \sum_{s\in F(b)} \pi\left(s\right)$. Let $b \in \mathbf{A}^0$ (i.e., b is a default action). Because $\beta_S = 1 = \sum_{s\in S} \pi\left(s\right)$ the preceding argument implies that, for all $a \in \mathbf{A}$, $\beta_{F(a)} \sum_{s\in F(a)} \pi\left(s\right) = 1$. Define $\pi\left(s \mid F\left(a\right)\right) = \pi\left(s\right) / \sum_{s\in F(a)} \pi\left(s\right)$.

Next observe that since all actions are implementable at $\left(a^0,\hat{\mathbf{x}}\right)$, for all $a \in \mathbf{A}$, $\left(a^0,\hat{\mathbf{x}}_{F(a)}\mathbf{z}\left(a\right)\right) \hat{\sim} \left(a,\hat{\mathbf{x}}_{F(a)}\right)$. Invoking the fact that $u_s\left(\hat{x}_s\right) = \kappa\left(a^0\right) = v\left(a^0\right) = 0$, the representation in equation (52) implies: $\kappa\left(a\right) + v\left(a\right) = \sum_{s\in S-F(s)} \pi\left(s\right) u_s\left(z_s\left(a\right)\right)$. But $v\left(a\right) = \sum_{s\in S-F(s)} \pi\left(s\right) u_s\left(z_s\left(a\right)\right)$ hence $\kappa\left(a\right) = 0$. Thus equation (52) imply that, for all $\left(a,\mathbf{x}_{F(a)}\right), \left(b,\mathbf{x}_{F(b)}\right) \in \mathbf{A}\times\mathbb{X}$, $\left(a,\mathbf{x}_{F(a)}\right) \hat{\succsim} \left(b,\mathbf{x}_{F(b)}\right)$ if and only if

$$\sum_{s\in F(a)} \pi\left(s \mid F\left(a\right)\right) u_s\left(x_s;a\right) + v\left(a\right) \geq \sum_{s\in F(b)} \pi\left(s \mid F\left(b\right)\right) u_s\left(x_s;b\right) + v\left(b\right). \tag{55}$$

This completes the proof that $(i) \Rightarrow (ii)$.

The proof that $(ii) \Rightarrow (i)$ follows by the same arguments as the proof of the same implications in Theorem 2. The proof of parts (b) and (c) follows by the corresponding arguments in the proof of Theorem 2. $\square$

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