Network Games

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Abstract. In a variety of contexts – ranging from public goods to job search to information collection – a person's well being depends on her own action as well as on the actions taken by her neighbors, i.e., those in close proximity. The current paper develops a general theoretical framework to analyze such strategic interactions when neighborhood structure, modeled in terms of an underlying network of connections, affects payoffs. Our goal is to understand how location within a fixed network, as well as changes in the overall network, affect individual behavior and social outcomes.

The framework we propose has two distinctive features: we allow for a fairly general class of payoffs, and we allow for incomplete information regarding the underlying network structure.

Our analysis provides an array of results characterizing how the nature of games (strategic substitutes vs complements and positive vs negative externalities), and the level of information (incomplete vs complete) shape individual behavior and payoffs.

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1. Introduction

In a variety of contexts – ranging from public goods to job search to information collection – a person's well being depends on her own action as well as on the actions taken by those in close proximity, i.e., her neighbors.[1] The pattern of neighborhoods is often formalized in terms of a network of relations. We wish to understand how individual behavior varies with location within a network as well as how changes in the network structure – increasing the number of connections or redistributing connections – affects individual behavior and welfare.

The paper develops a general framework to address these questions. There are two distinctive features of this framework. First, we allow for a fairly general class of payoffs. Second, we allow for the possibility of incomplete information about the underlying network structure.

In a network game, individual incentives depend on the actions of her neighbors since they alter the marginal returns to own actions. The nature of this neighborhood effect has two aspects: the first aspect is whether a game exhibits strategic substitutes or complements. This will determine how a change in a neighbor's actions affects incentives for own actions. The second and equally important aspect of the neighborhood effect is the expectations concerning the magnitude of neighbors' actions. Indeed, a neighbor's action depends on her neighbors, her neighbors' neighbors, and so on. Therefore, information on network structure plays an important role in shaping optimal individual actions. In reality, networks are complicated objects and it is reasonable to suppose that an individual will only have imperfect knowledge of the details of the structure. This observation motivates a study of the role of incomplete information in network games.[2]

We study information issues in terms of each player's knowledge of the number of her own and others' connections, i.e., the degrees, in the network. Suppose that $P(k)$ is the

[1] For empirical work on network effects see Coleman (1966, 1994), Conley and Udry (2004), Granovetter (1994), Topa (2001), and Glaeser, Sacerdote, and Scheinkman (1996), among others.

[2] There is also some empirical work which supports this assumption see, e.g., Kumbasar, Romney and Batchelder (1994), Bondonio (1998), and Casciaro (1998).

probability that a node has degree k in a network. The analysis in this paper concentrates on two polar cases: *Incomplete Information (II)* pertains to an environment in which each player knows the degree distribution $P(k)$ and how many neighbors she herself has (i.e., her degree). In contrast, under *Complete Information (CI)* players possess complete knowledge of the prevailing network.

When information is complete, we focus on the Nash equilibria of the induced network game; on the other hand, if players' information is partial, we model the situation using Harsanyi's apparatus and look for a Bayesian Nash Equilibrium of the induced game of incomplete information. In this case players' types are identified with their degree.[3]

Our *first* set of results, Propositions 3.1-3.2 relate to a comparison of equilibrium actions and payoffs of players who differ in their location in a given network. We show that under incomplete information, in every (symmetric) equilibrium,[4] actions are increasing (decreasing) in player degree if payoffs satisfy strategic complements (substitutes). We also show that, in every equilibrium, payoffs are increasing (decreasing) in degree in games with positive (negative) externalities. These results help us in understanding how the nature of strategic game determines whether well connected players work harder or free ride on their less connected cohort and if having more connections is good or bad for personal payoffs.

We then examine the role of limited network information in obtaining the above results. We find that if players have greater information about the network – e.g., they know their own degree and also the degree of their neighbors or they have complete information about the network – then they can use details of the network to condition their behavior and this leads to the possibility of multiple equilibria, some of which satisfy the above monotonicity properties while others may not satisfy them.

The *second* set of results, Propositions 4.1-4.5, are about effects of adding links in a network. In the case of complete information this can be done directly, while in the case of incomplete information this is done using first order stochastic dominance (FOSD) rela-

[3]Intermediate cases in which players are informed of a wider circle of neighbors yield similar qualitative results as those of the II case, though involve some technical complications discussed in Section 6.

[4]Existence of such equilibrium is established by Proposition 2.1.

tions across degree distributions. Generally speaking, adding links, or altering the degree distribution in a FOSD sense, has a positive effect on network games with complementarities, regardless of the information structure. For any equilibrium in an initial network there exists an equilibrium in the denser network which has higher actions. In games with positive externalities and with complete information, adding links generates higher welfare. In the case of incomplete information, payoffs of every degree player increase under a FOSD shift.

Games with strategic substitutes are harder to analyze as the effects of added links work in conflicting directions: given that equilibrium is monotonically decreasing in actions, an increase in probability of higher degree neighbors means a lower average action. However, in a game with strategic substitutes this implies that a player should increase actions. An equilibrium must balance these two forces: a lower average action from neighbors along with a higher action from any degree player. These complications lead us to restrict attention to two classes of games: binary action games[5] and continuous action games with quadratic linear payoffs. Under incomplete information, symmetric equilibrium actions of each degree player increase while the expected average action of a neighbor falls with FOSD shifts in degree distribution. Welfare implications for each degree player follow directly: in games with positive externalities, if expected action of each neighbor falls then expected payoff falls as well.

The *third* set of results, Propositions 5.1-5.2 relate to the effects of increasing heterogeneity across players in a network. The analysis here is complicated since increasing heterogeneity leads (roughly speaking) to an increase in low and high degree neighbors and the effects of this on individual incentives depend on the curvature of the payoff function and the details of the shift in degree distribution. To make progress, we are obliged to consider binary action games and also to concentrate on changes in the underlying degree distribution that shift weight to the extreme degree values, while preserving the mean degree (a strong version of

[5] In a binary action game, equilibria take on a particularly simple structure. There is a threshold degree $\hat{t}$ with all degrees $t < \hat{t}$ choosing action 1 (0) and all degrees $t > \hat{t}$ choosing action 0 (1) in strategic substitute (complement) games.

mean preserving spread). We find that when comparing equilibria across networks, the type of strategic game plays a significant role. In particular, in games with complements, with low thresholds, a strong mean preserving spread leads to an increase in threshold along with a corresponding fall in the probability of an arbitrary neighbor choosing the action 1. By contrast, in games with strategic substitutes such a mean preserving spread leads to a fall in threshold along with an increase in the probability of any random neighbor choosing the action 1. Thus, in games with complements (substitutes) the payoff of every degree player falls (rises) under a strong mean preserving spread of the degree distribution.

We now place the paper in the context of the literature. The main contribution of the current paper lies in the development and analysis of a general framework to study the effects of social interactions on individual behavior. Three aspects of the framework: the general nature of payoffs, the general network structure, and the allowance for varying levels of information are worth emphasizing and contrasting with the extant literature in the field. Almost all the existing work on network games to date – see, e.g., Ballester, Calvó-Armengol, and Zenou (2005), Bramoullé and Kranton (2005), Galeotti (2005), Galeotti and Vega-Redondo (2005), Goyal and Moraga-Gonzalez (2001) – has assumed complete information and worked with specific formulations both with regard to payoff functions and with regard to the network structures.[6] We now extend the discussion regarding Bramoullé and Kranton (2005) and Galeotti and Vega-Redondo (2005) as they help clarify the scope of our paper.

Bramoullé and Kranton (2005) consider a game where players search for valuable information and information is freely shared among neighbors. Players' utilities depend on a sum of their own efforts and efforts of neighbors. They assume that efforts of players are strategic substitutes and that each player has complete network information. They find that there is multiplicity of equilibria and that the comparative statics within and across networks are ambiguous. By contrast, Propositions 3.1-3.2 show that if information is incomplete then equilibria are monotone in actions and payoffs. Moreover, Proposition 4.4

[6] In particular, regular networks (in which all players have the same degree) and core-periphery structures (the star network is a special case of such structures) have been extensively explored in the literature.

shows that the effects of adding links (in a binary version of their game) are clear cut: every degree player chooses actions with greater probability, receives lower expected externalities from their neighbors, and earns lower payoffs. These results highlight the important role of network information in shaping behavior.

Galeotti and Vega-Redondo (2005) analyze a network game in which payoffs are a product of neighbors' actions and own action and players have incomplete information about the architecture of the network. We note that their payoff function violates the assumption on payoffs we make: We allow for all payoff functions that respect the following property: the payoff of a degree $k+1$ player facing a profile of actions $\{x_1, x_2, ..., x_k, 0\}$ is the same as the payoff of a degree k player who faces the profile $\{x_1, x_2, ..., x_k\}$. Our assumption facilitates welfare comparisons and is satisfied by most of the models that have been studied to date. However, it is not satisfied if payoffs are a product of actions of neighbors (as in the Galeotti and Vega-Redondo paper) or if they depend on the average of neighbors' actions.

The paper also relates to a strand of papers in the computer science literature on *graphical games* (see, e.g., Kearns, Littman, and Singh, 2001, and Kakade, Kearns, Langford, and Ortiz, 2003). While the underlying model tackled in that literature is very close to ours, the focus is very different. The literature in computer science is concerned with efficient algorithms for finding Nash equilibria. Our results are complementary in that they provide a characterization of the equilibria that these algorithms ultimately reach.

The rest of the paper is organized as follows: Section 2 develops the theoretical model. Section 3 presents results on equilibrium behavior as a function of location within a network. Sections 4 and 5 examine the effects of changing networks on equilibrium behavior; Section 6 considers the case of intermediate levels of knowledge between incomplete and complete information, while Section 6 concludes. Most of the proofs are relegated to an Appendix.

2. The General Model

This section presents the main elements of our theoretical framework: the network relations between players, the nature of strategies and payoffs, the information a player has about the network relations, and the equilibrium concepts.

Networks: The connections between a finite set of players $N = \{1, \dots n\}$ are described by an undirected network. That network is represented by a symmetric matrix $g \in \{0,1\}^{n \times n}$, with $g_{ij} = 1$ denoting that i and j are connected. We choose the convention that $g_{ii} = 0$ for all i. Let G denote the set of all possible non-directed networks with n vertices.

Let $N_i(g) = \{j | g_{ij} = 1\}$ denote the set of direct neighbors of i. For any integer $k \geq 1$, let $N_i^k(g)$ be the *k-neighborhood* of i in g; that is, all the players that can be reached from i by paths of length no more than k. So, inductively $N_i^1 = N_i$ and $N^k = N_i^{k-1} \cup (\cup_{j \in N_i^{k-1}} N_j)$. The *degree*, $k_i(g)$, of player i is the number of i's direct connections:

$$k_i(g) = |N_i(g)|.$$

We occasionally use $\bar{k}$ to denote the maximum degree in a network. We denote the degree distribution of the network by P, where $P(k)$ is the frequency of nodes with degree k.[7] Note that the induced distribution of neighbors' degrees for any particular node is given by

$$\tilde{P}(k) = \frac{kP(k)}{\langle k \rangle}, \tag{1}$$

where $\langle k \rangle = E_P[k]$ is the average degree in the network. This is the standard formalization of the idea that a randomly chosen link is likely to point to a node of a certain degree in proportion to that node's degree. This distribution is only applicable in settings where there is no correlation between the degree of nodes, which we assume in some of the results that follow.

We study the effects of changing the network on the behavior of players. The differences in degree distributions are expressed in terms of first order stochastic dominance (FOSD), second order stochastic dominance (SOSD), and mean preserving spread (MPS) relations.

[7] In this paper, we focus on undirected networks; in some applications such as learning from other's actions, it is possible that player i observes j but the converse is not true. Directed networks are more appropriate for such applications. We conjecture that most of the arguments we develop in this paper will carry over to the setting of directed networks under appropriate independence assumptions on the distribution of links.

Strategies and Payoffs: Each player i takes an action x_i in X, where X is a compact subset of $[0,1]$. Without loss of generality, we assume throughout that $0,1 \in X$. We consider both discrete and connected action sets X. The payoff of player i when the profile of actions is $x = (x_1, ..., x_n)$ is given by:

$$u_i(x, g) = v_{k_i(g)}(x_i, x_{N_i(g)})$$

where $x_{N_i(g)}$ is the vector of actions taken by the neighbors of i.

So, the payoff of a player depends on her own action, as well as on the actions that her direct neighbors take. Note also that the payoff function depends on the player's degree, but not on his or her identity, so that two players who have the same degree have the same payoff function. We also assume that v_k depends on the vector $x_{N_i(g)}$ in an anonymous way, so that if x' is a permutation of x (both k-dimensional vectors) then $v_k(x_i, x) = v_k(x_i, x')$ for any x_i. If X is a connected action set then we assume that v_k is differentiable in all arguments and concave in own action.

In order to make comparisons of actions and payoffs across players of different degrees, we make the following assumption, Assumption A, for any x_i and k-dimensional vector x:

Assumption A $v_{k+1}(x_i, (x, 0)) = v_k(x_i, x)$.

Thus adding a link to a neighbor who chooses action 0 is similar to not having an additional neighbor.

This assumption is useful in drawing welfare conclusions. However, it is an important restriction, as it implies that adding new neighbors is akin to increasing the action that a player perceives being played by neighbors. Assumption A seems appropriate in situations such as information sharing where it is the total information gathered by neighbors and shared with a player that is important. This assumption also seems to be appropriate in the context of adoption of technologies with local network effects: for instance, when contemplating a switch to a new computer editing technology, a player's payoff may reasonably be affected by the number of her neighbors who choose that technology (see Morris, 2000, and Jackson and Yariv, 2005, for an extended analysis of such setups).

However, if a player cares about the average action of his or her neighbors, rather than the absolute levels, then this condition is violated. Similarly, this assumption is also violated if payoffs are a product of the actions of neighbors, as in Galeotti and Vega-Redondo (2005).[8] In our view neither case is the "right" one; rather they should be seen as capturing different applications.

We study how individual behavior depends on the type of game that players are engaged in. We shall say that a game exhibits *strategic complements* if it satisfies increasing differences. That is, for all k, $x_i > x_i'$, and $x \geq x'$: $v_k(x_i, x) - v_k(x_i', x) \geq v_k(x_i, x') - v_k(x_i', x')$. Similarly, the game is said to exhibit *strategic substitutes* if it satisfies decreasing differences. That is, for all k, $x_i > x_i'$, and $x \geq x'$: $v_k(x_i, x) - v_k(x_i', x) \leq v_k(x_i, x') - v_k(x_i', x')$. These conditions are strict if the payoff inequalities are strict whenever $x \neq x'$.

We say that a game exhibits *positive externalities* if for each v_k, and for all $x \geq x'$, $v_k(x_i, x) \geq v_k(x_i, x')$, and *negative externalities* if $v_k(x_i, x) \leq v_k(x_i, x')$. The game exhibits *strict* externalities (positive or negative) if the corresponding payoff inequalities are strict whenever $x \neq x'$.

The following applied example illustrates the scope of Assumption A as well as clarifies different kinds of externalities that arise.

Example 2.1. *Information sharing as a local public good (Bramoullé and Kranton, 2005)*

Players search for lowest prices and the price quotations are shared with immediate neighbors. The benefits to each player depends on her own efforts and the search efforts of her direct neighbors. Suppose each player receives benefit from search according to a function $f(\cdot)$ where $f(0) = 0$, $f' > 0$ and $f'' < 0$. In particular, the payoffs to a player i in a network with k neighbors are given by

[8] They use the payoff function: $u_i(x_i, x) = x_i \prod_{j \in N_i} x_j - \alpha \frac{x_i^2}{2}$, where $x_j \in \mathbb{R}_+$ for all j.

$$v_k\left(x_i, \sum_{j=1}^{k} x_j\right) = f(x_i + \lambda \sum_{j=1}^{k} x_j) - cx_i \tag{2}$$

where $c > 0$ is the marginal cost of effort. It is easy to check that under the assumptions on $f(\cdot)$, this game satisfies Assumption A and the (strict) strategic substitutes property. Moreover, v_k in an increasing (decreasing) function of $\sum_{j=1}^{k} x_j$ if λ is positive (negative).[9] The following two special cases of this example will run through the paper:

2.1.I Quadratic form: $v_k(x_i, \sum_{j=1}^{k} x_j) = \frac{1}{4}(1 - \bar{c} + x_i - \sum_{j=1}^{k} x_j)^2 - \alpha\frac{x_i^2}{2}$;

2.1.II Best shot public goods: here $X = \{0, 1\}$, where we interpret 1 as acquiring information and 0 as not acquiring information. Assume further that $f(0) = 0$, $f(x) = 1$ for all $x \geq 1$ and $0 = c(0) < c(1) < 1$ (so that acquiring 1 piece of information suffices).

The Best-Shot game is a good metaphor for many situations in which there are significant spill-overs between players' actions. For instance, consumers learn from relatives and friends (Feick and Price, 1987), in research and development, innovations often get transmitted between firms, and similarly in agriculture, experimentation is often shared amongst farmers (Foster and Rosenzweig, 1995, Conley and Udry, 2004).

More generally, several economic applications that have been previously studied – such as collaboration among local monopolies studied by Goyal and Moraga-Gonzalez (2001), human capital investment and other social decisions (the decisions to smoke, buy some product, etc.) as modeled in Calvo- Armengol and Jackson (2004, 2005ab) and coordination games in networks as studied in Chwe (2000) and Ellison (1993)) – all satisfy Assumption A, and exhibit strategic complements. Crime networks explored in Ballester, Calvó-Armengol, and Zenou (2005), also satisfy Assumption A and exhibit strategic substitutes or complements, depending on values of the parameters.

Information Structures: In understanding behavior, it is important to keep track of what a player knows about the network in which she resides. At one extreme a player knows

[9] Bramoullé and Kranton (2005) study the case where $\lambda = 1$.

the entire network – her own degrees, the degrees of her neighbors, and the degrees of the neighbors of her neighbors and so on. On the other extreme she knows nothing more than the degree distribution of the network. In between there is a whole range of knowledge levels, which can be parameterized by a number $d \geq 0$, which reflects how far players' knowledge reaches into the network. For concreteness, we shall focus on two information structures that already raise many of the interesting considerations.

1. *Incomplete Information (II)* Each player knows the degree distribution $P(\cdot)$ and her own degree. This corresponds to $d = 1$.

2. *Complete Information (CI)* Every player knows the entire network structure, including the degree of each of her neighbors, that of her neighbors' neighbors and so on. This corresponds to $d \geq n - 1$.

In Section 6 we come back to the cases of $d = 0$ and $1 < d < n - 1$.

Under the Incomplete Information case, all players believe that the prevailing network g has been drawn stochastically from a family of networks G so that the following two properties are satisfied:

(A) The probability of any given node having k links in g is $P(k)$.

(B) The degrees $k_i(g)$ and $k_j(g)$ displayed by *any* two players i and j (even if they happen to be neighbors) are stochastically independent.

In order for a mechanism to exist that guarantees (A) and (B), we must take $n \to \infty$. An example of some such mechanism is provided by so-called configuration model in the theory of random networks, e.g. Bender and Canfield (1978) or Bollobás (1980). For example, if there are just three nodes, then it is possible for one of them to have degree 2 or three of them to have degree 2, but it is impossible for exactly two of the nodes to have degree 2. This necessitates some correlation in degree. In this sense, all of our results may be interpreted either as requiring some bounds on rationality of the players' beliefs, or as holding approximately for a large enough population.[10]

The strategic implications of the two information structures we consider can be analyzed within the usual Harasanyi framework by a suitable specification of the *type spaces* of players, $\mathcal{T}_i$. That is, given the type $t_i \in \mathcal{T}_i$ revealed to any given player i, her beliefs are simply obtained as the posterior induced by such t_i and the prior satisfying (A) and (B) above. More precisely, the two information structures correspond to the following specifications:

II: $\mathcal{T}_i = \{0, 1, ..., n-1\}$ for all i, and the type $t_i(g)$ revealed to i when any given g prevails is $t_i(g) = k_i(g)$.

CI $\mathcal{T}_i = G$ for all i, and the type $t_i(g)$ revealed to i when any given g prevails is $t_i(g) = g$.

Equilibrium A strategy of player i is a mapping $\sigma_i : \mathcal{T}_i \to \Delta(X)$, where $\Delta(X)$ is the set of distribution functions on X. A strategy profile is denoted by $\sigma = \{\sigma_1, \sigma_2, .., \sigma_n\}$. In the games we study, payoffs depend only on own actions and the actions of immediate neighbors. The expected payoffs to player i, facing a strategy profile $\sigma = \{\sigma_i, \sigma_{-i}\}$, can be written as:

$$U_i(\sigma, t_i) = \int_{g \in G} \left[\int_{x_i, x_{N_i(g)} \in X^{k_i(g)+1}} v_k(x_i, x_{N_i(g)}) d\sigma(g) \right] dP(g|t_i), \tag{3}$$

where $d\sigma$ is the measure on x induced when g is the realized network and players employ the profile of strategies σ.

Definition 2.1. *An* equilibrium *is a profile of strategies σ such that for all i, $t_i \in \mathcal{T}_i$, and $x_i \in X_i$,*

$$U_i(\sigma, t_i) \geq U_i(x_i, \sigma_{-i}, t_i). \tag{4}$$

[10]Nonetheless, we note that all the results we present for the case of strategic complements (strategic substitutes) hold when there is positive (negative) degree correlation.

In the case of incomplete information, we say that a profile of strategies is *symmetric* if $\sigma_i = \sigma_j$ for all i and j, and an equilibrium is said to be symmetric if it is in symmetric strategies.

We say that an equilibrium is in pure strategies if for every $i \in N$ and for every type $t_i \in \mathcal{T}_i$, $\sigma_i(t_i)$ places probability 1 on some element in X.

Proposition 2.1. *There exists an equilibrium in the network games defined above. In the case of incomplete information there always exists a symmetric equilibrium. If the game is of strategic complements, then it has an equilibrium in pure strategies (that can be chosen to be symmetric in the case of incomplete information).*

The proof of the first two claims is standard and omitted.[11] The proof for the case of strategic complements follows from arguments in Proposition 4.2 below.

3. Comparing Choices Within a Network

This section studies the way in which location within a network affects individual behavior. We first study the setting with incomplete information and derive a number of monotonicity results. In the case of strategic complements we show that in every symmetric equilibrium players with more neighbors choose higher actions, while in the case of strategic substitutes, the opposite pattern obtains and players with more neighbors choose a lower action. In a game with positive externalities, we show that in every symmetric equilibrium payoffs are increasing in degree of players, while the opposite pattern obtains in the case of negative externalities. We then turn to a consideration of complete information and examine how changing information alters equilibrium behavior and outcomes.

In the game with incomplete information, a symmetric strategy profile, say σ, is a function of degree, i.e. σ_k is a strategy of every player with degree k. We say that a symmetric strategy

[11] For example, existence of a symmetric equilibrium follows from Jackson, Simon, Swinkels and Zame (2002) – see remark (ii) following their Theorem 2, and note that the games here are a special case where communication is unnecessary as the outcome is single-valued.

profile $\{\sigma_k\}$ is *monotone increasing* whenever σ_k FOSD $\sigma_{k'}$, for each $k > k'$. A *monotone decreasing* symmetric strategy is defined analogously.

Proposition 3.1. *In a game of incomplete information every symmetric equilibrium is monotone increasing (decreasing) if the payoffs satisfy the strict strategic complements (substitutes) property.*

The strictness is important for the result. For instance, if players were completely indifferent between all actions, then non-monotone equilibria would clearly be possible.

The intuition of the proposition is as follows: consider the strategic complements case. Consider a player with degree $k+1$. Suppose all of her neighbors follow the symmetric equilibrium strategy, but her $(k+1)$'th neighbor chooses the minimal 0 action. Assumption A implies that her best response would be identical to the equilibrium best response of a degree k player. However, in any non-trivial equilibrium, the $(k+1)$'th neighbor would be choosing, on average, a positive action. Strict complementarities imply that our player best responds with strictly higher actions than her k degree peers. Analogous reasoning applies to the strategic substitutes case.

It is important to note that, in addition to conclusions about equilibrium play, we can also draw conclusions about welfare differences across players. The following result shows that equilibrium payoffs satisfy a monotonicity property in the incomplete information case.

Proposition 3.2. *In a game with incomplete information, every symmetric equilibrium has monotone increasing (decreasing) payoffs if payoffs exhibit positive (negative) externalities.*

We emphasize that under positive externalities, players with more neighbors earn higher payoffs irrespective of whether the game exhibits strategic complements or substitutes. Thus, as in the game with strategic substitutes higher degree players exert lower effort, and nevertheless earns a higher payoff. For this class of games, there is a clear advantage to being well connected and the above results may be interpreted as saying that well connected players exploit network connections to free ride on poorly connected peers.

The intuition behind Proposition 3.2 is as follows. Consider the case of positive externalities and look at a $k+1$ degree player. Suppose, as before, that all of her neighbors follow the symmetric equilibrium strategy, but her $(k+1)$'th neighbor chooses the minimal 0 action. Assumption A on payoffs implies that our player would be able to replicate the expected payoff of a k degree player by simply using the strategy of the degree k player. However, if there is a positive probability that the $(k+1)$'th neighbor chooses a positive action then positive externalities imply a higher expected payoff for our $k+1$ degree player. Thus, the $(k+1)$ degree player can assure herself an expected payoff which is at least as high as that of any k degree player.

Propositions 3.1 and 3.2 establish a clean relationship between degree and effort levels and payoffs for a broad class of games. These results are, however, in marked contrast to results obtained by Bramoullé and Kranton (2005) in the local public goods example. They find that there exist equilibria in which higher degree individuals choose higher actions and earn lower payoffs as compared to lower degree players.[12] The local public goods game satisfies strategic substitutes and positive externalities and so our results establish that actions are falling in degree while payoffs are increasing in degree in any equilibrium. We now examine the role of network information in explaining the difference in results.

The simplest way to see the important role of information is to examine equilibria in the best shot game presented in section 2. Consider a star network with, say, 5 players. In such a network there is an equilibrium in which the center (of degree 4) chooses 1 and all the spokes choose 0. In this equilibrium clearly the central player earns a lower payoff as well. Consider now a setting in which players are of either degree 1 or 4. Now, a degree 4 player expects the same level of average effort from any of her 4 neighbors as a degree 1 player expects from her single neighbor. Given that efforts are strategic substitutes, this would directly imply that any "central" player of degree 4 would choose a lower effort as compared to that chosen by the average degree 1 player.

[12]Our results also differ from the results of Galeotti and Vega-Redondo (2005); as discussed earlier in the paper, this is because their payoff function does not satisfy Assumption A.

Note that in the above 5 player network with complete information, there also exists an equilibrium in which the central player chooses 0 while all the peripheral players choose 1. This equilibrium exhibits the decreasing monotone action profile which is consistent with Proposition 3.1.

These observations allow us to make a general point: greater information on a network – such as the knowledge of neighbors' degrees – allows a player to condition behavior on finer details of the network and this will generally lead to the possibility of multiple equilibria, some of which will exhibit monotonicity properties as identified in Propositions 3.1 and 3.2, while others may not. In section 6 we explore intermediate levels of information between incomplete and complete information and show that there exist equilibria which satisfy action monotonicity on average.

4. Comparing Choices Across Networks: Adding Links

This section examines the effects of adding links in a network on individual behavior and social outcomes. In the case of complete information, we can proceed by actually adding links and asking how this alters equilibrium. In the incomplete information case, we work through first order stochastic dominance of the (conditional) degree distribution.

In the case of strategic complements, the intuition is straightforward. As we add links (or shift the degree distribution), players have (on average) more neighbors, leading to higher overall actions, and thus higher best responses. Thus, starting at one equilibrium, after we "add" links, we can find a new equilibrium with higher actions for all players. The case of strategic substitutes is more subtle. Adding a link to a network leads those players directly involved to lower their actions. However, this then leads their neighbors to raise their actions, which feeds back on their own actions, etc. Thus the new equilibrium will have to balance opposing forces. We show that in games where the action set is binary adding links, or taking a FOSD degree distribution, leads to higher equilibrium effort from every player but at the same time also leads to lower expected effort from each of the neighboring players. A similar result obtains in continuous action games when individual payoffs depend on expected effort of the neighbors (and do not depend on higher order moments of the effort distribution).

4.1. Strategic Complements. We start with the case of incomplete information. In this case the natural analogue of adding links to a network is to consider a degree distribution P which first order stochastically dominates another degree distribution another distribution P'. We compare equilibrium under P with equilibrium under P'.

In this connection it is worth noting the distinction between the degree distribution and conditional degree distribution. What is central to the results that follow is that, as the distribution shifts, a player perceives that his neighbors will have higher degrees in the sense of first order stochastic dominance. This shift corresponds to a shift in the *conditional* degree distribution and not the absolute degree distribution. We note that first order dominance in the degree distribution does not imply a similar relation for the conditional degree distribution.[13]

We say that an equilibrium σ dominates an equilibrium σ' if for every i and type t_i, $\sigma_i(t_i)$ FOSD $\sigma'_i(t_i)$.

Proposition 4.1. *Consider a game with incomplete information and strategic complements and suppose that $\tilde{P}$ FOSD $\tilde{P}'$. Then for every equilibrium σ' under P' there exists a symmetric equilibrium σ under P which dominates it.*

To get some intuition for this result consider the case of strict complements and a symmetric equilibrium σ' under P'. Proposition 3.1 assures us that this equilibrium is, in fact, monotone. In particular, as we shift weight to higher degree neighbors by switching to the degree distribution P any player's highest best response to the original equilibrium profile would be at least as high as the supremum of her original strategy's support. We can now iterate this best response procedure. Since the action set is compact, this process converges

[13] To see this, consider a degree distribution P' in which degree 2 or 10 arise probability 1/2 each and a distribution P in which degrees 8 or 10 arise with probability 1/2 each. Clearly P FOSD P'. Next consider the conditional degree distributions, $\tilde{P}$ and $\tilde{P}'$. Under $\tilde{P}'$, the probability that a neighbor has degree 10 is 5/6, while under $\tilde{P}$, the same probability is 5/9. Thus, $\tilde{P}$ does not FOSD $\tilde{P}'$. If $\tilde{P}$ FOSD $\tilde{P}'$, and average degree under P is higher than average degree under P', then P FOSD P'. However, as shown above, the converse does not hold.

and it is easy to see that the limit is a symmetric equilibrium which dominates the original one.

We next consider the complete information case. With complete information we are faced with a multiplicity of equilibria; the following proposition shows that these equilibria are nicely ordered.

Proposition 4.2. *In a complete information game with strategic complements, there exists a pure strategy Nash equilibrium and the set of pure strategy Nash equilibria forms a lattice.*

Proof: Start with the action profile $(0, 0, ..., 0)$. Consider each player's best response to this profile. The resulting profile of best responses must be weakly higher for all players. We can now iterate. At each step the best responses are weakly higher by the complementarity condition. Given compactness, we converge to some action profile which constitutes an equilibrium.

Given any two pure strategy equilibria, listed as vectors of actions, consider the point-wise maximum vector and the point-wise minimum vector. The profile of best responses to the point-wise maximum vector is at least as high as the point-wise maximum (again, given complementarities). Now iterate. We converge monotonically upwards to some point which is (weakly) above both of the pure strategy equilibria we started with. Similarly, for the minimum vector, there is a pure strategy equilibrium which is below both of the original equilibria. Thus, we have a lattice structure. ■

The next result derives a complete information analogue to Proposition 4.1 and shows that equilibria can be nicely ranked as we add links to a network. Let g be a network with $g_{i,j} = 0$ and define $g' = g + g_{i,j}$.

Proposition 4.3. *Consider a complete information game with strategic complements. For any pure strategy equilibrium σ under g there exists a pure strategy equilibrium under g' that dominates it. Moreover, if X is connected, the game is of strict strategic complements, and σ is interior, then there exists an equilibrium σ' under g' in which all players in the component of i and j play strictly higher actions.*

Proof: Consider the first statement. Fix an equilibrium σ under network g. If σ is an equilibrium under g' then we are done. Otherwise, consider best responses to σ; by strategic complements, there exists best responses which are weakly higher than the original profile. Iterations of best responses lead to weakly higher profiles at each iteration stage and, from compactness of X, converge. The limit strategy profile (say) σ' is an equilibrium under g'. This completes the proof of the first statement.

If X is connected, there are strict complementarities, and the starting equilibrium is interior, then each iteration leads to a strictly higher profile of actions, and in particular, repeating the iterations generates a strictly higher equilibrium profile under g'. ■

Taken together Propositions 4.1- 4.5 show that in games with strategic complements, adding links (or considering FOSD degree distributions), leads to higher equilibrium actions. We now examine the effects of adding links on equilibrium payoffs.

Welfare effects: For games with positive externalities the results are as follows. In the incomplete information case, we know that if $\tilde{P}$ FOSD $\tilde{P}'$, then for every equilibrium σ' under P' there exists an equilibrium σ under P in which players' actions are all higher. So if P FOSD P' and $\tilde{P}$ FOSD $\tilde{P}'$ then it follows that for every equilibrium payoff profile under P' there is a corresponding profile under P which dominates it. In the complete information case a similar argument allows us to obtain a domination result in payoffs when links are added to a network.

4.2. Strategic Substitutes. As was noted above, the analysis for games with strategic substitutes is more complicated. We start with a thorough analysis of the binary action game. Here we show that adding links (or changing the degree distribution in the sense of FOSD) leads to equilibria which dominate equilibria in the initial network. However, this increase is sustained with a fall in expected actions from each of the neighbors under the new network. We also show that these results hold in continuous action games if payoffs have the quadratic linear structure.

As usual we start with the case of incomplete information. We note that under strict substitutes, symmetric equilibria are unique and fully characterized by a threshold: there exists $t \in \{1, 2, ...\}$, such that all $t_i < t$ choose 1, all $t_i > t$ choose 0, and $t_i = t$ may mix between 0 and 1.

The following result shows that FOSD changes in (conditional) degree distributions have clear cut effects on equilibrium behavior in such games.

Proposition 4.4. *Consider an incomplete information network game with strict strategic substitutes and $X = \{0, 1\}$. There is a unique symmetric equilibrium characterized by a threshold. Furthermore, if $\tilde{P}$ FOSD $\tilde{P}'$, then the threshold is higher but the probability that any given neighbor chooses 1 is lower under P.*

The intuition behind this result is as follows: consider a symmetric threshold strategy with threshold t'. As the distribution of neighbors' degrees shifts, in the FOSD sense, each player believes that it is more likely that her neighbors will have a higher degree. This means that the neighbor is less likely to choose 1. Since the game is one with strategic substitutes, each player's incentives to choose 1 increase and the first part of the result follows. Of course, if ultimately the probability of each player choosing 1 goes up, then the incentives to choose 1 for each player are lower, which would generate an inconsistency. The second part of the result now follows.

Proposition 4.4 relies on the observation that the incentives of a player depend on the expected probability that each of her neighbors provides the public good. With this observation, we can extend Proposition 4.4 to specific continuous action games; namely, games in which the expected marginal payoff of a player can be written as a function of her own action and the average effort of each of her neighbors. Network games with quadratic linear utilities – an example is the local public goods example given in section 2 above – satisfy this property.

In this case the expected marginal payoff to a degree k player can be written as: $\alpha + \beta x_i(k) + \gamma k \tilde{x}$, where α, β, and γ are parameters, and $\tilde{x}$ is the expected action of each of the neighbors. Concavity in own actions and strategic substitutes imply that both $\beta < 0$ and

$\gamma < 0$. For an interior solution we require that $\alpha > 0$. With these observations we can now write, after some algebra, an interior equilibrium action for a degree k player as:

$$x_i(k) = \frac{-\alpha}{\beta}\left[\frac{k\beta}{\beta + \gamma\tilde{k}} + \frac{k\gamma(\tilde{k}-1)}{\beta + \gamma\tilde{k}} - (k-1)\right] \tag{5}$$

where $\tilde{k}$ is the conditional expected degree of a neighbor. It is now easily verified that if $\tilde{P}$ FOSD $\tilde{P}\prime$, then for every degree k equilibrium action levels under P are higher than effort levels under P' while the conditional expected action of every neighbor is lower under $\tilde{P}$.

Complete Information and the Best-Shot Game Once we move from incomplete to complete information, even in the $\{0,1\}$ case we no longer have unique equilibria. These can vary in complicated ways. In order to get some handle we look at a class of $\{0,1\}$ games known as "Best-Shot" games, which were mentioned in Section 2. In these games it is a best response to choose the action 0 if any neighbor chooses 1, and it is a best response to choose 1 if all neighbors choose 0.

As it turns out, equilibria of the best-shot game are closely linked with set theoretic objects termed independent sets. An *independent set* of a network g is a set $I \subseteq N$ such that for any $i, j \in I$, $g_{ij} \neq 1$, i.e., no two players in I are directly linked. A *maximal independent set* is an independent set that is not contained in any other independent set. The following Lemma is helpful.

Lemma 4.1. *(Bramoullé and Kranton, 2005) There is a one to one mapping between the pure strategy equilibria of a best-shot game and the set of maximal independent sets of g.*

To get some intuition for this result, consider a pure strategy equilibrium and pick any two players who choose the action 1. From the structure of the game, they cannot be neighbors. Furthermore, any player who chooses the action 0 must have at least one neighbor who chooses the action 1. In particular, the set of players who choose the action 1 is a maximal independent set of g.

What happens as we add links to a network? Consider the following simple example with three players. With no links, the only equilibrium is $(1,1,1)$. With one link (between the first two players), there are two equilibria $(1,0,1)$ and $(0,1,1)$. With two links (between 1 and 2 and 2 and 3), there are two equilibria $(1,0,1)$ and $(0,1,0)$. With three links there are three equilibria $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. So, for any equilibrium in some network, after adding links, we can find an equilibrium in which a subset of players chooses the action 1.

The Best-Shot game played on the empty network (no links) prescribes all players to choose the action 1 in the unique equilibrium, while the Best-Shot game played on the full network (in which every pair of players has a direct link) is characterized by a set of pure equilibria, each of which has precisely one player choosing the action 1. As it turns out, the above example generalizes and the set of equilibria transition nicely between these two extremes as links are added. Recall that g is a network with $g_{i,j} = 0$, while $g' = g + g_{i,j}$.

Proposition 4.5. *Consider the best-shot game with complete information. Consider any pure strategy equilibrium σ of g'. Either σ is an equilibrium under g, or there exists an equilibrium under g in which a strict superset of players chooses 1. Moreover, there are equilibria under g that are not equilibria when the link is added.*

Proposition 4.5 shows that in best-shot games, we can nicely order the equilibrium sets as we add links to a network. The set of players choosing 1 is "shrinking" according to a well-defined (partial) order.

Welfare: In the case of incomplete information, FOSD shifts in degree distributions have ambiguous effects on welfare. However, in some instances we can derive clear cut results. Suppose the network game exhibits positive externalities. On the one hand, if $\tilde{P}$ FOSD $\tilde{P}'$ we know that for each degree player the expected payoff is lower under P as compared to P' (this is because the probability that each neighbor chooses 1 decreases). If P' FOSD P then it follows from the above observation, and the fact that expected payoff is increasing in degree (for positive externality games, see Proposition 3.2), that welfare is lower under

P as compared to P'. However, if the ordering of degree distributions and conditional distributions does not have this pattern, then average payoffs can go either way.

In the case of complete information Proposition 4.5 allows us to infer welfare changes when we alter a network by adding links in the case of best shot games. The payoff of players increases as we add links (comparing equilibria as in the proposition).

5. Comparing Behavior Across Networks: Redistributing Links

We now turn to comparisons across networks where we make changes which alter the variance in degrees across players. In a setting with incomplete information such changes are modeled in terms of shifts in distribution in the sense of second order stochastic dominance shifts. When we make such shifts in distribution, we are changing the mix of players who are playing higher actions and those playing lower actions. The best response of players will depend on how strong these shifts are and on how concave or convex the payoff functions are. Thus, it is difficult to obtain general results without further assumptions on payoff functions.

We now turn to a discussion of the binary action game, where $X = \{0, 1\}$, for both the cases of complements and substitutes. We also assume strictness, and confine attention to the case of incomplete information.

Both for the case of strategic substitutes and strategic complements, symmetric equilibria can be represented as threshold equilibria.[14] In the case of strategic complements there can be multiple symmetric equilibria, while in the case of strategic substitutes there is a unique symmetric equilibrium.

The results here make use of the notion of strong mean preserving spread which is defined as follows.

Definition 5.1. *We say that P is a* strong MPS *of P' if they have the same mean and there exists L and H such that $P(k) \geq P'(k)$ if $k < L$ or $k > H$, and $P(k) \leq P'(k)$ otherwise.*

[14] In particular, for the strategic complements case, there is a threshold t such that all types $t' > t$, choose 1, while all types $t < t'$ choose 0. For the strategic substitutes case, as before, a threshold equilibrium is such that types above the threshold choose 0, and below it choose 1.

While strong MPS is commonly used in the literature, we note that it is indeed a stronger notion than that of standard MPS since if P strong MPS P', then P results from a shift of weight to the extremes by increasing the probabilities under P' of *all* sufficiently low and sufficiently high k and decreasing *all* the probabilities under P' of the remaining intermediate k.[15]

As it turns out, restricting attention to such consistent weight shifts to the extremes allows us to draw conclusions for both classes of games. In fact, for strategic complements,

Proposition 5.1. *Consider a game with strict strategic complements and* $X = \{0, 1\}$, *and let* $\tilde{P}$ *be a strong MPS of* $\tilde{P}'$, *with corresponding* L *and* H. *Consider a symmetric equilibrium under* P' *with threshold* $t' \leq L$. *Then there exists a new equilibrium under* P *with threshold* $t > t'$ *and there is a lower probability of any given neighbor choosing* 1. *Consider a symmetric equilibrium under* P' *with threshold* $t' \geq H$. *Then there exists a new equilibrium under* P *with threshold* $t < t'$ *and there is a higher probability of any given neighbor choosing* 1.

The analogous proposition for the case of strategic substitutes is given by:

Proposition 5.2. *Consider a game with strict strategic substitutes,* $X = \{0, 1\}$, *and let* $\tilde{P}$ *be a strong MPS of* $\tilde{P}'$, *with corresponding* L *and* H. *If the unique symmetric equilibrium threshold* $t' \leq L$, *then* $t < t'$ *but there is a higher probability of any given neighbor choosing* 1. *If the unique symmetric equilibrium threshold* $t' \geq H$, *then* $t > t'$ *but there is a lower probability of any given neighbor choosing* 1.

In the Appendix, we provide a proof of Proposition 5.2 only, as the proof of Proposition 5.1 follows precisely the same line of arguments.

The intuition is as follows: consider any $t' \leqslant L$. If we start with an equilibrium profile from the network game with underlying degree distribution P', it turns out that under P, the number of t''s neighbors who choose 1 goes up (in the sense of FOSD). With strategic

[15]As an example of P, P' such that P is a MPS of P', but P is not a strong MPS of P', consider the following: $P(1) = 0.35, P(10) = 0.13, P(100) = 0.271$, and $P(1000) = 0.249$, which is an MPS of P' defined by: $P'(1) = P'(10) = P'(100) = P'(1000) = 0.25$. Clearly, P is not a strong MPS of P'.

substitutes this makes t' more inclined to choose the action 0, and the threshold goes down. In order to maintain a lower threshold, it must be the case that overall, the probability that any given neighbor chooses 1 goes up.

We next discuss continuous action games with quadratic linear payoffs. In such games, the expected marginal returns are a function on own effort, the degree of the player and the average effort of each neighbor. In such games incentives do not depend on higher order moments of the degree distribution and so second order stochastic dominance shifts that are mean preserving do not have any impact on equilibrium actions.

Welfare: We conclude by discussing welfare effects of second order shifts in degree distribution. As before, consider the case of positive externalities (the results for negative externalities case are the converse). For any given degree, lower probability of neighbors choosing the action 1 leads to a lower payoff in both the complements and substitutes cases. This corresponds to the case where degree distribution changes imply a lower t' in the complements case and higher t' in the substitutes case. As we are changing the mix of types, and expected payoffs differ across types of players, the change in average payoff is ambiguous.

6. Intermediate Information Structures

So far we have focused on the polar cases of incomplete information and complete information of the underlying network. If no player has any information at all regarding the realized network (not even her own degree), other than the underlying degree distribution, the analysis becomes far less interesting. In that case, all players are symmetric when choosing their actions and the comparative statics within a network become irrelevant. The comparisons across networks follow directly from the consequences of shifts of the degree distributions on expected payoffs and end up generating similar results to those produced for the incomplete information case.

A more substantial question arises with intermediate information structures in which each player knows the degree of some of her neighbors in addition to her own degree. As it turns out, the qualitative insights are very similar to those generated for the incomplete information case when we consider shifts across networks. When considering comparative statics within a network, the knowledge of degrees of further removed neighbors adds a slight technical complication. Indeed, it is then quite possible that two players with the same degree, say 1, will face neighbors with different degrees and therefore face different actions, which will in turn make them choose different actions themselves. Does degree then become irrelevant as soon as we move from own degree knowledge to the next knowledge level?

Perhaps having a higher degree does make a difference on average: it is possible that some higher degree player makes a greater effort than a lower degree player in a game with strategic substitutes, due to their different neighborhoods, but this should wash out in the average. This idea leads to the concept of average monotone strategy. Suppose, to fix ideas, that every player knows her own degree, the degree of each of her neighbors and the degree distribution at large $P(k)$. Given a pure strategy profile $\{x_{t_i}\}$, define the average effort of degree k players as

$$A^k(\{x_{t_i}\}) = \sum_{l_1,\ldots,l_k} \prod_{l_j} \tilde{P}(l_j) x_{k,l_1,l_2\ldots,l_k}. \tag{6}$$

We shall say that a pure strategy profile $\{x_{t_i}\}$ is average monotone increasing in degree if $A^{k+1}(.) \geq A^k(.)$ for all $k \geq 1$, while it is average monotone decreasing if $A^{k+1}(.) \leq A^k(.)$, for all k.

One possible conjecture is that in games with strategic substitutes all equilibria should be monotone decreasing while in games with strategic complements all equilibria should be monotone increasing. Is this true? The answer is no, as we now illustrate.

Example 6.1. *Non average monotone equilibrium with strategic complements*

Let g be a network where nodes have either degree 1 or degree 2. Consider a game with strategic complements where $X = \{0, 1\}$. Players care only about their action and the total of the other players' actions in their neighborhood, so the payoffs can be written as

$v(0,0) = 0$, $v(0,1) = 1/2$, $v(0,2) = 3/4$, $v(1,0) = -1$, $v(1,1) = 1$, $v(1,2) = 3$. It is readily seen that for any P, the following is a symmetric equilibrium: type (1,1)'s play 1, type (1,2)'s play 0 and all type 2's play 0. For large $P(k)$, on average degree 1 players will put in more effort as compared to degree 2 players.

Similar examples can be constructed for the case of strategic substitutes as well. Thus, under local information there are equilibria which violate the monotonicity properties derived in Propositions 3.1 and 3.2. This leads us to examine a weaker claim: does there always exist an equilibrium which possesses these monotonicity properties? We start by showing that in the example above, there exist equilibria with the desired monotone properties.

Example 6.2. *Average Monotone Equilibria with Strategic Complements*

We continue with example 6.1. It can be checked that for any P, the following is a symmetric equilibrium: type (1,1)'s play 1, type (1,2)'s play 1 and all type 2's play 1. That is, all players choose 1 and the equilibrium is average monotone.

What has happened is that the change from incomplete to local information has introduced a multiplicity of equilibria. Some of the equilibria are average monotone, while others are not. However, there does always exist at least one equilibrium which is average monotone, as the following Proposition illustrates.

Proposition 6.1. *Under local information there exists a symmetric equilibrium which is average monotone increasing (decreasing) if payoffs satisfy strategic complementarities (substitutes).*

The intuition of this result is simple. Consider the case of strategic complements and start from a situation where each player chooses action 0. Let each player play her highest best response. Starting from the new profile, players with higher degree will have a higher best response than low degree players. This generates a new profile that is monotone in degree.

Consider now two players with the same degree but suppose that player 1's neighbors are of higher degree than player 2's neighbors. It follows that player 1 has higher best responses than player 2. Similarly, if player 1 has degree $k+1$, player 2 has degree k and the k neighbors of player 2 are less connected than the k neighbors of player 1. Thus, player 1 has a higher best response than player 2. Iterating these arguments the process converges to an average monotonic symmetric equilibrium.

7. Conclusions

Social interactions are modeled as a network of relationships and a player's payoff depends on her own action and the actions of her neighbors. We wish to understand how location within a fixed network as well as changes in overall network structure affects individual behavior.

The paper develops a general theoretical framework in which these questions can be addressed. In particular, the paper makes two innovations: we allow for a fairly general class of payoffs (which have as special cases practically all the models studied so far) and we allow for incomplete information about network structures (in contrast to most existing work which assumes complete network information).

Our results yield a number of insights about how the nature of the game (strategic substitutes vs complements and positive vs negative externalities) and the level of information (incomplete vs complete) shape individual behavior in networks.

8. Appendix - Proofs

Proof of Proposition 3.1: We present the proof for the case of strategic complements; The strategic substitutes case is analogous and omitted. Let $\{\sigma_k^*\}$ be a symmetric equilibrium of the network game. If $\{\sigma_k^*\}$ is a trivial profile with all degrees choosing action 0 with probability 1, the claim follows directly. Therefore, from now on, we shall assume that the equilibrium is non-trivial and that there is some k' and some $x' > 0$ such that $x' \in \mathbf{supp}(\sigma_{k'}^*)$.

Consider any $k \in \{0, 1, ..., n\}$ and let $x_k = \sup[\mathbf{supp}(\sigma_k^*)]$. If $x_k = 0$, it trivially follows that $x_{k'} \geqslant x_k$ for all $x_{k'} \in \mathbf{supp}(\sigma_{k'}^*)$ with $k' > k$. So let us assume that $x_k > 0$. Then, for any $x < x_k$, the assumption of (strict) strategic complements implies that

$$v_{k+1}(x_k, x_{l_1}, ..., x_{l_k}, x_s) - v_{k+1}(x, x_{l_1}, ..., x_{l_k}, x_s) \geqslant v_k(x_k, x_{l_1}, ..., x_{l_k}) - v_k(x, x_{l_1}, ..., x_{l_k})$$

for any x_s, with the inequality being strict if $x_s > 0$. Averaging over all types, the fact that at least $x_k > 0$ implies that

$$U(x_k, \sigma^*_{-i}, k+1) - U(x, \sigma^*_{-i}, k+1) > U(x_k, \sigma^*_{-j}, k) - U(x, \sigma^*_{-j}, k).$$

On the other hand, note that from the choice of x_k,

$$U(x_k, \sigma^*_{-j}, k) - U(x, \sigma^*_{-j}, k) \geqslant 0$$

for all x. Combining the aforementioned considerations we conclude:

$$U(x_k, \sigma^*_{-i}, k+1) - U(x, \sigma^*_{-i}, k+1) > 0$$

for all $x < x_k$. This in turn requires that if $x_{k+1} \in \mathbf{supp}(\sigma^*_{k+1})$ then $x_{k+1} \geqslant x_k$, which of course implies that σ^*_{k+1} FOSD σ^*_k. Iterating the argument as needed, the desired conclusion follows, i.e. $\sigma^*_{k'}$ FOSD σ^*_k whenever $k' > k$. ■

Proof of Proposition 3.2: We present the proof for positive externalities; The proof for negative externalities is analogous and thereby omitted. The claim is obviously true for a trivial equilibrium in which all players choose the action 0 with probability 1. Consider a non-trivial equilibrium σ^*. Suppose $x_k \in \mathbf{supp}(\sigma^*_k)$ and $x_{k+1} \in \mathbf{supp}\left(\sigma^*_{k+1}\right)$. By assumption,

$$v_{k+1}(x_k, x_{l_1}, ..., x_{l_k}, 0) = v_k(x_k, x_{l_1}, ..., x_{l_k})$$

for all $x_{l_1}, ..., x_{l_k}$.

Since the payoff structure satisfies positive externalities, it follows that for any $x > 0$,

$$v_{k+1}(x_k, x_{l_1}, ..., x_{l_k}, x) \geqslant v_k(x_k, x_{l_1}, ..., x_{l_k}).$$

Looking at expected utilities, it follows that

$$U(x_k, \sigma^*_{-i}, k+1) \geqslant U(x_k, \sigma^*_{-j}, k).$$

Since σ^*_{k+1} is a best response in the network game being played,

$$U(x_{k+1}, \sigma^*_{-i}, k+1) \geqslant U(x_k, \sigma^*_{-j}, k)$$

and the result follows. ■

Proof of Proposition 4.1: Let $\{\sigma_i(t)\}$ be an equilibrium of the network game with underlying network characterized by P'. We first show that there exists an equilibrium in the game with degree distribution P' which dominates $\{\sigma_i(t)\}$ and is monotone. Indeed, start with the (symmetric) profile of actions prescribing each player to use her 1 action with probability 1. Now consider the best response profile for all players, placing a probability 1 on the highest possible action for each player who is indifferent. Clearly, we are left with a profile that dominates $\{\sigma_i(t)\}$. Furthermore, from strategic complementarities and Assumption A, the profile is monotone. Continuing iteratively in this manner, we converge to a symmetric pure equilibrium profile characterized by $\{x_k\}$ (each player i uses the strategy $\tilde{\sigma}_i(t)$, where $\tilde{\sigma}_i(k) = x_k$ for all i) which dominates $\{\sigma_i(t)\}$ and is monotone.

Since $\{x_k\}$ is a monotonic sequence, strategic complementarities than guarantee that for any $x \geqslant x_k$:

$$\sum_{l_1,\ldots,l_{k-1}} \prod_{j=1}^{k-1} \tilde{P}(l_j)[v_k(x, x_{l_1}, ..., x_{l_k}) \geqslant \sum_{l_1,\ldots,l_{k-1}} \prod_{j=1}^{k-1} \tilde{P}'(l_j)[v_k(x, x_{l_1}, ..., x_{l_k}).$$

In particular, if players are playing the symmetric profile $\{x_k\}$ in the network game with underlying degree distribution P, there is best response of each degree k player which is at least as high as as x_k. Consider the profile of best responses (and, as before, upon indifference, choose the highest best response to be played with probability 1). The new profile dominates $\{x_k\}$ and is monotone. Proceeding iteratively in that way, we converge to a symmetric equilibrium profile in the network game with degree distribution P that dominates the original equilibrium $\{\sigma_i(t)\}$. ■

Proof of Proposition 4.4: Proposition 3.1 tells us that actions are monotone decreasing in games with strategic substitutes. This together with strict strategic substitutes implies that in any symmetric equilibrium, there is a unique degree threshold t and that for all degrees lower than t, a player chooses 1, for all degrees greater than t a player chooses 0, while the player may mix for degree t.

Suppose that equilibrium has threshold t' under P', where x is the probability of a player of degree t' choosing the action 0. Since by assumption $\tilde{P}$ FOSD $\tilde{P}'$, from monotone decreasing action property of equilibrium strategy it follows that the equilibrium threshold under P cannot be lower than t'. For any player of degree k, the probability that $l \in \{0, 1, ..., k\}$ of the neighbors choose the action 1 is given by:

$$\binom{k}{l}\left(1 - \tilde{P}'(t')x - \sum_{k=t'+1}^{n-1} \tilde{P}'(k)\right)^{l}\left(\tilde{P}'(t')x + \sum_{k=t'+1}^{n-1} \tilde{P}'(k)\right)^{k-l}.$$

Since $\tilde{P}$ FOSD $\tilde{P}'$,

$$\tilde{P}(t')x + \sum_{k=t'+1}^{n-1} \tilde{P}(k) \geqslant \tilde{P}'(t')x + \sum_{k=t'+1}^{n-1} \tilde{P}'(k),$$

and so the (binomial) distribution of the number of neighbors choosing the 1 action under P' FOSD that under P.

From strict strategic substitutes, the threshold t under P must be weakly higher, $t \geqslant t'$ and if $t = t'$ then the probability of choosing the action 0 of a player of degree t is $y \leqslant x$. Suppose that the probability that any neighbor choosing 1 strictly increased. Then,

$$\tilde{P}(t)y + \sum_{k=t+1}^{n-1} \tilde{P}(k) < \tilde{P}(t')x + \sum_{k=t'+1}^{n-1} \tilde{P}(k),$$

and it would be a strict best response for t' to choose 0, in contradiction. ■

Proof of Proposition 4.5: Let us show the second statement first. Start a set with both i and j in it. This forms a (possibly non-maximal) independent set of g. Add a node that keeps it an independent set. Iterate until a maximal independent set is reached. This is $S\prime$ which is a maximal independent set of g, but not of $g+ij$. Now, let us show that there any maximal independent set of $g+ij$ which is not a maximal independent set of g is a subset of a maximal independent set of g. Consider any maximal independent set S of $g+ij$. It can have at most one of i and j in it. If it has neither in it, then it is also clearly a maximal independent set of g. Suppose that i is in S but that j is not in S. If some neighbor of j, other than i, is in S, then S is a maximal independent set of g. So consider the case where the only neighbor of j that is in S is i. Consider $S\prime$ which is S union j. Then $S\prime$ is a maximal independent set of g. ■

Proof of Proposition 5.2: Let us treat the case where $t' \leq L$, as the other case is similar. Start with the threshold t'. Change to P from P'. Consider type t'. Under the original strategies, the number of t''s neighbors who choose 1 goes up in the sense of FOSD. To see this, note that the probability that any given neighbor provides is $\sum_{k<t'} kP'(k)/ < k > +t'mP'(t')/ < k >$ originally (where m is t''s mixing probability), and then $\sum_{k<t'} kP(k)/ < k > t'mP(t')/ < k >$ after the change. As the mean has not changed, and $P(k) \geq P'(k)$ for all $k \leq t' \leq L$, the claim follows directly. Now, suppose to the contrary of the Proposition that the threshold went up. Then the fraction of t''s neighbors choosing 1 would further increase above $\sum_{k<t'} kP(k)/ < k > t'mP(t')/ < k >$. But then it would be a strict best response for t' to choose 0, contradicting the supposition. Thus, the threshold is lower. Suppose that the probability that any neighbor choosing 1 went down instead of up. Then it would be a strict best response for t' to choose 1, which contradicts the fact that the threshold is lower. ■

Proof of Proposition 6.1: The proof for the strategic complements case is presented. The idea of the proof is to construct a best-response sequence, and shows that it is monotonically increasing over time as well as with regard to types. Convergence of the sequence follows from the compactness of the action set. The limit constitutes an equilibrium, with the desired properties, given that process is generated by a best-response decision rule.

At each stage t players take the action profile in stage $t-1$ given, say x^{t-1}, and choose an action to maximize their expected payoff. Specifically, type $(k, l_1, ..., l_k)$ at t chooses $x_{k,l_1,...,l_k}$ to maximize

$$\sum_{s_1^1,...,s_{l_1-1}^1} \prod_{j=1}^{l_1-1} \tilde{P}(s_j^1) ... \sum_{s_1^k,...,s_{l_k-1}^k} \prod_{j=1}^{l_k-1} \tilde{P}(s_j^k) v_k(x_i, x^{t-1}_{l_1,k,s_1^1,..,s_{l_1-1}^1}, ..., x^{t-1}_{l_k,k,s_1^k,..,s_{l_k-1}^k}) \tag{7}$$

Similarly, type $(k+1, l_1, ..., l_k, l_{k+1})$ at t chooses $x_{k+1,l_1,...,l_k,l_{k+1}}$ to maximize

$$\sum_{s_1^1,...,s_{l_1-1}^1} \prod_{j=1}^{l_1-1} \tilde{P}(s_j^1) ... \sum_{s_1^{k+1},...,s_{l_{k+1}-1}^{k+1}} \prod_{j=1}^{l_{k+1}-1} \tilde{P}(s_j^{k+1}) v_{k+1}(x_i, x^{t-1}_{l_1,k+1,s_1^1,..,s_{l_1-1}^1}, ..., x^{t-1}_{l_{k+1},k+1,s_1^{k+1},..,s_{l_{k+1}-1}^{k+1}}) \tag{8}$$

Start at all players play 0 and let each type choose the highest best response. Since every player is best responding to neighbors who all choose 0's, we can set the best response as $x^1_{t_i} = x^1_{t'_i}$, for all types $t, t'_i \in \mathcal{T}_i$. It follows then that $x^1_{t_i} \geq x^0_{t_i}$, for all $t_i \in \mathcal{T}_i$. Let $t_i = \{k, l_1, l_2, ...l_k\}$ and $t'_i = \{k', l'_1, l_2, ...l'_k\}$. We say that $t_i \geq t'_i$ either if $k = k'$ and $\{l_1, l_2, ...l_k\} \geq \{l'_1, l'_2, ...l'_k\}$, or if $k > k'$ and $\{l_1, l_2, ...l_{k'}\} \geq \{l'_1, l'_2, ...l'_{k'}\}$. It follows by construction that $x^1_{t_i} \geq x^1_{t'_i}$, for all $t_i \geq t'_i$.

Next we suppose that these inequalities hold for all time periods until and including period $\tilde{t} \geq 1$: $x^{\tilde{t}}_{t_i} \geq x^{\tilde{t}-1}_{t_i}$, for all $t_i \in \mathcal{T}_i$ and $x^{\tilde{t}}_{t_i} \geq x^{\tilde{t}}_{t'_i}$, for all $t_i \geq t'_i \in \mathcal{T}_i$. We wish to show that these inequalities also hold for time $\tilde{t}+1$.

First note that $x^{\tilde{t}+1}_{t_i}$ is a best response to the vector $\{x^{\tilde{t}}_{t_i}\}$ the profile in period $\tilde{t}$, while $x^{\tilde{t}}_{t_i}$ is a best response to the vector $\{x^{\tilde{t}-1}_{t_i}\}$ the profile in period $\tilde{t}-1$; by the induction hypothesis we know that $x^{\tilde{t}}_{t_i} \geq x^{\tilde{t}-1}_{t_i}$, for all $t_i \in \mathcal{T}_i$. From strict strategic complements and concavity of payoffs v_k it then follows that $x^{\tilde{t}+1}_{t_i} \geq x^{\tilde{t}}_{t_i}$, for all t_i.

Next consider degree monotonicity property: First take two players i and i', with same degree k but suppose player 1 has neighbors with higher degree than player 2: $\{l_1, ...l_k\} \geq$

$\{l'_1, ..., l'_k\}$. It then follows that for those neighbors with $l_k = l_{k'}$, the neighbor actions are the same, while for those $l_k > l'_k$, given the induction hypothesis, $x^{\tilde{t}}_{t_i} \geq x^{\tilde{t}}_{t'_i}$, for all $t_i \geq t'_i$, the neighbors actions are higher for player i. From strategic complements and concavity then this implies that the best response of player i is weakly higher than that of player i'. Second, consider two players i and i' with $k > k'$ and $\{l_1, ...l_k\} \geq \{l'_1, ..., l'_{k'}\}$. Again, applying the induction hypothesis, $x^{\tilde{t}}_{t_i} \geq x^{\tilde{t}}_{t'_i}$, for all $t_i \geq t'_i$, and using strategic complements it follows that the best response of i is weakly higher than the best response of player i'. Combining these observations yields that $x^{\tilde{t}+1}_{t_i} \geq x^{\tilde{t}+1}_{t'_i}$, for all $t_i \geq t'_i \in \mathcal{T}_i$.

Monotonicity in t along with compactness of action set implies convergence. The limit is an equilibrium, given the best response rules governing the dynamics. Since the strategy configuration is monotone in type of players for every time t it is also monotone in the limit. Let $\{x^*_{t_i}\}$ be the equilibrium strategy profile.

In this equilibrium average action of degree k and degree $k+1$ players is, respectively:

$$A^k(\{x^*_{t_i}\}) = \sum_{l_1,\ldots,l_k} \prod_{l_j} \tilde{P}(l_j) x^*_{k,l_1,l_2\ldots,l_k} \tag{9}$$

$$A^{k+1}(\{x_{t_i}\}) = \sum_{l_1,\ldots,l_k} \prod_{l_j} \tilde{P}(l_j) \sum_{l_{k+1}} \tilde{P}(l_{k+1}) x^*_{k+1,l_1,l_2\ldots,l_k,l_{k+1}}. \tag{10}$$

We have shown that $x^*_{t_i} \geq x^*_{t'_i}$ for $t_i \geq t'_i$. It then follows that for fixed $\{l_1, l_2, ...l_k\}$,

$$x^*_{k,l_1,l_2\ldots,l_k} \leq x^*_{k+1,l_1,l_2\ldots,l_k,l_{k+1}}. \tag{11}$$

This observation yields us the inequality $A^k(\{x^*_{t_i}\}) \leq A^{k+1}(\{x_{t_i}\})$. Since k was arbitrary, that shows that there exists an average monotone equilibrium under strategic complements.

■

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