

Self-fulfilling traps with money as a medium of exchange

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Abstract

A model of an economy with random matching and monetary exchanges is shown to have two equilibrium regimes. In the first, agents find an opportunity for a sale in each period and there is full-employment. In the second, agents attempt to increase their balances because of a lack of opportunities for sales which is in turn induced by the saving for higher money balances, and there is unemployment. In any period of the high regime, self-fulfilling expectations can plunge the economy into the low equilibrium but a recovery from the low to the high equilibrium may not be possible.

Keywords: Money, liquidity trap, paradox of thrift, multiple equilibria, strategic complementarity

1 Introduction

In the current crisis, the demand for precautionary saving and the reduction of consumption has played an important contributing role. Uncertainty about employment and income raises the motive for saving and the lower demand for goods feeds into the uncertainty. The mechanism, which has some relation with the paradox of thrift, is analyzed here in a model of general equilibrium with money as the medium of exchange.

In any contemporary economy, exchanges are between goods and money. Money is liquid as it can be used to trade any good. Agents increase their money balances through sales and use these balances to buy consumption goods. Both inflows and outflows of money are subject to random micro-shocks but in a “standard” regime of economic activity, these shocks can be assumed to be relatively small and agents can afford to keep a relatively low level of money inventories. Such a regime depends on individual expectations. If agents expect that opportunities for sales are subject to a larger uncertainty, they reduce their consumption to accumulate more money as a precaution. But the reduction of consumption by some agents may increase the sale uncertainty of others and raise the demand for money. The higher demand for money (liquidity) may be self-fulfilling.

In this paper, the sudden increase of the demand for money shifts the economy from an equilibrium with a regime of high consumption to another equilibrium with a regime of low consumption where agents attempt to accumulate higher money balances and there is insufficient aggregate demand and output.

Money is valuable because agents are spatially separated. The spatial separation of agents has been the foundation of models of money since Samuelson (1958). A important assumption in that model is that agents cannot enter in bilateral credit agreements because they are temporally separated in different generations and meet only once. That fundamental property has been embodied by Townsend (1980) in a setting with infinitely lived agents who are paired along the two opposite lanes of a “turnpike”, each selling and consuming with his *vis à vis* on alternate days and carrying money from a day of production to the next day for consumption. The central property again is that any two agents are paired at most once. That property is preserved here in an adaptation of the Townsend model in which agents are in a continuum of mass one and matched pair wise in each period with two other agents, one customer for the produced good and one supplier of the consumption good.

There is no search in the model, because search introduces separate issues of efficiency and is not the essential property for the existence of money¹. The search process is implicit in Green and Zhou (2002)² because agents may not find a good to their taste of consumption in their current match and thus may find a randomly long time gap between opportunities for sales and consumption.

The focus on the essential property of money as in the turnpike model of Townsend will enable us to use the model as a simple analytic representation of a liquidity crisis as one of two possible equilibria.

In the full-employment equilibrium, there is not more than one period between a sale and a consumption. Money is necessary for consumption but because a stable and high inflow of cash is expected, a relatively low level of money balance is sufficient to maintain a high level of consumption. No agent is cash constrained for consumption and producers can always sell.

In an equilibrium with unemployment, a producing agent cannot sell when he is matched with an agent who has no money. Because of the probability of no sale, agents attempt to accumulate money. But the higher balances for some agents must result in smaller or no balances for others because the endogenous money price of goods is the same in the two equilibria and the total quantity of money is not affected by the regime of activity.

In the model presented here, when the economy is in a full-employment stationary equilibrium, a negative shock of expectations is sufficient to push the economy to an equilibrium with unemployment: the fear of smaller opportunities for sale induces agent to keep money: if they do not have an urgent need to consume they choose to save, but this act of saving reduces the opportunity of another agent to sell his production. The two equilibria with and without full employment are not symmetric: in the stationary equilibrium with unemployment a jump of optimism may not be sufficient to nudge the economy into a recovery: in full employment all agents who don't have a high need for consumption can shift to saving. In the economy with unemployment, agents who are liquidity constrained cannot jump to consumption even if they become optimistic about the future.

¹The role of money or credit with search has been analyzed by Diamond and Yellin (1990), Diamond (1990), Shi (1995), Trejos and Wright (1995), among others.

²See also Green and Zhou (1998), Zhou (1999a).

The model is presented in Section 2. In order to simplify the analysis, agents are constrained in the maximum of cash they can hold, by assumption. Since the higher demand for cash is what generates an inefficient equilibrium, the restriction should not limit the validity of the properties. The assumption is relaxed later in the paper.

The stationary equilibrium with full employment and low demand for money is presented in Section 3. In general, the optimal consumption rule depends on the dynamics of the economy. The dynamics of the two regimes of high and low consumption are first analyzed in Section 4. In the following section, these two regimes are shown to be equilibria under suitable parameter conditions. Section 6 shows that the steady state of the low regime with unemployment may be a trap out of which no optimism about the future can lift the economy. Section 7 shows that there can be multiple stationary equilibria with unemployment because of the a strategic complementarity between individuals' saving in money. A higher saving generates more unemployment which provides more incentive for precautionary saving.

2 The model

There is continuum of infinitely lived agents, indexed by $i \in [0, 1)$. The utility of agent i in period t is $u(x_{it}, \theta_{it})$, where $\theta_{it} \in \{0, 1\}$ are i.i.d. random variables that represent shocks to the utility of consumption. When $\theta_{it} = 1$, agent i has a higher need to consume in period t than when $\theta_{it} = 0$. If $\theta_{it} = 0$, the agent is, in period t , of the *low type*, and if $\theta_{it} = 1$, the agent is of the *high type*. The probability of the high type is exogenous and equal to α , ($0 < \alpha < 1$), which is known by all agents.

To simplify the exposition, we assume that the utility function in period t is given by

$$u(x_t, \theta) = \begin{cases} 1 & , \quad \text{if } x_t \geq 1, \\ -c\theta_t & , \quad \text{if } x_t < 1. \end{cases} \quad (1)$$

For example, a high type agent has to make some repair (material or bodily) to avoid a penalty c . It will be shown later that the properties of the model do not depend on the indivisibility properties of the utility function. The welfare of agent i in any period, say period 0, is the discounted expected sum of the utilities of consumption in the future periods:

$$U_i = E \left[\sum_{t \geq 0} \beta^t u(x_{it}, \theta_{it}) \right], \quad \text{with} \quad \beta = \frac{1}{1 + \rho} < 1. \quad (2)$$

Agents produce goods which they sell, and consume goods produced by others. Goods are not storable. In each period, an agent meets two other agents, one buyer and one seller, according to a process of random matching that is defined by the function

$$\phi_t(i) = \begin{cases} i + \xi_t, & \text{if } i + \xi_t < 1, \\ i + \xi_t - 1, & \text{if } i + \xi_t \geq 1. \end{cases} \quad (3)$$

Any agent i can sell his product to agent $j = \phi_t(i)$ and consume the good produced by agent $\phi_t^{-1}(i)$.

The variables $\xi_t \in (0, 1)$ are either random with a density or a constant number that is not a rational multiple of π . The process embodies the absence of a double coincidence of wants and implies that a agent has zero probability to find the same match in a future period³. The present setup with no centralized market is similar to Townsend's turnpike that fits a circle of infinite diameter with random pairing of atomistic agents between the two lanes (Townsend, 1980).

In order to simplify the demand for money, it is assumed that each agent is like a two-headed household: at the beginning of period t , say a day, one head of household i goes out with some cash (if there is any in the household) to buy a consumption good from a randomly matched supplier $\phi_t^{-1}(i)$. The second head stays at home to service the customer $\phi_t(i)$: if that customer buys, he produces and sells one unit of the good at no cost. The two heads meet at the end of the day to consume if a purchase has been made. A setup with a single person who buys and sells with a random order during the day may be analyzed in later work.

To summarize, in each period t events proceed in the following sequence:

1. Each agent i first learns his type, *i.e.* the value of $\theta_{i,t}$. The probability of the high type ($\theta_{i,t} = 1$) is equal to α .
2. Each agent decides to carry a quantity of money m "to the market" (which is not centralized). In order to have strictly optimal strategies, it is assumed that carrying m entails a vanishingly small cost that is proportional to m . The decision about m has to take place at the beginning of the day, before the eventual production and sale during the day.

³One could use other matching functions ϕ_t provided that they satisfy the property that for any subset \mathcal{H} of $[0, 1)$, $\mu(\mathcal{H}) = \mu(\phi_t(\mathcal{H}))$, where μ is the Lebesgue-measure on $[0, 1)$. The property is required for a uniform random matching of all agents.

3. For each agent, the seller produces either 0 or 1 (since agents demand 0 or 1 in the utility function). The production is cost-free⁴. The seller posts a price p and there is no bargaining. One can assume that the seller produces instantly, only after he knows whether he has a buyer.

We will consider equilibria where all sellers post the same price, p , in the same period. That price is publicly known in a rational expectation equilibrium. Suppose an agent decides to consume in a given period: he carries an amount of money \tilde{m} to the market. Because of the small transportation cost, any $\tilde{m} > p$ is inferior to $\tilde{m} = p$, and to carry $\tilde{m} < p$ would be equivalent to decide on no purchase. Hence, for a posted seller's price p , buying agents bring each $\tilde{m} = p$ to the market.

Given the price p , no seller has an incentive to deviate and post another price: if he posts $\tilde{p} > p$, he cannot make a sale since customers bring an amount of cash equal to p ; any price $\tilde{p} < p$ does not increase sales: a consuming agent expects in equilibrium that when he comes to the shop to buy the good, the price will be p . Some consumers who would not buy at the price p might be willing to buy at a lower price, but the probability that a given customer would meet a single seller with a lower price is 0. Hence, a consumer decides either to consume and bring an amount of cash equal to p or not consume and therefore bring no cash to the matched seller.

Lemma 1

In an equilibrium where all sellers post a price p in a given period, all agents who consume in that period carry an amount $m = p$ to buy goods. A seller who deviates from posting p gets a strictly smaller payoff.

3 Steady state equilibrium with full-employment

In a full-employment equilibrium, by definition all agents produce and sell, (and hence consume). When some agents do not sell (and others do not consume), there is unemployment.

Assume that each agent has a quantity of money at least equal to \bar{m} at the beginning of the period and that sellers post a price $p < \bar{m}$. If all agents carry an amount of

⁴One could assume that the agent could produce $x \in [0, 1]$, costlessly, with the maximum capacity of 1, and remove the indivisibility of the utility function.

money $\bar{m} = p$ to their matched seller, all agents consume and sell in the period. Each agent has the same amount of cash at the end of the period. No consumer has an incentive to deviate by hoarding: he would just miss the utility of consumption in the period. From Lemma 1, no seller has an incentive to deviate. We have the following result.

Lemma 2 *Assume that the quantity of money of each agent is bounded below by \bar{m} . Then any price $p \leq \bar{m}$ that is constant over time determines a steady state with full employment.*

There is a continuum of equilibrium price levels. That property is the same as in Green-Zhou (1998, 2002) because of the similar price determination. The property of a continuum of price equilibria is of no special interest here. The main property is that for all these prices there is full employment. All the equilibria with different p have the same real allocation of resources which is the socially optimal allocation.

In a full employment equilibrium, money is needed for transaction, but there is no precautionary motive since agents are sure that they will be able to sell and replenish their cash at the end of the period.

Using Lemma 1, we will consider only equilibria with a price that is constant over time. That price will be normalized to 1. All money holdings in interval $I_k = [k, k+1)$ generate the same opportunities for trade. An agent with an amount of money $m \in I_k$ at the beginning of a period is defined to be in *state k*. In state 0, an agent is liquidity constrained and cannot consume. We first make the technical assumption that money holdings are bounded: there is N arbitrary such that $m < N + 1$. This assumption restricts the hoarding capability of agents. It is not harmful in a model that generates an equilibrium with excess hoarding. It is shown in the Appendix (Proposition 9) that in a steady state equilibrium with unemployment and no restriction on holding money, the distribution of money has a bounded support.

Let $\Gamma(t)$ be the vector of the distribution of agents at the beginning of period t across states

$$\Gamma(t) = (\gamma_0(t), \gamma_1(t), \dots, \gamma_N(t))',$$

where $\gamma_k(t)$ is the mass of agents in state k . Any distribution of money must satisfy the two constraints:

$$\sum_0^N \gamma_k(t) = 1, \quad \sum_{k=1}^N k\gamma_k(t) = M. \quad (4)$$

In a steady state with full employment, there is no need for precautionary saving

toward consumption in future periods because each agent knows that at the end of the day, a sale is made for sure and the cash balance that has been reduced in the morning by the expenditure for consumption is replenished at the end of the day for the next one. In general, the demand for money will depend on the type of the agent (high or low) and the opportunities of future sales as determined by the path of the probabilities of making a sale in period t . That path depends on the consumption function of agents in the future. We consider two consumption functions that define each a regime, high or low. We will show later that under some conditions, each of the two regimes is an equilibrium.

4 Dynamics in two regimes

The high regime is defined by the consumption function that any agent who is not liquidity constrained (*i.e.*, in state 0), consumes. In that regime, consumption is at its highest possible level. The low regime is defined by the lowest possible rational consumption function: an agent saves (hoards) when he is not in a high state (with a higher need for consumption), and is not in state 0 (with a liquidity constraint) or state N (the maximum level of money holding). The purpose of saving is to self-insure for a later day with a higher consumption need against a lack of sale opportunities. It makes little sense *a priori* to save when there is a current higher consumption need, because of the discounting of the future. Hence the definition of the consumption in a low regime. The optimality of the two consumption functions in a given period will depend on the utility of money balances that depends in turn on the dynamics of the economy beyond that period. That optimality will be analyzed in the next section.

4.1 The high regime

At the beginning of the first period, period 0, the distribution of money, $\Gamma(0)$, is given. Since all agents except those in state 0 consume, and the matching is independent of the money balance, each agent faces the same probability $\pi(t)$ of not making a sale in period t and being unemployed. The probability $\pi(t)$ is equal to the fraction of agents in state 0, $\gamma_0(t)$. The evolution of the distribution of money is given by

$$\Gamma(t+1) = H(\pi_t).\Gamma(t), \quad \text{with} \quad \pi_t = \gamma_0(t), \quad (5)$$

where the transition matrix H is equal to

$$H(\pi) = \begin{pmatrix} \pi & \pi & 0 & 0 & 0 & \dots \\ 1 - \pi & 1 - \pi & \pi & 0 & 0 & \dots \\ 0 & 0 & 1 - \pi & \pi & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & 0 & 0 & 1 - \pi & \pi \\ 0 & \dots & 0 & 0 & 0 & 1 - \pi \end{pmatrix}. \quad (6)$$

For example, in the first line, the agents who are in state 0 at the end of period t are in that state either (i) because they met another agent in state 0, with probability π , and that determines H_{11} , or (ii) because they were in state 1 at the beginning of period t , consumed but met with a buyer with no cash and therefore got no inflow of cash which determines $A_{12} = \pi$. Likewise for the other elements of the matrix H .

From Lemma 2, given the price $p = 1$, there is a steady state full-employment equilibrium if and only if the aggregate quantity of money, M , is at least equal to one. The next result (proven in the Appendix) shows that in this case, the high regime where all non liquidity-agents consume converges to full employment.

Proposition 1

In the high regime (when all agents who are not liquidity-constrained consume), the distribution of money $\Gamma(t)$ converges to a limit. Let M be the aggregate quantity of money.

- *If $M \geq 1$, for any initial distribution of money, the rate of unemployment converges to zero. At the limit, no agent is liquidity-constrained.*
- *If $M < 1$, the rate of unemployment converges to $\pi^* = 1 - M$. The distribution of money converges such that $\gamma_0^* = \pi^*$, $\gamma_1^* = 1 - \gamma_0^*$, and $\gamma_k^* = 0$ for $k \geq 2$.*

In the case where $M \geq 1$, the economy may have liquidity-constrained agents at the beginning of time. But a diffusion process of money takes places that reduces gradually the mass of constrained agents to zero. The real economy at the limit is invariant to the initial distribution of money since it has full employment. The limit distribution

of money, which has no incidence on the real economy, does depend on the initial distribution of money. As a particular case, any distribution with full employment and $\gamma_0 = 0$ is invariant through time.

The case with three states

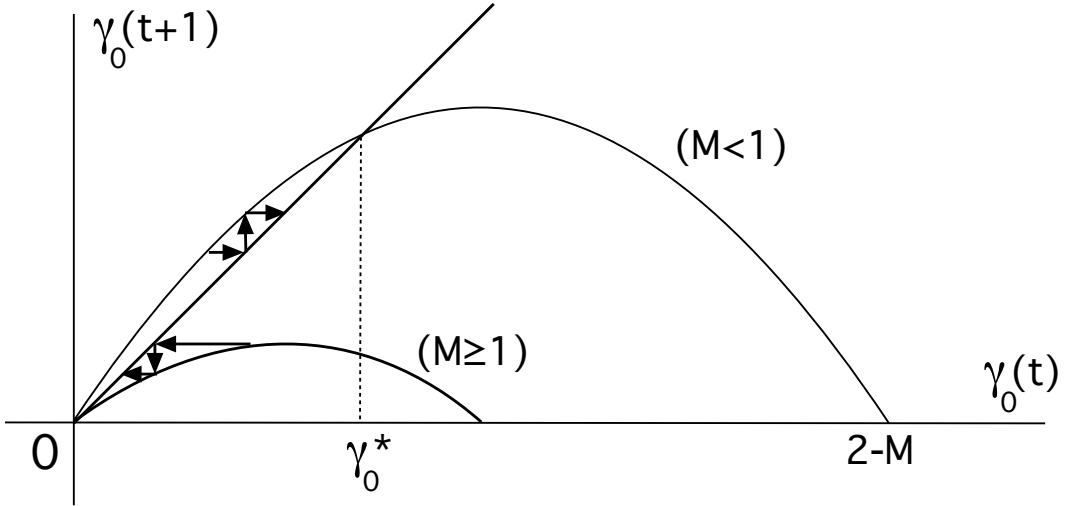
When there are three states, the quantity of money is bounded by 2. Using the equations of the quantity of agents and of money, $\sum_{k=1}^2 \gamma_k = 1$ and $\sum_{k=1}^2 k\gamma_k = M$, to eliminate $\gamma_1(t)$ and $\gamma_2(t)$, the dynamics can be expressed in function of $\gamma_0(t)$:

$$\gamma_0(t+1) = \gamma_0(t)(\gamma_0(t) + \gamma_1(t)),$$

which is equivalent to

$$\gamma_0(t+1) = \gamma_0(t)(2 - M - \gamma_0(t)). \quad (7)$$

The evolution of $\gamma_0(t)$ is represented in Figure 1 for the cases $M \geq 1$ and $M < 1$. When $1 \leq M < 2$, at the limit, $\gamma_0^* = \pi^* = 0$ and the distribution of money in states 1 and 2 is determined uniquely by the unit mass of agents and the quantity of money. When M increases, γ_2^* increases and γ_1^* decreases.



When $M < 1$, the economy converges to a steady state with unemployment and the fraction γ_0 of liquidity constrained agents tends to a positive value. When $M > 1$, the economy converges to full-employment with no liquidity constrained agents in the limit.

Figure 1: Dynamics of the liquidity constrained agent in the regime of high consumption (three states).

4.2 The low regime

In the low regime, consumption is generated by the fraction α of the agents in states 1 to $N - 1$, and all agents in state N . As the fraction of agents who consume is $1 - \pi_t$,

$$\pi(t) = 1 - \alpha \sum_{k=1}^{N-1} \gamma_k(t) - \gamma_N(t), \quad (8)$$

which can be written as

$$\pi(t) = 1 - \alpha(1 - \gamma_0(t)) - (1 - \alpha)\gamma_N(t). \quad (9)$$

The value of $\pi(t)$ is equal to zero if and only if $\gamma_0(t) = 0$ and $\gamma_N(t) = 1$ which is equivalent to $\gamma_N(t) = 1$ because $M \leq N$. (All individuals are in the highest state). We have the following result.

Proposition 2

In the low regime, with $M < N$, there is unemployment in all periods: $\pi(t) > 0$ for all $t \geq 1$.

The evolution of the distribution of money is now given by

$$\Gamma(t+1) = L(\pi_t) \cdot \Gamma(t), \quad \text{with} \quad \pi_t = \gamma_0(t), \quad (10)$$

where the transition matrix $L(\pi)$ takes a form that depends on N .

For $N = 2$,

$$L(\pi) = \begin{pmatrix} \pi & \alpha\pi & 0 \\ 1 - \pi & (1 - \alpha)\pi + \alpha(1 - \pi) & \pi \\ 0 & (1 - \pi)(1 - \alpha) & 1 - \pi \end{pmatrix}. \quad (11)$$

For example in the first line, the mass of liquidity constrained agents $\gamma_0(t+1)$ comes from the constrained agents (not consuming) who do not make sale, with the probability π that is equal to L_{11} , and the agents in state 1 of the high type and do not make a sale, with the joint probability $\alpha\pi$ equal to L_{12} . Likewise for the other elements of the matrix L .

For $N \geq 3$,

$$L(\pi) = \begin{pmatrix} \pi & \alpha\pi & 0 & 0 & \dots & 0 \\ 1 - \pi & a & \alpha\pi & 0 & \dots & 0 \\ 0 & b & a & \alpha\pi & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & b & a & \alpha\pi & 0 \\ 0 & \dots & 0 & b & a & \pi \\ 0 & 0 & 0 & \dots & b & 1 - \pi \end{pmatrix}, \quad (12)$$

with

$$\begin{cases} a = (1 - \alpha)\pi + \alpha(1 - \pi), \\ b = (1 - \pi)(1 - \alpha), \end{cases}$$

and where the middle lines are omitted for $N = 3$.

The dynamics of the economy are completely specified by equation (10) where the matrix $L(\pi)$ is given in (11) or (12), and π_t in (8).

The stationary economy

Let e be the row-vector with $N + 1$ components equal to 1. One verifies that $e.L(\pi) = e$ for any π . (For any distribution of money, the matrix transition matrix L keeps the total quantity of money invariant). Fix a value of π . The matrix $L(\pi)$ has an eigenvalue equal to 1 and that eigenvalue is of order 1. There is a unique vector Γ^* such that $B(\pi).\Gamma^* = \Gamma^*$ and $e.\Gamma^* = 1$. The vector Γ^* defines a stationary distribution of money holdings that depends on the probability π . The total amount of money is equal to

$$M = \sum_{k=1}^N k\gamma_k^*.$$

We thus have a function from π to the total amount of money in the economy and that function can be inverted. The next result which is proven in the Appendix is intuitive: there is a positive relation between money and employment.

Proposition 3

The low regime with money balances bounded by N has a unique steady state. In that steady state, the rate of unemployment is a function of the aggregate quantity of money M that is strictly decreasing from 1 to 0 when M increases from 0 to N .

The proposition is illustrated in Figure 2 for two values of N , $N = 2$ and $N = 3$ and in each case for two values of α , low and high. One can observe that for a given quantity

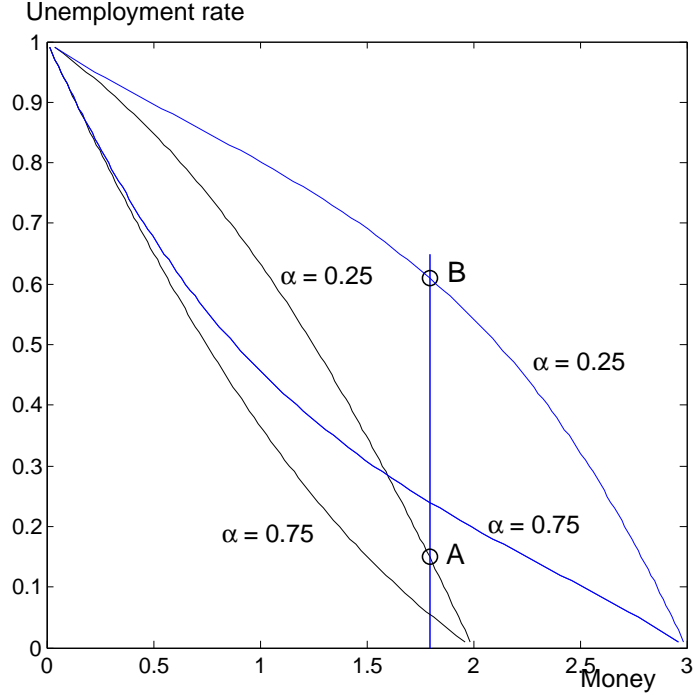


Figure 2: Money and unemployment in the stationary economy under the low regime for $N = 2$ and $N = 3$. (If $M \geq N$, the unemployment rate is 0). For each value of N , two cases are presented with low ($\alpha = 0.25$) and high ($\alpha = 0.75$) consumption. For a quantity of money equal to 1.7944, the unemployment rate is equal to 0.15 when $N = 2$ and to 0.6105 when $N = 3$.

of money, the rate of unemployment rises with N . When N rises from, say, N_1 to N_2 , agents who consume when in state N_1 in the first economy save in the second economy if they are of the low type. The unemployment rate rises with the fraction of agents who save in the aggregate.

As in the previous section, we analyze in more details the case of three states with $N = 2$.

The case with three states

As for the high regime, we can eliminate $\gamma_1(t)$ and $\gamma_2(t)$ because the mass of agents is equal to one and the total quantity of money equal to M , and characterize the dynamics by the evolution of $\gamma_0(t)$.

Let $S = 2 - M$. From the constraints on the distribution of money $(\gamma_0(0), \gamma_1(0), \gamma_2(0))$, we have $-2\gamma_0(t) + S = \gamma_1(t) \geq 0$. Hence, any initial value of the fraction of agents in state 0, $\gamma_0(0)$, must satisfy the condition

$$\gamma_0(0) \leq S/2. \quad (13)$$

The analysis which are presented in the Appendix, shows that for $t \geq 0$,

$$\begin{aligned} \gamma_0(t+1) &= P(\gamma_0(t)), \quad \text{with} \\ P(x) &= -(1-2\alpha)^2 x^2 + (1-2\alpha)^2 Sx + \alpha(1-\alpha)S^2. \end{aligned} \quad (14)$$

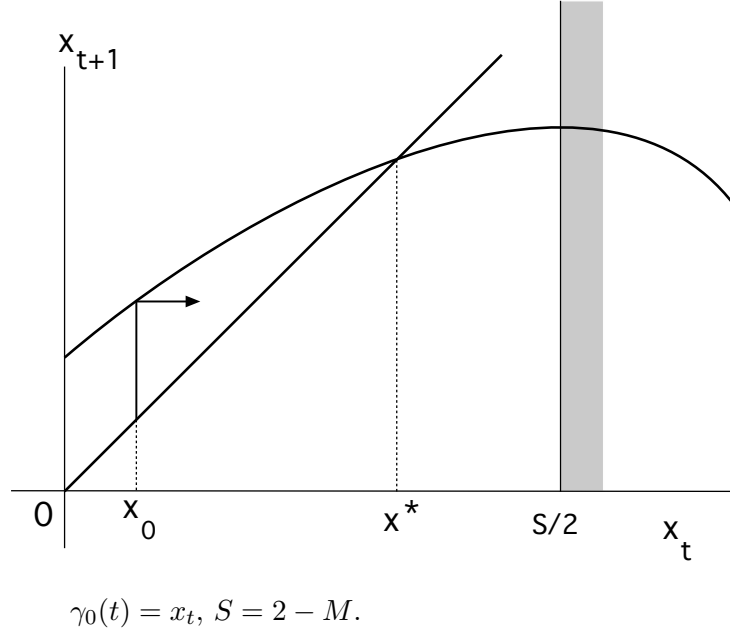


Figure 3: Dynamics of the fraction of liquidity-constrained agents in the low regime

The polynomial $P(x)$ has its maximum at $x = S/2$. One verifies⁵ that $P(S/2) < S/2$. Since $P(x)$ is increasing on the interval $[0, S/2]$, there is for $x > 0$ a unique value x^* such that $P(x^*) = x^*$ and $x^* < S/2$. For any admissible value of x_0 which must be in the interval $[0, S/2]$ by (13), the sequence $x_{t+1} = P(x_t)$ converges to x^* monotonically. The evolution of x_t is represented in Figure 3.

⁵ $P(S/2) = \left(\frac{(1-2\alpha)^2}{4} + \alpha(1-\alpha) \right) S^2$. Since $S < 1$, this expression is strictly smaller than $S/2$.

During the transition, the value of $\gamma_0(t)$ varies monotonically towards its steady state. The unemployment rate, $\pi(t)$, is a linear function of $\gamma_0(t)$

$$\pi(t) = -(1 - 2\alpha)\gamma_0(t) + (1 - \alpha)S, \quad (15)$$

and it converges to the limit π^* that is determined by the equation

$$M = \psi(\pi^*) = 2 - \frac{\pi^*(1 - \pi^*(1 - 2\alpha))}{1 - \alpha - \pi^*(1 - 2\alpha)}. \quad (16)$$

The function $\psi(\pi)$ has a negative derivative and is strictly decreasing, as shown already in Proposition 3. Let ϕ be its inverse function with $\pi = \phi(M)$. One verifies that

$$\phi(0) = 1, \quad \phi(2) = 0, \quad \phi(1.5 - \alpha) = 0.5. \quad (17)$$

We have the following result.

Proposition 4

In a low regime, with $N = 2$ and $M < 2$, the distribution of money converges to a stationary distribution and the unemployment rate converge to a limit that is a decreasing function $\phi(M)$ such that $\phi(0) = 1$, $\phi(1) = \pi_1^ > 0$, and $\phi(2) = 0$.*

(i) On the dynamic path, $\gamma_0(t)$ is a monotone function of time and the unemployment rate, $\pi(t)$, is an increasing (decreasing) function of $\gamma_0(t)$ when $\alpha > 1/2$, ($\alpha < 1/2$).

(ii) In the special case where $\alpha = 1/2$, the distribution of money and the unemployment rate are constant for all periods with $\gamma_0 = (S/2)^2$.

The result has a simple interpretation. In the low regime, all agents consume with probability α except for the agents with the highest balance who consume with probability 1 and the agents with lowest balance who consume with probability 0. Therefore, comparing an agent in state 0 with an agent in any state except the highest (N)⁶, the first reduces aggregate demand by α . On the other hand, an agent in the highest state, N , increases aggregate demand (compared to other agents) by $1 - \alpha$. The two effects are found in equation (9).

When the economy switches to the low regime from an initial position of full employment, aggregate demand falls and the mass of unemployment agents, γ_0 , gradually

⁶When $N = 2$, there is only one such middle state, the state 1, but the argument is general.

increases, and that effect increases the unemployment rate over time. As agents save, the mass of agents in the highest state N increases and that effect reduces the unemployment rate over time. When the “propensity to consume” α is high, the short-fall created by the liquidity constrained agents dominates the contribution of the agents in the highest state and the unemployment rate increases over time.

The previous analyses covers all plausible cases of the consumption function when $N = 2$. Agents in state 0 cannot consume and if they make a sale, they save; agents in state 2 consume since they cannot accumulate more money, by assumption. The consumption functions are differentiated by the behavior of agents in state 1. Since it is absurd to think that the high type would save and the low type consume, we have covered all cases. In the next two sections, it is assumed that there are three states for individual money balances and $N = 2$. That restriction is lifted in Section 7.

5 Optimal consumption functions

So far, we have considered how the distribution of money and the unemployment rate depends on the consumption function. We now determine which consumption function is optimal. We assume that there are 3 states ($N = 2$). Let $U_k(t)$ be the utility of an agent in state k who consumes, with $k = 1, 2$. That utility is the same for the high and the low types. Let U_0 the utility of an agent in state 0 at the beginning of the period *before* he learns his type. (Recall that such an agent cannot consume).

Let ζ be the probability of consumption of an agent who is in state 1. (Recall that the behavior of agents in state 0 or 2 is determined, no consumption in state 0 and consumption in state 1). In the high regime, $\zeta = 1$ and in the low regime, $\zeta = \alpha$. We now analyze the evolution of the utility levels $U_k(t)$ in the low regime.

5.1 Equilibrium in the low regime

Consider an agent in state 2, with the highest balance $m \in I_2$. He consumes, gets a utility of 1 from that consumption to which he adds the discounted value of the utility in the next period. By standard backward induction, his utility in period t is

$$U_2(t) = 1 + \beta \left((1 - \pi)U_2(t + 1) + \pi \left((1 - \alpha)W_1(0, t + 1) + \alpha W_1(1, t + 1) \right) \right), \quad (18)$$

where $W_1(\theta, t + 1)$ is the utility of an agent at the beginning of period $t + 1$ of type θ with $m \in I_1$. (The time index is omitted for π in the next three equations). For

example, the agent makes in the current period no sale with probability π in which case he has $m \in I_1$ in the next period and with probability α he is of the type $\theta = 1$, in which case his utility is $W_1(1, t + 1)$.

In the low regime, a low type in state 1 does not consume and has the same distribution of money at the end of the period as an agent who is in state 2 and consumes. An agent with a high type in state 1 consumes and has a utility $U_1(t)$. Hence

$$W_1(0, t) = U_2(t) - 1, \quad W_1(1, t) = U_1(t). \quad (19)$$

Substituting in (18),

$$U_2(t) = 1 + \beta \left((1 - \pi)U_2(t + 1) + \pi((1 - \alpha)(U_2(t + 1) - 1) + \alpha U_1(t + 1)) \right). \quad (20)$$

Likewise, for an agent in state 1 who consumes,

$$U_1(t) = 1 + \beta \left((1 - \pi) \left((1 - \alpha)(U_2(t + 1) - 1) + \alpha U_1(t + 1) \right) + \pi U_0(t + 1) \right), \quad (21)$$

where $U_0(t + 1)$ is utility in period $t + 1$, before knowing the type in that period, of a liquidity-constrained agent (in state 0) who cannot consume. An agent in state 0 has to pay an additional penalty if he turns to be of the high type, and has the same distribution of money at the end of the period as an agent in state 1 who consumes (but he does not have the benefit of that consumption). Hence,

$$U_0(t) = -\alpha c - 1 + U_1(t). \quad (22)$$

Equations (20), (21) and (22) determine $(U_0(t), U_1(t), U_2(t))$ as a recursive function of $(U_0(t + 1), U_1(t + 1), U_2(t + 1))$. Let U be the vector of utilities $U = (U_0, U_1, U_2)'$. The three recursive equations form a system

$$U(t) = \beta A(\pi(t))U(t + 1) + B(\pi(t)), \quad (23)$$

where $A(\pi)$ is a 3×3 matrix and $B(\pi)$ is a 3×1 vector:

$$A(\pi) = \begin{pmatrix} \pi & (1 - \pi)\alpha & (1 - \pi)(1 - \alpha) \\ \pi & (1 - \pi)\alpha & (1 - \pi)(1 - \alpha) \\ 0 & \pi\alpha & 1 - \pi\alpha \end{pmatrix}, \quad B(\pi) = \begin{pmatrix} -\alpha c - \beta(1 - \pi)(1 - \alpha) \\ 1 - \beta(1 - \pi)(1 - \alpha) \\ 1 - \beta\pi(1 - \alpha) \end{pmatrix} \quad (24)$$

The value of $\pi(t)$ is a function of the distribution of money in period t , and is determined by (8).

The consumption function of the low regime is optimal under the following conditions for an agent in state 1:

- (i) the utility of a low type agent who saves, $U_2 - 1$, is greater than that of consumption, U_1 : $U_2(t) - U_1(t) \geq 1$;
- (ii) the utility of a high type agent who consumes, U_1 , is greater than that of saving, $U_2 - 1 - c$. Hence, $U_2(t) - U_1(t) \leq 1 + c$.

Combining the two conditions, the necessary and sufficient condition is

$$1 \leq U_2(t) - U_1(t) \leq 1 + c. \quad (25)$$

Note that with probability one, the inequalities are strict. Using the equations (23) and (24), it is shown in the appendix that the difference $X(t) = U_2(t) - U_1(t)$ satisfies the difference equation

$$X_t = \beta \left(\alpha(1 - 2\pi_t) + \pi_t \right) X_{t+1} + \beta \pi_t (1 + \alpha c) + \beta(1 - \alpha)(1 - 2\pi_t). \quad (26)$$

We begin with the steady state. Let X^* be the stationary solution of the previous difference equation. Simple algebra shows that the inequality $X^* \geq 1$ is equivalent to $\pi^* \alpha c / \rho \geq 1$. Furthermore, for any parameter and value of π , $X^* < 1 + c$: the right inequality in (25) is satisfied in the steady state. and that inequality has also a simple interpretation: it were not satisfied, a high type would prefer to save in order to reduce his cost of no consumption while a high type in the future over the cost of no consumption in the present. Because of the discounting, this cannot be true. We have proven the following result.

Proposition 5

When $N = 2$, the low regime steady state is an equilibrium if and only if

$$\frac{\pi^* \alpha c}{\rho} \geq 1. \quad (27)$$

where π^ is the rate of unemployment as described in Proposition 4.*

The result has a simple interpretation: the discounted expected value of the cost of unemployment measured as the product of the probability of the high type and the penalty for not consuming in the high type must be greater than one.

For the dynamic path, we have only sufficient conditions that are established in the appendix using the difference equation (26).

Proposition 6

When $N = 2$, the low regime consumption is optimal under the following sufficient conditions:

- $\pi^* \alpha c > \rho$ if $\alpha \leq \frac{1}{2}$ and $\gamma_0(0) < \gamma_0^*$, or $\alpha > 1/2$ and $\gamma_0(0) > \gamma_0^*$, where γ_0 is the initial mass of agents in state 0 and π^*, γ_0^* are values in the steady state of the low regime.
- for values of α and $\gamma_0(0)$ that do not satisfy one of the previous conditions, $\pi(0)\alpha c > \rho$, with $\pi(0) = -(1 - 2\alpha)\gamma_0(0) + (1 - \alpha)(2 - M)$.

In the second item of the proposition, the unemployment rate rises on the dynamic path and the condition $\pi(0)\alpha c > \rho$ implies the inequality (25) in the steady state.

When the economy is initially at full employment, $\gamma_0(0) = 0$, and the previous conditions are simpler

$$\begin{cases} \pi^* \alpha c > \rho, \text{ if } \alpha \leq \frac{1}{2}, \\ \pi_0 \alpha c > \rho, \text{ with } \pi_0 = (1 - \alpha)(2 - M), \text{ if } \alpha > \frac{1}{2}. \end{cases} \quad (28)$$

5.2 Equilibrium in the high regime

We assume that $M > 1$ which is the most interesting case. The consumption function of the high regime is optimal in period t if an agent in state 1 and of the low type prefers to consume rather than save, that is if $X_t = U_2(t) - U_1(t) \leq 1$. The analysis of the low regime can be used here if we replace α by 1 in the previous equations. The difference equation (26) takes now the form

$$X_t = \beta(1 - \pi_t)X_{t+1} + \beta\pi_t(1 + c). \quad (29)$$

The stationary solution is equal to $\beta\pi^*(1 + c)/(1 - \beta(1 - \pi^*))$, and since $\pi^* = 0$ (Proposition 1), it is equal to 0. The consumption function is trivially optimal near the steady state. Since the high regime converges to the steady state with full employment (Proposition 1), there is T such that if $t > T$, $X_t < 1$. Using (29), if $X_{t+1} < 1$, a sufficient condition for $X_t < 1$ is that $c < \rho$. By induction, for any t , $X_t < 1$.

Proposition 7

If $M > 1$ and $c < \rho$, for any initial distribution of money, the dynamic path in the high regime is an equilibrium.

Note the condition $c < \rho$ in the proposition is strong and the result holds for any distribution of money. The high regime could be an equilibrium under a weaker assumption for particular distributions of money in the first period. For example, if $M \geq 1$ with a uniform distribution, a stationary equilibrium with full employment is sustainable, for any value of c (Proposition 1). By continuity, if the initial money distribution is not too different from a full-employment distribution, the high regime can be a equilibrium that converges to full employment.

6 Liquidity Trap

Suppose that the economy is a stationary equilibrium with full employment. There exists a value \bar{c}_1 such that if $c > \bar{c}_1$, one of the inequalities in (28) is satisfied and by Proposition 6, the low regime path is also an equilibrium: an exogenous shift of (perfect foresight) expectations towards pessimism can push the economy on the path with an employment rate that converges to a positive value. The switch to the low regime is self-fulfilling.

Suppose now that the economy is in the stationary equilibrium of the low regime with $\pi^* \alpha c > \rho$ (Proposition 5). Can an exogenous change of animal spirits lift the economy out of that state and set the economy on a path back to full employment?

Let period 0 be the first period in which the consumption is higher than in the low regime. In that period, we must have $X_0 \leq 1$. We now show that if c is sufficiently large, this inequality cannot be satisfied.

Using (26) with $\alpha = 1$ and $X_1 \geq 0$, (more money is better), $X_0 \geq \beta \pi(0)(1 + c)$, where $\pi(0)$ is the unemployment rate in period 0. If $\alpha = 1$, the only agents who do not consume are the liquidity-constrained in state 0. Hence $\pi(0) = \gamma_0^*$, where γ_0^* is the mass of liquidity-constrained agents in the steady state of the low regime and depends only on M , (Proposition 4). X_0 cannot be smaller than 1 if $\beta \gamma_0^*(1 + c) > 1$, which is equivalent to

$$\gamma_0^*(1 + c) > 1 + \rho. \tag{30}$$

If c is sufficiently large (to satisfy $\pi^* \alpha c > \rho$ and (30)), then $X_1 > 1$ and there is no first period in which agents in state 1 with a low type shift to consumption instead

of saving. The stationary equilibrium in the low regime is the only equilibrium. We have proven previous discussion is summarized by the next result.

Proposition 8

There exists a value \bar{c} such that if $c > \bar{c}$,

- (i) if the economy is at or near the full-employment stationary equilibrium, a shift of expectations can push the economy to a low regime path with an unemployment rate that converges to a strictly positive value;*
- (ii) if the economy is at the stationary equilibrium of the low regime with positive unemployment, that is the unique equilibrium.*

The previous result shows the existence of a liquidity trap equilibrium. In that equilibrium, agents attempt to accumulate money balances because of the uncertainty of future exchanges. There is an asymmetry between the high regime with full employment and the low regime that leads to a liquidity trap. In any period, a switch from the high to the low regime can occur, but if the economy has been sufficiently long in a low regime, the economy may not switch back to a path toward full employment and the low regime may be the only equilibrium with a permanent positive unemployment rate.

7 Multiple stationary equilibria with unemployment

Proposition 5 has shown that under the constraint that agent cannot have money balances in excess of $N = 2$, there is a stationary equilibrium with unemployment. That equilibrium is unique by Proposition 3. When N is higher than 2, there may be multiple stationary equilibria with unemployment. The intuition is straightforward. Assume that agents are not constrained in the amount of money they can hold. Consider the example of Figure 2. For a given quantity of money M^* , the unemployment rate in the stationary state with $N = 2$ is equal to $\pi_2 = 0.15$, (point A in Figure 2). For the same quantity of money, in the stationary state with $N = 3$, the unemployment rate is higher at $\pi_3 = 0.6105$, (point B). But an agent will accumulate more money balances when the unemployment rate is higher. There is a strategic complementarity between the monetary savings.

It remains to show that each stationary state is an equilibrium, that the consumption function where agents do not accumulate a balance higher than $N = 2$ ($N = 3$) is

optimal when the unemployment rate is equal to π_2 (π_3). That is shown in a numerical example for $M^* = 1.7944$, with the other parameters $C = 1.8$, $\alpha = 0.25$, $\beta = 0.9$.

Assume first that the unemployment rate is $\pi_2 = 0.15$. If an agent saves (when of the low type) up to a maximum of money equal to 2, his utility vector⁷ U_i is presented in the second column of Table 7. The differences between the utility levels are presented in the next column. One verifies that the condition (25) is satisfied with $U_2 - U_1 = 1.0208 > 1$: when the agent has one unit of money (in state 1), he prefers to save when he is of the low type.

If the agent saves in state 2 (when of the low type) and accumulates balances up to 3, he gets the utility levels in the fourth column of the table (using (40) and (41) in the appendix). A necessary condition for that consumption to be optimal is $U_2 - U_1 \geq 1$ and $U_3 - U_2 \geq 1$, which is not met by the values in the table. Other consumption functions for accumulation above 3 would also be sub-optimal.

	Max M= 2		Max M=3	
State	Utility	Difference	Utility	Difference
0	5.7721		5.1081	
1	7.6221	1.85	6.9581	1.85
2	8.643	1.0208	7.8633	0.9051
3			8.7038	0.8406

Table 1: Utilities for two consumption function in the stationary equilibrium with unemployment rate $\pi = 0.15$

Consider now the second case with the unemployment rate $\pi_3 = 0.6105$. Using the same arguments as for the previous case, the results in Table 7 show that the consumption function that accumulates balances up to $M = 3$ is optimal. That consumption function generates the distribution of money that itself generates the unemployment rate equal to π_3 : the economy is in equilibrium.

⁷Recall that U_0 is the utility in state 0 before knowing the type, and for $i \geq 1$, U_i is the utility of consumption.

	Max M= 3		Max M=4	
State	Utility	Difference	Utility	Difference
0	-0.689		-0.8456	
1	1.161	1.85	1.0044	1.85
2	2.6653	1.5043	2.4491	1.4447
3	4.0193	1.354	3.5962	1.1472
4			4.558	0.9618

Table 2: Utilities for two consumption function in the stationary equilibrium with unemployment rate $\pi = 0.6105$

8 Conclusion

When the economy is in a liquidity trap, a uniform lump-sum distribution of money can make the switch to a high regime possible. However, if expectations remain of a low regime, that regime may still be an equilibrium. The policy has some effect however because we have seen that in the stationary equilibrium of the low regime, the rate of unemployment is inversely related to the money supply. There is probably a sufficient quantity of money expansion that can eliminate the unemployment⁸.

The reduction of the price level by policy to a new value that is still an equilibrium value has the same effect as an expansion of money in this model. That equivalence may not hold if agents are able to borrow from financial institutions, an issue that will be examined subsequently.

⁸If individual money balances are bounded by some number N , we have seen that if $M \geq N$, there is full-employment. Without upper-bound, the property presumably holds also.

APPENDIX: Proofs

Proposition 1

Let $S_k(t) = \sum_{j=k}^N \gamma_j(t)$. Given the matrix $H(\pi)$, for any n with $2 \leq n \leq N$,

$$S_k(t+1) = S_k(t) - \pi(t)\gamma_k(t) \leq S_k(t).$$

Any such sequence $S_k(t)$ is monotone decreasing, bounded below by 0 and therefore converges.

Since, $\gamma_N(t) = S_N(t)$ and for $2 \leq k \leq N-1$, $\gamma_k(t) = S_k(t) - S_{k+1}(t)$, $\gamma_k(t)$ converges for any $2 \leq k \leq N$.

Because $\gamma_1(t) = M - \sum_{k=2}^N k\gamma_k(t)$ and $\gamma_0(t) = 1 - \sum_{k=1}^N \gamma_k(t)$, $\gamma_1(t)$ and $\gamma_0(t)$ also converge. Let γ_0^* be the limit of $\gamma_0(t)$.

If $\gamma_0^* > 0$, since $\pi(t) = \gamma_0(t)$, we must have, by induction from $k = N$, and using the expression of the matrix H , $\gamma_k^* = 0$ for $k \geq 2$. In this case, $M = \gamma_1^* = 1 - \gamma_0^* < 1$. Therefore if $M \geq 1$, then $\gamma_0^* = 0$. If $M < 1$, then for any t , $\gamma_0(t) \geq 1 - M$ which concludes the proof. \square

To prove Proposition 3, we first establish some Lemmata.

Lemma 3

For any $\pi \in (0, 1)$, the matrix $L(\pi)$, defined in (12) has an eigenvalue equal to 1 that is of order 1.

Recall that the matrix $L(\pi)$ is square and of dimension $N+1$. Call e the row-vector of ones and of dimension $N+1$. Any distribution Γ has the sum of its components equal to 1. Since the matrix $L(\pi)$ is a function in the set of distributions of dimension $N+1$, $e.L(\pi) = 1$ (which can also be verified directly). Hence, 1 is an eigenvector of the matrix $L(\pi)$. To show that it is of order 1, consider the matrix

$$L(\pi) - I = \begin{pmatrix} \pi - 1 & \alpha\pi & 0 & 0 & \dots & 0 \\ 1 - \pi & a - 1 & \alpha\pi & 0 & \dots & 0 \\ 0 & b & a - 1 & \alpha\pi & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & b & a - 1 & \alpha\pi & 0 \\ 0 & \dots & 0 & b & a - 1 & \pi \\ 0 & 0 & 0 & \dots & b & -\pi \end{pmatrix}, \quad (31)$$

where I is the identity matrix of dimension $N + 1$. Let Δ_j be the determinant for the rows and columns $N - j + 1$ to N of this matrix. We want to prove that the determinant Δ_{N-1} of dimension $N - 1$ is different from 0.

Replacing the first row by the sum of all rows and using $a + b - 1 = -\alpha\pi$, $\Delta_2 = \pi(1 - a - b) = \alpha\pi$, and for $2 \leq j \leq N - 2$, $\Delta_{j+1} = -\alpha\pi\Delta_j$. Hence Δ_{N-1} is not equal to 0 and the unit eigenvalue is of order 1.

Lemma 4

For any $\pi \in (0, 1)$, and Γ such that $\sum_k \gamma_k = 1$, $L^t(\pi)\Gamma$ tends to the eigenvector with sum of the components equal to one that is associated to the unit eigenvalue of $L(\pi)$ as t tends to infinity.

We will replace $L(\pi)$ by L in the proof. Since the sum of the elements of L in each column is equal to 1, for any vector v , $|Lv| = |\sum_{ij} L_{ij}v_j| = |\sum_j v_j| \leq \sum_j |v_j|$. The matrix L is contracting. Furthermore if there are two non identical distributions $v \neq w$ with $\sum_j v_j = \sum_j w_j = 1$, then $|L.(v - w)| = |\sum_j (v_j - w_j)| < \sum_j |v_j - w_j|$.

The sequence $\Gamma^t = L^t\Gamma$ has at least one accumulation point, $\tilde{\Gamma}$, because it belongs to a compact. Let Γ^* be the eigenvector associated to the unit eigenvalue of $L(\pi)$ such that $\sum_k \gamma_k^* = 1$. Using the definition of $\tilde{\Gamma}$, the difference $\tilde{\Gamma} - \Gamma^*$ can be approximated arbitrarily closely by $L^k\tilde{\Gamma} - L^k\Gamma^* = L^{k-1}L.(\tilde{\Gamma} - \Gamma^*)$. If $\tilde{\Gamma} \neq \Gamma^*$, by the previous paragraph, $|\tilde{\Gamma} - \Gamma^*| \leq |L.(\tilde{\Gamma} - \Gamma^*)|$, a contradiction. Therefore, the sequence $\Gamma^t = L^t\Gamma$ has the limit Γ^* . \square

From Lemma 3 and Lemma 4, we have immediately

Lemma 5

For any $\pi \in (0, 1)$, the eigenvector $\Gamma^(\pi)$ that is associated to the unit-eigenvalue of $L(\pi)$ such that $\sum_k \gamma_k^*(\pi) = 1$ has strictly positive components: $\gamma_k(\pi) > 0$ for $k = 0, 1, \dots, N$.*

We denote by \succeq the ordering according to first-order stochastic dominance. Comparing to distributions Γ and $\tilde{\Gamma}$,

$$\Gamma' \succeq \Gamma \quad \text{if and only if for any } K < N \quad \sum_{k=0}^{k=K} \gamma'_k \geq \sum_{k=0}^{k=K} \gamma_k.$$

Lemma 6

For any distribution of money Γ , let $\tilde{\Gamma}(\pi) = L(\pi)\Gamma$, where $L(\pi)$ is the transition matrix given in (12). If $\pi' \geq \pi$, then $\tilde{\Gamma}(\pi') \succeq \tilde{\Gamma}(\pi)$.

The case $n = 2$ is trivial. Assume $N \geq 3$. For $K = 1$, the inequality is verified because $\gamma_0 > 0$. For $K = 2$,

$$\tilde{\gamma}_0(\pi) + \tilde{\gamma}_1(\pi) = \gamma_0 + \left(\pi(1 - \alpha) + \alpha\right)\gamma_1 + \alpha\pi\gamma_2,$$

which is non-decreasing in π .

Likewise, for $K \leq N - 1$,

$$\sum_{k=0}^K \tilde{\gamma}_k(\pi) = \sum_{k=0}^{K-1} \gamma_k(\pi) + \left(\pi(1 - \alpha) + \alpha\right)\gamma_K + \zeta_K \pi \gamma_{K+1}, \quad (32)$$

with $\zeta_K = \alpha$ if $K \leq N - 2$, and $\zeta_K = 1$ if $K = N - 1$.

The right-hand side in (32) is non-decreasing in π . \square

The result has an intuitive interpretation. When π increases, the aggregate demand falls which shifts distribution of money “to the left”.

Lemma 7

Let there be two distributions Γ^2 and Γ^1 such that $\Gamma^2 \succeq \Gamma^1$. For any $\pi \in (0, 1)$,

$$L(\pi)\Gamma^2 \succeq L(\pi)\Gamma^1.$$

To prove that the first component of $L(\pi)\Gamma^2$ is at least equal to that of $L(\pi)\Gamma^1$, use $\gamma_0^2 + \gamma_1^2 \geq \gamma_0^1 + \gamma_1^1$ and take the difference

$$\begin{aligned} \pi\gamma_0^2 + \alpha\pi\gamma_1^2 - \pi\gamma_0^1 - \alpha\pi\gamma_1^1 &= \pi(\gamma_0^2 - \gamma_0^1) + \alpha\pi(\gamma_1^2 - \gamma_1^1) \\ &\geq (1 - \pi\alpha)(\gamma_0^2 - \gamma_0^1), \end{aligned}$$

which is non negative because $\gamma_0^2 \geq \gamma_0^1$.

A similar argument is applied for the sum of the first two components of $L(\pi)\Gamma^j$, $j=1,2$. In general, the difference between the sum of the first k components, $3 \leq k \leq N - 1$ is

$$\begin{aligned} &\sum_{j=0}^{k-1} (\gamma_j^2 - \gamma_j^1) + ((1 - \alpha)\pi + \alpha)(\gamma_k^2 - \gamma_k^1) + \alpha\pi(\gamma_{k+1}^2 - \gamma_{k+1}^1) \\ &\geq (1 - \alpha\pi) \sum_{j=0}^{k-1} (\gamma_j^2 - \gamma_j^1) + ((1 - \alpha)\pi + \alpha - \alpha\pi)(\gamma_k^2 - \gamma_k^1) \\ &\geq (1 - \alpha - (1 - \alpha)\pi) \sum_{j=0}^{k-1} (\gamma_j^2 - \gamma_j^1) = (1 - \alpha)(1 - \pi) \sum_{j=0}^{k-1} (\gamma_j^2 - \gamma_j^1) \end{aligned}$$

which is non negative. \square

The proof of the following result is left as an exercise.

Lemma 8

If $\Gamma^2 \succeq \Gamma^1$, then the quantity of money in the distribution Γ^2 is not strictly greater than in the distribution Γ^1 .

We can now prove Proposition 3.

Proposition 3

Using Lemma 5, we can define $\Gamma^*(\pi)$ as the steady state distribution associated to π such that $\sum_k \gamma_k^* = 1$. For any t , $L(\pi)^t \Gamma^* = \Gamma^*$. Take $\pi' > \pi$ and define the sequence $\Gamma^t = L(\pi')^t \Gamma^*$. By Lemma 6,

$$\Gamma^1 = L(\pi') \Gamma^* \succeq L(\pi) \Gamma^* = \Gamma^* = \Gamma^0.$$

By Lemma 7, if $\Gamma^t \succeq \Gamma^{t-1}$,

$$\Gamma^{t+1} = L(\pi') \Gamma^t \succeq L(\pi') \Gamma^{t-1} = \Gamma^t.$$

The sequence Γ^t defines an increasing sequence of distribution in the sense of first-order stochastic dominance and, by Lemma 4, converges to the eigenvector of $L(\pi')$ and defines the distribution of money in the steady state associated to $L(\pi')$. By Lemma 8, the quantity of money in that distribution is not strictly smaller than in the distribution $\Gamma^*(\pi)$. The quantity of money in the distribution $\Gamma^*(\pi)$ is a continuous function of π because $\Gamma^*(\pi)$ is continuous. The proof is concluded by taking the limits for $\pi \rightarrow 0$ and $\pi \rightarrow 1$. \square

The low regime with $N = 2$

The transition matrix L in the case $N = 2$ is given in (12). For any t , the quantity of money is $M = 2(1 - \gamma_0(t) - \gamma_1(t)) + \gamma_1$,

$$\gamma_1(t) = -2\gamma_0(t) + S, \quad \text{with} \quad S = 2 - M, \quad (35)$$

and the rate of unemployment, $\pi(t)$, is equal to $\gamma_0(t) + (1 - \alpha)\gamma_1(t)$. Hence,

$$\pi(t) = -(1 - 2\alpha)\gamma_0(t) + (1 - \alpha)S, \quad (36)$$

and

$$\begin{aligned}\gamma_0(t+1) &= \pi(t)\left(\gamma_0(t) + \alpha(-2\gamma_0(t) + S)\right), \\ &= \left((2\alpha - 1)\gamma_0(t) + (1 - \alpha)S\right)\left((1 - 2\alpha)\gamma_0(t) + \alpha S\right),\end{aligned}$$

or

$$\gamma_0(t+1) = -(1 - 2\alpha)^2\gamma_0(t)^2 + (1 - 2\alpha)^2S\gamma_0(t) + \alpha(1 - \alpha)S^2. \quad (37)$$

The value of γ_0 characterizes the distribution of money because γ_1 and γ_2 can be derived from γ_0 using the mass one of agents and the quantity M of money.

Proposition 5

We introduce the differences

$$Z_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix}, \quad \text{with} \quad \begin{cases} X_t = U_2(t) - U_1(t), \\ Y_t = U_2(t) - U_0(t). \end{cases}$$

We can write

$$\begin{cases} X_t = g.U(t), & \text{with} \quad g = \begin{pmatrix} 0 & -1 & 1 \end{pmatrix}, \\ Y_t = h.U(t), & \text{with} \quad h = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}. \end{cases}$$

Using (23),

$$\begin{cases} X_t = g.U(t) = \beta g.A(\pi_t)U(t+1) + g.B(\pi(t)), \\ Y_t = h.U(t) = \beta h.A(\pi_t)U(t+1) + h.B(\pi(t)). \end{cases}$$

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \beta \begin{pmatrix} \alpha(1 - 2\pi_t) & \pi_t \\ \alpha(1 - 2\pi_t) & \pi_t \end{pmatrix} \cdot \begin{pmatrix} X_{t+1} \\ Y_{t+1} \end{pmatrix} + \begin{pmatrix} \beta(1 - \alpha)(1 - 2\pi_t) \\ 1 + \alpha c + \beta(1 - \alpha)(1 - 2\pi_t) \end{pmatrix}.$$

Hence,

$$\begin{cases} X_t = \beta\alpha(1 - 2\pi_t)X_{t+1} + \pi_t\beta Y_{t+1} + \beta(1 - \alpha)(1 - 2\pi_t), \\ Y_t = X_t + 1 + \alpha c. \end{cases}$$

Substituting for Y_{t+1} in the first equation,

$$X_t = \beta\left(\alpha(1 - 2\pi_t) + \pi_t\right)X_{t+1} + \beta\pi_t(1 + \alpha c) + \beta(1 - \alpha)(1 - 2\pi_t),$$

with the stationary solution

$$X^* = \beta \frac{\pi^*(1 + \alpha c) + (1 - \alpha)(1 - 2\pi^*)}{1 - \beta(\alpha(1 - 2\pi^*) + \pi^*)}. \quad (38)$$

□

Proposition 6

Assume that (25) holds strictly. Since $X^* > 1$, on the path of the low regime that converges to the steady state, $X_t > 1$ for t sufficiently large.

Suppose that $X_{t+1} > 1$. Then using the equation of backward induction (26), a sufficient condition for $X_t > 1$ is

$$\pi_t \alpha c > \rho. \quad (39)$$

From Proposition 4, π_t varies monotonically on the transition path. It decreases with time if $\alpha < 1/2$ and $\gamma_0(0) < \gamma_0^*$, or $\alpha > 1/2$ and $\gamma_0(0) > \gamma_0^*$. If π_t is decreasing, then $X_t > 1$ for all t .

In the other parametric cases, π_t is increasing and if $\pi_0 \alpha c > \rho$, then (39) is satisfied for all t and since for some T , $X_t > 1$ for $t > T$, by backward induction, $X_t > 1$ for all t . The condition $\pi_0 \alpha c > \rho$ with $\pi_0 = \gamma_0(0) + (1 - \alpha)(\gamma_1(0) + \gamma_2(0))$. In this case, the steady state condition (27) holds. Using $\sum \gamma_k = 1$, the previous expression can be replaced by $\pi_0 = 1 - \alpha + \alpha \gamma_0(0)$.

If $\alpha = 1/2$, the economy is stationary for all periods and the necessary and sufficient condition for the optimality of consumption is (27). \square

The bounded distribution of money

We show here that the assumption of an arbitrary upper-bound on money holding can be removed. If the total quantity of money M is greater than 1, there can be a full-employment equilibrium. In that equilibrium, the distribution of money is stationary but somewhat irrelevant. The interesting case is the low regime steady state where the unemployment rate is positive.

In an equilibrium where agents save (when of the low type) up to the maximum balance N , the vector of utilities, U , for each state (amount of money) is obtained by generalizing equations (23) and (24):

$$(I - \beta A)U = B, \quad (40)$$

where I is the N -identity matrix and

$$A(\pi) = \begin{pmatrix} \pi & (1-\pi)\alpha & b & 0 & \dots & 0 \\ \pi & (1-\pi)\alpha & b & 0 & \dots & 0 \\ 0 & \pi & (1-\pi)\alpha & b & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \pi & (1-\pi)\alpha & b \\ 0 & 0 & 0 & \dots & \pi\alpha & 1-\pi\alpha \end{pmatrix}, \quad B(\pi) = \begin{pmatrix} -\alpha c - \beta b \\ 1 - \beta b \\ \vdots \\ 1 - \beta b \\ 1 - \beta\pi(1-\alpha) \end{pmatrix} \quad (41)$$

The next result shows N must be bounded.

Proposition 9

In a steady state equilibrium, the distribution of individual money balances has a bounded support. No agent holds an amount of money greater than $(1 + \alpha c)/(1 - \beta)$.

The result is proven by contradiction. Let U_k , $k \geq 0$ be defined as in the previous section: U_0 is the utility of an agent in state 0 at the beginning of the period, before knowing his type, and U_k is the utility of an agent who consumes in state k , $k \geq 1$.

For any N , define the vector

$$\Delta = (U_0, U_1 - U_0, U_2 - U_1, \dots, U_N - U_{N-1}).$$

For $k \geq 0$. $\Delta_k = U_k - U_{k-1} \geq 0$ because the utility is increasing with money. (There is free disposal).

$$\sum_{k=1}^N \Delta_k = U_N - U_0.$$

The value of U_N is bounded above by $1/(1 - \beta)$ which is the utility of an agent who consumes in every period and U_0 is bounded below by $-\alpha c/(1 - \beta)$ which is the utility of an agent who never consumes. Hence, $\sum_{k=1}^N \Delta_k$ is bounded above. Since $\Delta_k \geq 0$, there cannot be an infinite number of Δ_k that are at least equal to 1. Hence, there is K such that for $k > K$, $\Delta_k < 1$. In all the states $k > K$, the agent does not save. In a steady state with unemployment, there is no agent with strictly positive money balance when in state $k > K$. \square

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