Financially Constrained Arbitrage and Cross-Market Contagion *

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Abstract

We propose a continuous time infinite horizon equilibrium model of financial markets in which arbitrageurs have multiple valuable investment opportunities but face financial constraints. The investment opportunities, heterogeneous along different dimensions, are provided by pairs of similar assets trading at different prices in segmented markets. By exploiting these opportunities, arbitrageurs alleviate the segmentation of markets, providing liquidity to other investors by intermediating their trades. We characterize the arbitrageurs' optimal investment policy, and derive implications for market liquidity and asset prices.

Keywords: Financial constraints, arbitrage, liquidity, contagion.

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1 Introduction (preliminary)

The ongoing crisis has highlighted the importance of intermediary capital for the functioning of financial markets. Indeed, the large losses banks incurred in the subprime market has led them to cut their lending across the board, notably their financing of other intermediaries, causing liquidity to dry up in many otherwise unrelated markets. Central banks the world around struggled to deal with a combined banking liquidity and financial market liquidity crisis.

This paper develops a framework to examine the relation between intermediary capital, financial market liquidity and asset prices. The framework itself has three main features.

First, we model arbitrageurs as specialized investors able to exploit profitable trades that other, less sophisticated market participants cannot access directly as easily or quickly. Arbitrageurs are to be understood here as individuals and institutions responsible for providing liquidity in different financial markets. At the same time, arbitrage is assumed to require capital to which arbitrageurs have only limited access, i.e., they face financial constraints. These financial constraints, be they margin requirements, limited access to external capital or barriers to entry of new capital, affect the arbitrageurs' investment capacity.

Second, ours is a dynamic general equilibrium model. On the one hand, arbitrageurs' capital affects their ability to provide liquidity, which is ultimately reflected in asset prices. On the other hand, asset price movements determine arbitrage profits and, therefore, arbitrageurs' capital. This dynamic interaction shapes arbitrageurs' investment policies, asset prices and market liquidity.

Third, in our model, arbitrageurs face multiple arbitrage opportunities with different characteristics, across which they must allocate their scarce capital. This aspect is important to study the cross-sectional properties of arbitrageurs' optimal investment policy, as well as those of market liquidity and asset prices. In particular, it allows us to analyze phenomena of price contagion and liquidity linkages across markets.

We aim to analyze a number of questions relative to arbitrageurs' investment strategy. To start with, what is the optimal investment strategy of an arbitrageur with financial constraints? How is the need for risk management created by financial constraints resolved when there are multiple arbitrage opportunities with different characteristics? How does an arbitrageur's optimal investment policy respond to shocks to their capital?

More importantly, we are interested in questions about asset prices and market liquidity. Financial constraints lead to wealth effects creating price and liquidity linkages across markets. Which asset or trade characteristics make them more sensitive to changes in arbitrageurs' capital? How much time-variation in convergence spreads is explained by contagion vs. fundamentals? Is diversification of arbitrageurs effective despite contagion effects?[2]

Our model's building block is as follows.

We begin by modelling financial markets needing liquidity as in Gromb and Vayanos (2002, 2008). We consider two risky assets paying similar (possibly identical) dividends but traded in segmented markets. The demand by investors on each market for the local risky asset is affected by endowment shocks that covary with the asset's payoff. Since the covariances differ across the two markets, the assets' prices can differ. Said differently, the investors in the segmented markets would benefit from trading with each other to improve risk sharing. However, there is no liquidity due to the assumed segmentation.

This unsatisfied demand for liquidity creates a role for arbitrageurs. We model arbitrageurs as competitive specialists able to invest across markets and thus exploit price discrepancies between the risky assets. Doing so, they facilitate trade between otherwise segmented investors, providing liquidity to them. Arbitrageurs, however, face financial constraints in that their risky asset positions must be collateralized separately with a position in the riskfree asset. Given these constraints, the arbitrageurs' ability to provide market liquidity depends on their wealth. The arbitrageurs' wealth is to be understood as the pool of capital they can access frictionlessly. In that case, there is no distinction between arbitrageurs' internal funds and the "smart capital" they raise externally. If this total pool of capital is insufficient, arbitrageurs may be unable to provide perfect liquidity.

Based on that building block, we develop a continuous time general equilibrium model in which competitive arbitrageurs face at each point in time several arbitrage opportunities, i.e., multiple asset pairs as above. These opportunities are heterogeneous along different dimensions (e.g., volatility, market size, margin requirements). Due to their financial constraints, arbitrageurs face a complex investment problem. On the one hand, they must allocate their scarce capital across opportunities and over time. On the other hand, the performance of these investments affect their investment capacity.

To begin with, we study the case of riskless arbitrage, in which two assets in a pair pay the exact same dividends. In this case, we are able to derive all equilibrium variables in closed form. This allows us to draw many implications. The following are but examples.

Some implications are cross-sectional in nature, i.e., comparing variables across opportunities with different characteristics. For instance, we show that opportunities with higher margin requirements are more illiquid, offer higher excess returns and have higher risk premia. The intuition is that investment opportunities requiring arbitrageurs to tie up more capital as collateral must provide them with a greater reward, i.e., a higher excess return. Risk premia being the present

[2] We do not explicitly analyze the important welfare and policy issues raised by financially constrained arbitrage. See however Gromb and Vayanos (2002, 2008) as well as Gala, Gromb and Vayanos (2008).

value of future excess returns they must be higher for such opportunities. In our model, risk premia are a measure of the illiquidity arbitrageurs do not eliminate. Therefore, illiquidity is higher for opportunities with higher margin requirements.

Other implications involve comparative statics, in particular with respect to arbitrageur wealth. One way to interpret these results is as the effect of an unanticipated exogenous shock to arbitrageur wealth. For instance, we show that illiquidity and risk premia are more sensitive to arbitrageur wealth for opportunities with higher margin requirements. Intuitively, changes in arbitrageur wealth affect the excess return (current or future) per unit of collateral, and therefore impact more opportunities with higher collateral requirements.

Next, we analyze the case of risky arbitrage. There are two sources of arbitrage risk: fundamental risk and supply risk. Fundamental risk means that the assets in a given pair may not pay the exact same dividends. Fundamental shocks affect arbitrage profits, and therefore arbitrageur wealth and ultimately assets prices and liquidity. Supply risk means that the demand for liquidity may not be predictable. Shocks to the demand for liquidity affect risk premia both directly, i.e., holding arbitrageur wealth constant, and indirectly because changes in premia affect arbitrageur's profit and therefore arbitrageur wealth.

First, we show that the arbitrageurs' financial constraints create a linkage across otherwise independent assets, i.e., fundamental and supply shocks to one opportunity affect all opportunities' risk premia. The linkage goes through arbitrageur wealth. Indeed, a fundamental shock to one opportunity affects the dividend arbitrageurs derive from that opportunity, and hence arbitrageur's wealth. Similarly, a supply shock to one opportunity affects that opportunity's risk premium and hence the capital gains arbitrageurs realize from their investment in that opportunity. In both cases, a change in arbitrageur wealth affects all other opportunities' risk premia.

To derive further implications, we consider the effect of arbitrarily small shocks, i.e., we study equilibrium variables around the riskfree arbitrage equilibrium. We characterize the effect of different shocks on arbitrageurs' portfolios, market liquidity, the volatility of asset prices as well as the correlation between the prices of different assets.

We show that liquidity, volatility and correlations are generally non-monotonic in arbitrageur wealth. (Here an asset's liquidity is defined as the inverse of the impact a supply shock would have on the asset return.) If arbitrageur wealth is high, as arbitrageur wealth increases, liquidity increases, while price volatility decreases for all assets. As for correlations, they decrease for assets whose fundamentals are uncorrelated, and increase for assets whose fundamentals are correlated. Maybe more surprisingly, these relationships are reversed when arbitrage wealth is low. In that case, as arbitrageur wealth increases, liquidity decreases, while price volatility increases for all assets. At the same time, correlations increase for assets whose fundamentals are uncorrelated, and decrease for assets whose fundamentals are correlated.

The reason for this reversal is that for high levels of wealth, arbitrageurs are unlikely to be constrained and therefore their positions are not very sensitive to wealth. In that case, a drop in wealth does not reduce much the positions arbitrageurs take and therefore does not affect much the amount of wealth arbitrageurs put at risk. The opposite holds for low levels of wealth. In that case, a drop in wealth leads to a large reduction in arbitrageurs' positions, reducing the amount of wealth arbitrageurs put at risk.

Relation to the Literature (Incomplete)

We view our framework itself as a contribution. Indeed, we provide a model of limited arbitrage capturing both intertemporal aspects of arbitrage and allowing for a cross-section of investment opportunities with different characteristics: volatility, horizon, market size, etc. The model nests both riskfree arbitrage as well as arbitrage involving fundamental risk and supply risk. Yet, the model remains fairly tractable and flexible (e.g. closed for solutions), and has some standard features including an infinite horizon and stationarity.

Our analysis builds on the recent literature on the limits to arbitrage and, more particularly, on financially constrained arbitrage.[3,4] Gromb and Vayanos (2002) introduce a model of arbitrageurs providing liquidity across two segmented markets but facing collateral-based financial constraints. Their setting is dynamic, i.e., they consider explicitly the link between arbitrageurs' past performance and their ability to provide market liquidity, and how arbitrageurs take this link into account in their investment decision. They also conduct a welfare analysis. This paper extends the analysis by considering multiple investment opportunities.[5]

The Gromb and Vayanos (2002) model is extended to multiple investment opportunities in a static setting by Brunnermeier and Pedersen (2009) who show how financial constraints imply that shocks propagate and liquidity co-moves across markets. In contrast, ours is a dynamic setting. Kyle and Xiong (2001) obtain similar financial contagion effects driven by the wealth of arbitrageurs. These arise not from financial constraints but from arbitrageurs' logarithmic utility implying that their demand for risky assets is increasing in wealth.

The paper proceeds as follows. Section 2 presents the model. Sections 3 and 4 study riskless and risky arbitrage respectively. Section 5 concludes. The Appendix contains mathematical proofs.

[3] Alternative theories of the limits to arbitrage are generally based on incentive problems in delegated portfolio management or bounded rationality of investors.

[4] Here, we discuss the relation of our paper to only the closest literature. It is however connected to a broader set of contributions which, given binding (time) constraints, we intend to discuss in future versions. Gromb and Vayanos (2002, 2008) contain more detailed descriptions of the relevant literature.

[5] Also, the model is cast in an infinite horizon, rather than a finite horizon.

2 The Model

The universe of assets consists of a riskless asset and pairs of risky assets. The risky assets in a pair pay correlated dividends (Section 2.1), and are traded on segmented markets by agents we call outside investors (Section 2.2). Segmentation prevents gains from trade, thereby creating a role for arbitrageurs who, unlike outside investors, can access all assets and thus provide liquidity to outside investors. While arbitrageurs have better investment opportunities than outside investors, they face financial constraints limiting their ability to exploit these opportunities (Section 2.3).

2.1 Assets

Time is continuous and indexed by $t \in \mathbb{R}^+$. The riskless asset has an exogenous continuously compounded return r. There is a set $\mathcal{I}$ of risky assets, all in zero net supply.[1] For $t \in \mathbb{R}^+$, asset $i \in \mathcal{I}$'s price is $p_{i,t}$ and its cumulative dividend $D_{i,t}$ evolves according to

$$dD_{i,t} = D_i dt + \sigma_i dB_{i,t} + \sigma_i^f dB_{i,t}^f, \tag{1}$$

where D_i, σ_i, and σ_i^f are constants, and $B_{i,t}$ and $B_{i,t}^f$ are independent Brownian motions. The constant D_i is asset i's instantaneous expected dividend.

Assets come in pairs. We denote $-i$ the other asset in asset i's pair. Moreover, $B_{i,t}$ and $B_{i',t'}$ as well as $B_{i,t}^f$ and $B_{i',t'}^f$ are independent Brownian motions unless $i' = -i$ and $t = t'$. Instead, $B_{-i,t} = B_{i,t}$ and $B_{-i,t}^f = B_{i,t}^f$. Finally, we assume that the shock $dB_{i,t}$ has identical effects on the two assets, while $dB_{i,t}^f$ has opposite effects, i.e.,

$$\sigma_{-i} = \sigma_i \quad \text{and} \quad \sigma_{-i}^f = -\sigma_i^f. \tag{2}$$

This specification nests the case in which the dividends paid by assets i and $-i$ in a pair are correlated but different ($\sigma_i^f \neq 0$) and that in which they are identical ($\sigma_i^f = 0$). These cases will correspond to arbitrage opportunities with and without fundamental risk respectively.

2.2 Outside Investors

Outside investors form overlapping generations that each live for an infinitesimal time period. For them, the markets for all risky assets are segmented, i.e., each outside investor can form portfolios of only two assets: the riskless asset and a single and specific risky asset. We refer to investors born at time t and who can invest in risky asset i as (i, t)-investors.

[1] The assumptions of exogenous riskless return, and zero net supply assets are for simplicity. In particular, the latter ensures that arbitrageurs hold opposite positions in risky assets in the same pair.

Market segmentation is taken as given, i.e., (i,t)-investors are assumed to face prohibitively large transaction costs for investing in any risky asset other than asset i. These costs can be due to unmodelled physical factors (e.g., distance), information asymmetries or institutional constraints. Market segmentation is a realistic assumption in many contexts. In an international context, for example, it is well known that investors mainly hold domestic assets.

At each time t, the (i,t)-investors are competitive, form a continuum of measure μ_i, and all have the same constant absolute risk aversion coefficient a_i. Denoting $dw_{i,t}$ a change in an (i,t)-investor's wealth $w_{i,t}$, his preference over instantaneous mean and variance is therefore[6]

$$E_t(dw_{i,t}) - \frac{a_i}{2} Var_t(dw_{i,t}). \tag{3}$$

At each time t, the (i,t)-investors receive non-tradeable endowments that are correlated with asset i's dividend. Specifically, the endowment consists of $u_{i,t}$ shares of asset i, where $u_{i,t}$ evolves according to the Ornstein-Uhlenbeck process

$$du_{i,t} = \kappa_i^u (u_i - u_{i,t})dt + \sigma_i^u dB_{i,t}^u, \tag{4}$$

where κ_i^u, u_i and σ_i^u are constants and $B_{i,t}^u$ is a Brownian motion independent of $B_{i,t}$ and $B_{i,t}^f$.[7]

The coefficient $u_{i,t}$, which can be positive or negative, measures the extent to which the endowment covaries with asset i's dividend. If $u_{i,t}$ is large and positive, the shock and dividend are highly positively correlated, and thus the willingness of (i,t)-investors to hold asset i at time t is low. Conversely, if $u_{i,t}$ is large and negative, the shock and dividend are highly negatively correlated, and the (i,t)-investors keen to hold asset i at time t for hedging. We refer to $u_{i,t}$ as the (i,t)-investors' supply parameter to emphasize that their asset demand decreases with $u_{i,t}$. The model nests the case in which $u_{i,t}$ is stochastic ($\sigma_i^u \neq 0$) and that in which $u_{i,t}$ is deterministic ($\sigma_i^u = 0$). These cases will correspond to arbitrage opportunities with and without supply risk respectively.

We assume that (i,t)- and $(-i,t)$-investors are identical but for their supply parameters, i.e.,[8]

$$\forall i \in \mathcal{I}, \quad u_{-i,t} = -u_{i,t}. \tag{5}$$

Because their supply parameters differ, (i,t)- and $(-i,t)$-investors have different propensities to hold assets i and $-i$. However, they cannot realize the potential gains from trade due to market segmentation. This unsatisfied demand for liquidity creates a role for arbitrageurs.

[6]This property also arises under exponential utility or if initial wealth is the same across all generations. We show below that the logarithmic preferences of arbitrageurs generate similar preferences over instantaneous mean and variance, but with the weight on variance depending on wealth. We suppress wealth effects for the (i,t)-investors because unlike those for arbitrageurs, they are not central to our analysis.

[7]For consistency with the zero net supply assumption, endowments can be interpreted as positions in a different but correlated asset, e.g. labor income. This specification is standard in the market microstructure literature (O'Hara, 1995).

[8]That is, we assume $\mu_i = \mu_{-i}$ and $a_i = a_{-i}$. The assumed preferences make assuming $w_{i,t} = w_{-i,t}$ useless for the analysis.

Each (i,t)-investor's problem, $\mathcal{P}_{i,t}$, is to choose a holding $y_{i,t}$ of asset i at time t to maximize (3) subject to a dynamic budget constraint, derived as follows. At time t, an (i,t)-investor invests $y_{i,t}p_{i,t}$ in asset i and the rest of his wealth, $(w_{i,t} - y_{i,t}p_{i,t})$, in the riskless asset, from which he receives a return r. His total exposure to asset i is therefore $(y_{i,t} + u_{i,t})$, from which he receives a dividend $dD_{i,t}$ and a capital gain $dp_{i,t}$. Hence, his dynamic budget constraint is

$$dw_{i,t} = r(w_{i,t} - y_{i,t}p_{i,t})dt + (y_{i,t} + u_{i,t})(dD_{i,t} + dp_{i,t}). \tag{6}$$

Note that the evolution $du_{i,t}$ of the endowment does not affect $dw_{i,t}$: it only affects the endowment of the generation of $(i, t + dt)$-investors.

2.3 Arbitrageurs

We model arbitrageurs as specialists able to invest across markets to exploit price discrepancies between similar risky assets. Arbitrageurs are infinitely lived, competitive and form a continuum with measure 1.[9] We denote their wealth at time t as W_t. They maximize intertemporal utility of consumption (c_s at time $s \in \mathbb{R}^+$) under logarithmic preferences with discount factor β, i.e.,

$$E_t\left[\int_t^\infty \log(c_s)e^{-\beta(s-t)}ds\right]. \tag{7}$$

Unlike other investors, arbitrageurs can invest in all risky assets and in the riskless asset. Doing so, they facilitate trade between otherwise segmented investors, providing liquidity to them.

At time t, each arbitrageur invests $x_{i,t}p_{i,t}$ in asset i, from which he receives a dividend $dD_{i,t}$, and a capital gain $dp_{i,t}$, and the rest of his wealth, $\left(W_t - \sum_{i\in\mathcal{I}} x_{i,t}p_{i,t}\right)$, in the riskless asset, from which he receives a return r. Therefore, an arbitrageur's dynamic budget constraint is

$$dW_t = r\left(W_t - \sum_{i\in\mathcal{I}} x_{i,t}p_{i,t}\right)dt + \sum_{i\in\mathcal{I}} x_{i,t}(dD_{i,t} + dp_{i,t}) - c_t dt. \tag{8}$$

Arbitrageurs, however, face financial constraints in that their positions in any risky asset must be collateralized separately with a position in the riskfree asset. Specifically, we assume that to long or short one share of asset i, they must post some exogenous amount $m_i > 0$ of riskless collateral. Therefore, their investment capacity is limited by their wealth:

$$W_t \geq \sum_{i\in\mathcal{I}} m_i |x_{i,t}|. \tag{9}$$

[9] By fixing the measure of the arbitrageurs, we rule out entry into the arbitrage industry. This seems a reasonable assumption at least for understanding short-run market behavior.

To keep the model symmetric, we assume that $m_i = m_{-i}$. Given these constraints, the arbitrageurs' ability to provide market liquidity depends on their wealth. The arbitrageurs' wealth is to be understood as the pool of capital they can access frictionlessly. In that case, there is no distinction between arbitrageurs' internal funds and the "smart capital" they raise externally. If this total pool of capital is insufficient, arbitrageurs may be unable to provide perfect liquidity to markets.

Each arbitrageur's problem, $\mathcal{P}_t$, is to choose holdings $\{x_{i,s}\}_{i\in\mathcal{I},s\geq t}$ to maximize utility (7) subject to the dynamic budget constraint (8) and the financial constraint (9).

2.4 Equilibrium

Definition 1 *A competitive equilibrium consists of prices $p_{i,t}$, asset holdings of the (i,t)-investors, $y_{i,t}$, and of the arbitrageurs, $x_{i,t}$, such that given the prices, $y_{i,t}$ solves problem $\mathcal{P}_{i,t}$, and $\{x_{i,s}\}_{i\in\mathcal{I},s\geq t}$ solve problem $\mathcal{P}_t$, and the markets clear for all risky assets $i \in \mathcal{I}$ and all times $t \in \mathbb{R}^+$:*

$$\mu_i y_{i,t} + x_{i,t} = 0. \tag{10}$$

We define the risk premium $\phi_{i,t}$ of asset i at time t as the difference between the asset's expected payoff and its price, i.e.,

$$\phi_{i,t} \equiv E_t\left[\int_t^{\infty} e^{-r(s-t)} dD_{i,s}\right] - p_{i,t} = \frac{D_i}{r} - p_{i,t}. \tag{11}$$

Definition 2 *A competitive equilibrium is symmetric if for any asset pair $(i,-i)$ and time t, the risk premia are opposites, i.e., $\phi_{-i,t} = -\phi_{i,t}$, the arbitrageurs' positions in the two assets are opposites, i.e., $x_{-i,t} = -x_{i,t}$, and so are the outside investors' positions, i.e., $y_{-i,t} = -y_{i,t}$.*

Due to our model's symmetry, a symmetric competitive equilibrium can be shown to exist. Intuitively, risk premia are opposites because assets are in zero net supply and the supply shocks of (i,t)- and $(-i,t)$-investors are opposites. The arbitrageurs' positions are opposites because the risk premia are opposites, and the outside investors' are opposites because markets must clear.

Note that by symmetry, the midpoint between prices $p_{i,t}$ and $p_{-i,t}$ is the value they would both take absent dividend risk ($\sigma_i = \sigma_i^f = 0$), i.e.,

$$\frac{p_{-i,t} + p_{i,t}}{2} = E_t\left[\int_t^{+\infty} e^{-r(s-t)} dD_{i,s}\right] = \frac{D_i}{r}. \tag{12}$$

Therefore, asset i's risk premium is also one-half of the price wedge between assets $-i$ and i, i.e.,

$$\phi_{i,t} = \frac{p_{-i,t} - p_{i,t}}{2}. \tag{13}$$

Note also that arbitrageurs act as intermediaries. Suppose, for instance, that (i,t)-investors incur a positive supply shock, in which case $(-i,t)$-investors incur a negative one. Then arbitrageurs buy asset i from (i,t)-investors and sell asset $-i$ to $(-i,t)$-investors. Through this transaction arbitrageurs make a profit, while at the same time providing liquidity to the other investors.

Define the instantaneous return per share of asset i at time t in excess of the riskfree rate as

$$dR_{i,t} \equiv dD_{i,t} + dp_{i,t} - rp_{i,t}dt,$$

i.e., dividends and capital gains net of the riskfree return. Using (1) and (11), we have

$$dR_{i,t} = r\phi_{i,t}dt + \sigma_i dB_{i,t} + \sigma_i^f dB_{i,t}^f - d\phi_{i,t}. \tag{14}$$

3 Riskless Arbitrage

We start with the case in which there is no fundamental risk and no supply risk. No fundamental risk means that assets i and $-i$ pay identical dividends, i.e., $\sigma_i^f = 0$. No supply risk means that $u_{i,t}$ is deterministic, i.e., $\sigma_i^u = 0$. For simplicity, we assume $u_{i,t}$ to be constant over time, i.e., $u_{i,t} = u_i$.

We can show that a symmetric equilibrium exists in which the risk premia $\phi_{i,t}$, the outside investors' positions $y_{i,t}$, and the arbitrageurs' positions $x_{i,t}$ and total wealth W_t are all deterministic. Indeed, absent fundamental risk, the arbitrageurs' opposite positions in assets i and $-i$ ensures their wealth does not depend on dividends. Hence $\phi_{i,t}$ and $x_{i,t}$ are also independent of dividends. Since there is no other source of risk, $\phi_{i,t}$, $x_{i,t}$ and W_t are deterministic.[10]

3.1 Optimal Investment Policies

We first characterize outside investors' and arbitrageurs' optimal investment policies. As a first step, we denote by $\nu_{i,t}^{\phi}$ the drift of the premium $\phi_{i,t}$, i.e.,

$$d\phi_{i,t} \equiv \nu_{i,t}^{\phi} dt, \tag{15}$$

and by $\Phi_{i,t}$ the instantaneous expected excess return of asset i at time t, i.e.,

$$\Phi_{i,t} \equiv \frac{E_t(dR_{i,t})}{dt} = r\phi_{i,t} - \nu_{i,t}^{\phi}. \tag{16}$$

Absent arbitrageurs, $\Phi_{i,t}$ is positive for any asset $i \in \mathcal{A} \equiv \{j \in \mathcal{I} : u_j > 0\}$.

[10] Note that the prices $p_{i,t}$ are deterministic since they are ex dividend prices, but this property is not crucial.

3.1.1 Outside Investors

We study the (i,t)-investors' problem $\mathcal{P}_{i,t}$ for $i \in \mathcal{A}$. The budget constraint (6) becomes

$$dw_{i,t} = \left[rw_{i,t} + u_{i,t}\left(D_i - \nu^{\phi}_{i,t}\right) + y_{i,t}\Phi_{i,t}\right]dt + (y_{i,t} + u_{i,t})\sigma_i dB_{i,t}, \tag{17}$$

and each (i,t)-investors' objective is

$$\max_{y_{i,t}} \left[y_{i,t}\Phi_{i,t} - \frac{a_i\sigma_i^2}{2}(y_{i,t} + u_i)^2\right].$$

The first term is the (i,t)-investor's gain from their investment in asset i over and above the riskless rate if dividends are certain. This term is certain. The second term is the cost of bearing risk due to the uncertainty of dividends and of the supply shock. This inventory cost increases with the position in the risky asset, $y_{i,t}$, the magnitude of the supply shock, u_i, the volatility of dividends, σ_i, and the investor's risk aversion, a_i. At the optimum, the marginal inventory cost must equal the expected capital gain per unit of risky asset, i.e.,

$$\Phi_{i,t} = a_i\sigma_i^2(y_{i,t} + u_i). \tag{18}$$

This first-order condition determines the (i,t)-investors' demand $y_{i,t}$. By symmetry, problem $\mathcal{P}_{-i,t}$ yields the same first-order condition as $\mathcal{P}_{i,t}$.

3.1.2 Arbitrageurs

We study the arbitrageurs' optimization problem $\mathcal{P}_t$. We characterize an arbitrageur's optimal consumption and investment policy under the restriction that the prices of assets in the same pair are driven by symmetric processes, i.e., $\phi_{-i,t} = -\phi_{i,t}$. Using (1), (11), (15), (16) and symmetry, we can write the arbitrageurs' dynamic budget constraint (8) as

$$dW_t = \left(rW_t + 2\sum_{i\in\mathcal{A}} x_{i,t}\Phi_{i,t} - c_t\right)dt. \tag{19}$$

The first term is the arbitrageurs' return if they invest all their wealth in the riskless asset. The second term is the arbitrageurs' total gains over and above the riskfree rate. The arbitrageurs' positions in assets i and $-i$ being opposites, they eliminate all risk. Therefore, these gains are an instantaneous riskfree excess return. The third term is arbitrageurs' consumption.

Arbitrageurs reduce price wedges but do not change the sign of $\Phi_{i,t}$. Moreover, for all $i \in \mathcal{A}$, $u_i > 0$ implies that arbitrageurs long asset i, i.e., $x_i \geq 0$, and asset i yields a positive instantaneous excess return, i.e., $\Phi_{i,t} \geq 0$. By symmetry, the arbitrageurs' financial constraint (9) is

$$W_t \geq 2 \sum_{i \in \mathcal{A}} m_i x_{i,t}. \tag{20}$$

Proposition 1 *For symmetric price processes, each arbitrageur consumes a constant fraction β of his wealth, i.e., $c_t = \beta W_t$, and invests as follows.*

- *If $\Phi_{i,t} = 0$ for all $i \in \mathcal{I}$, the arbitrageur is indifferent between all investment policies satisfying the financial constraint.*
- *Otherwise, he invests only in opportunities $(i, -i)$ yielding the maximum excess return per unit of collateral, i.e., such that $i \in \arg\max_{j \in \mathcal{A}} \Phi_{j,t}/m_j$, up until his financial constraint binds.*

The intuition is as follows. From expression (19), an arbitrageur's optimization problem at all times is to maximize the excess return $2 \sum_{i \in \mathcal{A}} x_{i,t} \Phi_{i,t}$ under financial constraint (20). The solution is simple. Given their logarithmic utility, arbitrageurs consume a fixed fraction β of their wealth.[11] If all arbitrage opportunities yield no excess return, the arbitrageur's investment policy is a matter of indifference. If however some opportunities yield a strictly positive excess return, the arbitrageur should focus on those yielding the highest excess returns per unit of collateral, and invest up to the financial constraint in any (subset) of them. Establishing the "excess return on collateral" per each asset pair is also simple. Indeed, asset pair $(i, -i)$ yields an excess return $2\Phi_{i,t}$ but requires collateral of $2m_i$. Hence the excess return per dollar used as collateral is $\Phi_{i,t}/m_i$. Finally, since some opportunities offer arbitrageurs a riskfree return that strictly exceeds the riskfree rate, the arbitrageur should invest as much as possible, i.e., "max out" his financial constraint.

3.2 Equilibrium

Proposition 1 is a statement about optimal investment policies given asset prices, but can be rephrased in terms of the asset prices implied by these investment policies.

Corollary 1 *At any time t, there exists $\Pi_t \in [0, 1)$ such that arbitrageurs invest only in opportunities $(i, -i)$ with $\frac{a_i \sigma_i^2 u_i}{m_i} > \Pi_t$. Moreover, all opportunities in which arbitrageurs invest offer the same instantaneous excess return per unit of collateral, equal to Π_t, i.e.,*

$$\frac{\Phi_{i,t}}{m_i} = \Pi_t, \tag{21}$$

[11] An alternative interpretation of β is therefore a proportional cost of running the arbitrage business.

while opportunities in which they do not invest offer lower returns.

The equalization of "excess return on collateral" across opportunities in which arbitrageurs invest is a consequence of equilibrium: If returns differed, arbitrageurs would focus on those opportunities with the highest returns (Proposition 1), which would be a contradiction. Moreover if arbitrageurs do not invest in opportunity $(i,-i)$, the potential excess return $\Phi_{i,t}$ it offers is $a_i\sigma_i^2 u_i$ (from Eq. (18) with $y_{i,t}=0$). Hence arbitrageurs do not invest in opportunities with $\frac{a_i\sigma_i^2 u_i}{m_i} \leq \Pi_t$ as this would push their return below Π_t. Instead, they invest in opportunities if the supply parameter u_i, volatility of dividends σ_i, and investor risk aversion a_i are large, and the collateral required is low

Lemma 1 *At time t, arbitrageurs' position in opportunity $(i,-i)$ is*

$$x_{i,t} = \max\left\{\mu_i\left(u_i - \frac{m_i\Pi_t}{a_i\sigma_i^2}\right); 0\right\}. \tag{22}$$

We can now determine arbitrageur wealth dynamics, which depend on whether financial constraint (20) is binding. First, notice that optimal risk sharing (which would occur absent segmentation) would require $y_{i,t} = -y_{-i,t} = \mu_i u_i$. To provide this optimal level of liquidity to all outside investors at time t, arbitrageurs would have to post total collateral of

$$W_c \equiv 2\sum_{i\in\mathcal{A}} m_i\mu_i u_i.$$

Lemma 2 *Arbitrageur wealth evolves according to*

$$dW_t = (\Pi_t - (\beta - r))W_t dt. \tag{23}$$

The excess return Π_t per unit of collateral depends on wealth as follows:

- *If $W_t \geq W_c$, the financial constraint is slack, and arbitrageurs earn no excess return, i.e.,*

  $$\Pi_t = 0.$$

- *If $W_t < W_c$, the financial constraint is binding and arbitrageurs earn positive excess returns, i.e., $\Pi_t > 0$. Moreover, the lower arbitrageur wealth, the higher the excess returns. In particular, if arbitrageurs invest in all opportunities,*

  $$\Pi_t = B(W_c - W_t) \qquad \text{with } B \equiv \frac{1}{2\sum_{i\in\mathcal{A}}\frac{m_i^2\mu_i}{a_i\sigma_i^2}}. \tag{24}$$

If $W_t \geq W_c$, arbitrageurs have enough wealth to close all arbitrage opportunities, i.e., the financial constraint is slack. Therefore, they earn no excess return ($\Pi_t = 0$), and their wealth declines at rate $(\beta - r)$, i.e., the consumption rate β net of the riskfree rate r. If instead $W_t < W_c$, arbitrageurs do not close all opportunities, i.e., their financial constraint is binding. As a result, they earn the positive excess return $\Pi_t = B(W_c - W_t)$ per unit of wealth. Importantly, this return is larger when arbitrageur wealth is lower. Indeed arbitrageurs can intermediate less trades between outside investors and price discrepancies are larger.

To solve the dynamics in Lemma 2, we make the following assumptions.

Assumption 1 *Arbitrageur wealth converges to a steady-state value* $W \in (0, W_c)$*, i.e.,*

$$0 < \beta - r < BW_c. \tag{25}$$

If $r < \beta$, arbitrageurs are net dissavers when earning the riskfree rate, which they do if $W_t \geq W_c$. Hence $W < W_c$, where the financial constraint binds. If $\beta < r + BW_c$, arbitrageurs are net savers when facing the attractive returns the opportunities yield if W_t is small. Hence $W > 0$.

Note that the steady state the excess return per unit of collateral Π must be such that $dW_t = 0$. From Eq. (23), we have $\Pi = \beta - r$, which is larger if arbitrageurs are more impatient (large β). Indeed, to maintain a constant wealth despite consuming at a higher rate, arbitrageurs must earn a higher return on their investments. Said differently, if arbitrageurs consume more, they have less wealth to use as collateral and therefore price wedges remain wide, ensuring higher excess returns.

Assumption 2 *In the steady state, arbitrageurs invest in all arbitrage opportunities, i.e.,*

$$\min_{i \in \mathcal{A}} \frac{a_i \sigma_i^2 u_i}{m_i} > \beta - r. \tag{26}$$

Obviously, arbitrageurs also invest in all opportunities when their wealth exceeds the steady-state value and by continuity, when it is not far below.

Under these assumptions, we can characterize all equilibrium variables in closed form.

Proposition 2 *Define* $A \equiv BW_c - (\beta - r) > 0$*. Arbitrageur wealth converges monotonically towards its steady state value* $W = A/B$*. From time* t *onwards, arbitrageur wealth is as follows.*

- *If* $W_t < W_c$*, arbitrageur wealth remains strictly below* W_c *and is given by:*

$$W_s = \frac{W_t e^{A(s-t)}}{\frac{B}{A} W_t \left[e^{A(s-t)} - 1\right] + 1} < W_c \qquad \text{for} \quad s \geq t. \tag{27}$$

- *If $W_t \geq W_c$, arbitrageur wealth decreases and eventually drops strictly below W_c and follows the above dynamics from then on, i.e.,*

$$W_s = W_t e^{-(\beta-r)(s-t)} \geq W_c \qquad \text{for} \quad s \in [t, t_c], \tag{28}$$

$$W_s = \frac{W_c e^{A(s-t_c)}}{\frac{B}{A} W_c \left[e^{A(s-t_c)} - 1\right] + 1} < W_c \qquad \text{for} \quad s > t_c, \tag{29}$$

where t_c is the time when $W_t = W_c$ and is given by

$$W_t e^{-(\beta-r)(t_c-t)} = W_c. \tag{30}$$

We now turn to risk premia which, from Eq. (16), equal the present value of future instantaneous expected excess returns:

$$\phi_{i,t} = \int_t^\infty \Phi_{i,s} e^{-r(s-t)} ds. \tag{31}$$

From $\Phi_{i,s} = m_i \Pi_s = m_i B \max\{0, W_c - W_s\}$, we can derive the risk premia dynamics from that of arbitrageur wealth:

$$\phi_{i,t} = m_i B \int_t^\infty \max\{0, W_c - W_s\} e^{-r(s-t)} ds. \tag{32}$$

Proposition 3 *The risk premium of asset $i \in \mathcal{A}$ at time t is*

- *If $W_t < W_c$,*

$$\phi_{i,t} = m_i B \int_0^\infty \left[W_c - \frac{W_t e^{As}}{\frac{B}{A} W_t \left(e^{As} - 1\right) + 1} \right] e^{-rs} ds. \tag{33}$$

- *If $W_t \geq W_c$,*

$$\phi_{i,t} = m_i B \left(\frac{W_c}{W_t}\right)^{\frac{r}{\beta - r}} \int_0^\infty \left[W_c - \frac{W_c e^{As}}{\frac{B}{A} W_c \left(e^{As} - 1\right) + 1} \right] e^{-rs} ds. \tag{34}$$

- *As t increases, the risk premium $\phi_{i,t}$ converges monotonically towards its steady state value*

$$\phi_i = m_i \left(\frac{\beta - r}{r}\right).$$

Finally, we can derive the arbitrageurs' equilibrium positions (Lemma 1).

Proposition 4 *The arbitrageurs' position in asset $i \in \mathcal{A}$ at time t is as follows:*

$$x_{i,t} = \mu_i \left(u_i - \frac{m_i}{a_i \sigma_i^2} B \max\{W_c - W_t; 0\} \right). \tag{35}$$

3.3 Properties

Having derived all equilibrium variables in closed form, we can draw many implications. Some of these are cross-sectional in nature, i.e., comparing variables across opportunities with different characteristics. Others involve comparative statics with respect to arbitrageur wealth. These can be considered in two ways. First, because arbitrageur wealth varies over time out of steady state, the comparative statics results can be translated into time series predictions while the equilibrium is off the steady state. Alternatively, they can be interpreted as the effect of an unanticipated exogenous shock to arbitrageur wealth. These are also useful for the analysis of risky arbitrage.

Note that due to the model's symmetry, optimal risk-sharing, which would result from unconstrained trading, would imply $\phi_{i,t} = 0$.

Definition 3 *The risk premium $\phi_{i,t}$ is a measure of the illiquidity (i,t)- and $(-i,t)$-investors face.*

Corollary 2 *The risk premia are decreasing and convex in arbitrageur wealth, i.e., for all $i \in \mathcal{A}$*

$$\frac{\partial \phi_{i,t}}{\partial W_t} < 0 \qquad \textit{and} \qquad \frac{\partial^2 \phi_{i,t}}{\partial W_t^2} > 0.$$

Consider a drop in arbitrageur wealth. Intuitively, the risk premia should increase because arbitrageurs being poorer, they reduce their liquidity provision and allow prices to diverge. Moreover, when arbitrageur wealth is smaller, the return on arbitrageurs' wealth is larger and therefore a drop in arbitrageur wealth has a larger impact on future arbitrage wealth and thus on risk premia.

Corollary 3 *An asset's risk premium is increasing in its supply, and more so the lower arbitrageur wealth is, i.e., for all $i \in \mathcal{A}$*

$$\frac{\partial \phi_{i,t}}{\partial u_i} > 0 \qquad \textit{and} \qquad \frac{\partial^2 \phi_{i,t}}{\partial u_i \partial W_t} < 0.$$

Intuitively, $\phi_{i,t}$ increases with u_i since the discrepancy between the valuations of (i,t)- and $(-i,t)$-investors is larger. There is a mitigating effect. Indeed, the higher u_i, the higher the arbitrageurs' return and their future wealth. This tends to reduce future excess returns, and therefore the current risk premium. For low levels of W_t the mitigating effect is small, and therefore u_i has a large effect on $\phi_{i,t}$.

Corollary 4 *Illiquidity is higher for opportunities with higher margin requirements. These opportunities offer higher instantaneous excess returns and have higher risk premia, i.e., for all $(i,j) \in \mathcal{A}^2$*

$$m_i > m_j \qquad \Rightarrow \qquad \Phi_{i,t} > \Phi_{j,t} \qquad \textit{and} \qquad \phi_{i,t} > \phi_{j,t}. \tag{36}$$

Intuitively, investment opportunities requiring arbitrageurs to tie up more capital as collateral must provide them with a greater reward, i.e., a higher excess return. Risk premia being the present value of future excess returns, they must be higher for such opportunities.

Corollary 5 *Illiquidity and risk premia are more sensitive to arbitrageur wealth for opportunities with higher margin requirements, i.e., for all* $(i, j) \in \mathcal{A}^2$

$$m_i > m_j \qquad \Rightarrow \qquad \frac{\partial \phi_{i,t}}{\partial W_t} < \frac{\partial \phi_{j,t}}{\partial W_t} < 0. \tag{37}$$

Intuitively, changes in arbitrageur wealth affect the excess return (current or future) per unit of collateral, and therefore impact more strongly opportunities with higher collateral requirements.

Corollary 6 *Illiquidity and risk premia are more sensitive to the supply of other assets for opportunities with higher margin requirements, i.e., for all* $(i, j, k) \in \mathcal{A}^2$

$$m_i > m_j \qquad \Rightarrow \qquad \frac{\partial \phi_{i,t}}{\partial u_k} > \frac{\partial \phi_{j,t}}{\partial u_k} > 0$$

Intuitively, changes in supply affect the excess return (current or future) per unit of collateral, and therefore impact more strongly opportunities with higher collateral requirements.

Corollary 7 *Suppose* $W_t < W_c$. *Changes in arbitrageur wealth impact more strongly their position in opportunity* $(i, -i)$ *than in* $(j, -j)$ *if* $\frac{a_i \sigma_i^2}{m_i \mu_i} < \frac{a_j \sigma_j^2}{m_j \mu_j}$.

When arbitrageurs are unconstrained ($W_t > W_c$), they invest $x_{i,t} = \mu_i u_i$ in opportunity $(i, -i)$, independently of their wealth. Instead, when they are constrained ($W_t < W_c$), their wealth affect their positions. For example, following a drop in wealth, arbitrageurs are more constrained and reduce their investment in all opportunities. Investment is more wealth-sensitive for opportunities with higher collateral requirements because the excess returns that arbitrageurs require to invest in those opportunities are more affected by wealth changes (Corollary 4). Investment is less wealth-sensitive for opportunities where outside investors are more risk-averse or assets are riskier because outside investors for those opportunities have a more inelastic demand for insurance.

4 Risky Arbitrage

We now consider the possibility of arbitrage risk which in our model, stems from two sources: fundamental risk and supply risk. Fundamental risk means that assets i and $-i$ in a pair need not pay identical dividends, i.e., $\sigma_i^f \neq 0$. We assume $\sigma_i^f > 0$ for $i \in \mathcal{A}$. Supply risk means that asset i's supply $u_{i,t}$ is stochastic, i.e., $\sigma_i^u \neq 0$. We assume $\sigma_i^u > 0$ for $i \in \mathcal{A}$.[12]

We derive equilibrium conditions in Section 4.1, derive general properties of the equilibrium in Section 4.2, and characterize the equilibrium more fully for small arbitrage risk in Section 4.3.

4.1 Optimal Investment Policies

As we will see, an asset's risk premium is affected by the fundamental shocks and the supply shocks to *all* assets. Hence, for $i \in \mathcal{A}$, we denote the dynamics of the risk premium $\phi_{i,t}$ by

$$d\phi_{i,t} \equiv \nu_{i,t}^{\phi} dt + \sum_{j \in \mathcal{A}} \sigma_{i,j,t}^{f\phi} dB_{j,t}^f + \sum_{j \in \mathcal{A}} \sigma_{i,j,t}^{u\phi} dB_{j,t}^u. \tag{38}$$

Similarly, the instantaneous return of asset i is affected by *all* shocks to *all* assets because they affect the asset's risk premium. From Eqs. (14), (16) and (38), we have

$$dR_{i,t} = \Phi_{i,t} dt + \sigma_i dB_{i,t} + \sigma_i^f dB_{i,t}^f - \sum_{j \in \mathcal{A}} \sigma_{i,j,t}^{f\phi} dB_{j,t}^f - \sum_{j \in \mathcal{A}} \sigma_{i,j,t}^{u\phi} dB_{j,t}^u. \tag{39}$$

4.1.1 Outside Investors

We first characterize the optimal investment policies of outside investors. Using (1), (11), (16) and (38), we can write the (i,t)-investors' dynamic budget constraint (6) as

$$\begin{aligned} dw_{i,t} = & \left[r w_{i,t} + u_{i,t} \left(D_i - \nu_{i,t}^{\phi} \right) + y_{i,t} \Phi_{i,t} \right] dt \\ & + (y_{i,t} + u_{i,t}) \left[\sigma_i dB_{i,t} + \sigma_i^f dB_{i,t}^f - \sum_{j \in \mathcal{A}} \sigma_{i,j,t}^{f\phi} dB_{j,t}^f - \sum_{j \in \mathcal{A}} \sigma_{i,j,t}^{u\phi} dB_{j,t}^u \right]. \end{aligned} \tag{40}$$

The drift is the same as for riskfree arbitrage. The diffusion term captures the risky part of the return of asset i (Eq. (39)). Given this, the (i,t)-investors' objective is

$$\max_{y_{i,t}} \left[y_{i,t} \Phi_{i,t} - \frac{a_i}{2} (y_{i,t} + u_{i,t})^2 \left(\sigma_{i,t}^R \right)^2 \right]. \tag{41}$$

The first term is the expected excess return (i,t)-investors derive from their holding in asset i. The second term is a cost of bearing risk. It depends on asset i's instantaneous volatility computed as

$$\left(\sigma_{i,t}^R \right)^2 \equiv \frac{Var_t(dR_{i,t})}{dt} = \sigma_i^2 + \left(\sigma_i^f - \sigma_{i,i,t}^{f\phi} \right)^2 + \sum_{j \in \mathcal{A}/\{i\}} \left(\sigma_{i,j,t}^{f\phi} \right)^2 + \sum_{j \in \mathcal{A}} \left(\sigma_{i,j,t}^{u\phi} \right)^2. \tag{42}$$

[12] Assuming $\sigma_i^f > 0$ and $\sigma_i^u > 0$ is without loss of generality as we can replace $B_{i,t}^f$ and $B_{i,t}^u$ with their opposites.

At the optimum, the expected excess return of asset i to the marginal cost of risk-bearing, i.e.,

$$\Phi_{i,t} = a_i \left(\sigma^R_{i,t}\right)^2 (y_{i,t} + u_{i,t}). \tag{43}$$

Their first order condition determines the (i,t)-investors' demand $y_{i,t}$. By symmetry, problem $\mathcal{P}_{-i,t}$ yields the same first-order condition as $\mathcal{P}_{i,t}$.

4.1.2 Arbitrageurs

We characterize an arbitrageur's optimal consumption and investment policy under the restriction that the prices of assets in the same pair are driven by symmetric processes, i.e., $\phi_{-i,t} = -\phi_{i,t}$. Using (1), (11), (16), (38) and symmetry, the arbitrageurs' dynamic budget constraint (8) is

$$dW_t = \left(rW_t + 2\sum_{i\in\mathcal{A}} x_{i,t}\Phi_{i,t} - c_t\right) dt + 2\sum_{i\in\mathcal{A}} x_{i,t}\left(\sigma^f_i dB^f_{i,t} - \sum_{j\in\mathcal{A}} \sigma^{f\phi}_{i,j,t} dB^f_{j,t} - \sum_{j\in\mathcal{A}} \sigma^{u\phi}_{i,j,t} dB^u_{j,t}\right). \tag{44}$$

The drift is the same as for riskless arbitrage, i.e., arbitrageur wealth increases by the risk free return plus the expected excess returns provided by all opportunities net of the arbitrageurs' consumption. Now however, there are also diffusion terms because arbitrage is risky. Denote the respective diffusion coefficients for the fundamental shock $dB^f_{j,t}$ and the supply shock $dB^u_{j,t}$ as

$$\sigma^{fW}_{j,t} \equiv 2x_{j,t}\sigma^f_j - 2\sum_{i\in\mathcal{A}} x_{i,t}\sigma^{f\phi}_{i,j,t} \quad \text{and} \quad \sigma^{uW}_{j,t} \equiv -2\sum_{i\in\mathcal{A}} x_{i,t}\sigma^{u\phi}_{i,j,t}. \tag{45}$$

A fundamental shock to opportunity $(j,-j)$ means that assets j and $-j$ do not pay the exact same dividend. The "net dividend" affects arbitrageurs' profit and hence their wealth. This direct effect is captured by $2x_{j,t}\sigma^f_j$. At the same time, the shock affects all opportunities' risk premia, and hence the arbitrageurs' capital gains from their investments in these opportunities. This effect is captured by $-2\sum_{i\in\mathcal{A}} x_{i,t}\sigma^{f\phi}_{i,j,t}$. A supply shock to opportunity $(j,-j)$ means that (j,t)- and $(-j,t)$-investors are more eager to trade. Such a shock does not affect arbitrageur wealth directly but indirectly through its effect on the risk premia of all opportunities, which in turn affects arbitrageurs' capital gains and ultimately their wealth. This effect is captured by $-2\sum_{i\in\mathcal{A}} x_{i,t}\sigma^{u\phi}_{i,j,t}$.

For $i \in \mathcal{A}$, denote $2\hat{\Phi}_{i,t}$ the arbitrageurs' risk-adjusted return from opportunity $(i,-i)$. Indeed, their expected excess return from opportunity $(i,-i)$, $2\Phi_{i,t}$, must be adjusted for the fundamental and supply risk the opportunity entails. This is done by multiplying the arbitrageurs' coefficient of absolute risk aversion, equal to $1/W_t$ due to logarithmic utility, with the covariance of the return of opportunity $(i,-i)$ and that of the arbitrageurs' portfolio, i.e.,

$$2\hat{\Phi}_{i,t} \equiv 2\Phi_{i,t} - \frac{Cov_t(dR_{i,t} - dR_{-i,t}, dW_t)}{W_t dt}. \tag{46}$$

The covariance is obtained by summing over all fundamental and supply shocks the loading of $(i,-i)$'s return on each shock times the arbitrageurs' portfolio loading on that shock, i.e.,[13]

$$\hat{\Phi}_{i,t} = \Phi_{i,t} - \frac{1}{W_t}\left[(\sigma_i^f - \sigma_{i,i,t}^{f\phi})\sigma_{i,t}^{fW} - \sum_{j\in\mathcal{A}/\{i\}} \sigma_{i,j,t}^{f\phi}\sigma_{j,t}^{fW} - \sum_{j\in\mathcal{A}} \sigma_{i,j,t}^{u\phi}\sigma_{j,t}^{uW}\right].$$

We can now derive the arbitrageurs' optimal policy.

Proposition 5 *Denote* $\Pi_t \equiv \max_{i\in\mathcal{A}}\left|\hat{\Phi}_{i,t}/m_i\right|$. *Each arbitrageurs consumes a fraction* β *of his wealth, i.e.,* $c_t = \beta W_t$, *and his investment policy satisfies one of the following conditions.*

- *The financial constraint (9) is slack and* $\hat{\Phi}_{i,t} = 0$ *for all* i.
- *The financial constraint (9) is binding and for all* $i \in \mathcal{A}$,

$$x_{i,t} > 0 \Rightarrow \frac{\hat{\Phi}_{i,t}}{m_i} = \Pi_t \qquad and \qquad x_{i,t} < 0 \Rightarrow \frac{\hat{\Phi}_{i,t}}{m_i} = -\Pi_t. \tag{47}$$

Proposition 5 is Proposition 1's counterpart for risky arbitrage. When the financial constraint is slack, arbitrageurs close all opportunities. When the financial constraint is binding, arbitrageurs invest only in opportunities yielding the maximum return on collateral. There are however two differences with Proposition 1. First, the relevant return from opportunity $(i,-i)$ is the risk-adjusted return $\hat{\Phi}_{i,t}$, which depends both on prices and arbitrageur positions. Second, arbitrageurs can "short" some opportunities, i.e., long the pricier asset and short the cheaper one ($x_{i,t} < 0$ for $i \in \mathcal{A}$). This can be optimal for arbitrageurs for hedging their long positions in other opportunities.

4.2 Amplification and Contagion: Direct and Indirect Effects

Equilibrium prices and positions solve the first-order condition of outside investors (Eq. (43)) and arbitrageurs (Proposition 5). This system of equations is complex. Here we derive general properties of equilibrium. We assume that the conditions in Assumptions 1 and 2 hold for the values $u_{i,t}$.

Assumption 3 *Define* $W_{c,t} \equiv 2\sum_{i\in\mathcal{A}} m_i\mu_i u_{i,t}$. *We assume*

$$0 < \beta - r < BW_{c,t} \qquad and \qquad \min_{i\in\mathcal{A}} \frac{a_i\sigma_i^2 u_{i,t}}{m_i} > \beta - r.$$

[13] The arbitrageurs' logarithmic utility simplifies the analysis because risk is measured by the covariance with the arbitrageurs' portfolio and not with other state variables.

In equilibrium, the risk premium $\phi_{i,t}$ is a function of arbitrageur wealth W_t and the supply parameters $\{u_{j,t}\}_{j\in\mathcal{A}}$. Eq. (44) and Ito's Lemma imply

$$\sigma^{f\phi}_{i,j,t} = \frac{\partial \phi_{i,t}}{\partial W_t}\sigma^{fW}_{j,t} \qquad \text{and} \qquad \sigma^{u\phi}_{i,j,t} = \frac{\partial \phi_{i,t}}{\partial W_t}\sigma^{uW}_{j,t} + \frac{\partial \phi_{i,t}}{\partial u_{j,t}}\sigma^{u}_{j}. \tag{48}$$

As for riskfree arbitrage, arbitrageur wealth creates a linkage between the different opportunities even though their fundamentals are independent.

Lemma 3 *Fundamental and supply shocks to one opportunity affect arbitrageur wealth and the risk premia of all opportunities. More precisely, for $(i,j)\in\mathcal{A}^2$, the effect of a fundamental shock $dB^f_{j,t}$ to opportunity $(j,-j)$ on arbitrageur wealth and on asset i's the risk premium $\phi_{i,t}$ are respectively*

$$\sigma^{fW}_{j,t} = \frac{2x_{j,t}\sigma^f_j}{1+2\sum_{k\in\mathcal{A}} x_{k,t}\frac{\partial \phi_{k,t}}{\partial W_t}}, \tag{49}$$

$$\sigma^{f\phi}_{i,j,t} = \frac{\partial \phi_{i,t}}{\partial W_t}\frac{2x_{j,t}\sigma^f_j}{1+2\sum_{k\in\mathcal{A}} x_{k,t}\frac{\partial \phi_{k,t}}{\partial W_t}}. \tag{50}$$

The effect of the supply shock $dB^u_{j,t}$ on the same variables are respectively

$$\sigma^{uW}_{j,t} = -\frac{2\sum_{k\in\mathcal{A}} x_{k,t}\frac{\partial \phi_{k,t}}{\partial u_{j,t}}\sigma^u_j}{1+2\sum_{k\in\mathcal{A}} x_{k,t}\frac{\partial \phi_{k,t}}{\partial W_t}}, \tag{51}$$

$$\sigma^{u\phi}_{i,j,t} = \frac{\partial \phi_{i,t}}{\partial u_{j,t}}\sigma^u_j - \frac{\partial \phi_{i,t}}{\partial W_t}\frac{2\sum_{k\in\mathcal{A}} x_{k,t}\frac{\partial \phi_{k,t}}{\partial u_{j,t}}\sigma^u_j}{1+2\sum_{k\in\mathcal{A}} x_{k,t}\frac{\partial \phi_{k,t}}{\partial W_t}}. \tag{52}$$

To develop an intuition, assume that (as for riskless arbitrage) arbitrageurs long all opportunities ($x_{i,t}>0$ for all $i\in\mathcal{A}$), and risk premia decrease with arbitrageur wealth.

After a positive fundamental shock $dB^f_{j,t}$ to opportunity $(j,-j)$, asset j's dividend exceeds asset $-j$'s, and arbitrageurs receive the "net dividend" $2x_{j,t}\sigma^f_j dB^f_{j,t}$. This direct effect on wealth corresponds to the numerator in (49). Moreover, arbitrageurs being richer, risk premia decrease and the arbitrageurs realize capital gains $2\sum_{k\in\mathcal{A}} x_{k,t}\frac{\partial \phi_{k,t}}{\partial W_t}$. This indirect effect on wealth corresponds to the denominator in (49). Since $x_{k,t}>0$ and $\frac{\partial \phi_{k,t}}{\partial W_t}<0$, this indirect effect amplifies the direct effect. The indirect effect on the risk premium $\phi_{i,t}$ is (50).

A positive supply shock $dB^u_{j,t}$ to opportunity $(j,-j)$ means that (j,t)- and $(-j,t)$-investors are more eager to trade. Holding wealth constant, such a shock has the direct effect of increasing asset i's risk premium by $\frac{\partial \phi_{i,t}}{\partial u_{j,t}} \sigma_j^u dB_{j,t}^u$, the first term in (52). Due to the increase in risk premia, arbitrageurs realize a capital loss $2\sum_{k \in \mathcal{A}} x_{k,t} \frac{\partial \phi_{k,t}}{\partial u_{j,t}} \sigma_j^u$. Moreover, arbitrageurs being poorer, risk premia increase and the arbitrageurs' loss is amplified. The indirect effect on wealth is (51) and on the risk premium $\phi_{i,t}$ is the second term in Eq. (52).

4.3 Small Arbitrage Risk

In this section, we characterize the solution more fully when arbitrage risk is small ($\sigma_i^f \simeq 0$ and $\sigma_i^u \simeq 0$), and supply parameters are slowly mean-reverting ($\kappa_i^u \simeq 0$). Specifically, we study how an asset's liquidity, volatility and correlation with other assets depend on arbitrageur wealth. We also study arbitrageurs' positions.

4.3.1 Liquidity

Eq. (39) implies that the impact of a supply shock $dB_{i,t}^u$ to asset i at time t on the asset return $dR_{i,t}$ is $|\sigma_{i,i,t}^{u\phi}|$. Hence we define asset i's liquidity as

$$\lambda_{i,t} \equiv \frac{1}{|\sigma_{i,i,t}^{u\phi}|}. \tag{53}$$

All markets are less liquid than absent constraints. Indeed, arbitrageurs cannot absorb as much of the supply shocks as they otherwise would. Since the extent to which financial constraints bind depends on arbitrageur wealth, it is clear that liquidity should depend on arbitrageur wealth. We show that while it does indeed, more arbitrageur wealth does not always yield more liquid markets.

Proposition 6 *There exists $\epsilon > 0$ going to zero when $\{\sigma_j^f, \sigma_j^u, \kappa_j^u\}_{j \in \mathcal{A}}$ go to zero such that*

- *If $W_t > W_{c,t} + \epsilon$, asset i's liquidity $\lambda_{i,t}$ increases with arbitrageur wealth W_t.*
- *If $\underline{W} \leq W_t < W_{c,t} - \epsilon$, asset i's liquidity $\lambda_{i,t}$ decreases with arbitrageur wealth W_t.*

The intuition is as follows. Supply shocks affect risk premia directly but also indirectly through arbitrageur wealth (Lemma 3). The direct effect is weaker when arbitrageur wealth is high (Corollary 3).[14] The indirect effect, however, is a hump-shaped function of wealth. Indeed, at low levels of wealth, the financial constraint is binding and an increase in wealth triggers a sharp increase in

[14] Corollary 3 is for the case $\sigma_j^f = \sigma_j^u = \kappa_j^u = 0$, but by continuity, the result extends to small values of $(\sigma_j^f, \sigma_j^u, \kappa_j^u)$.

arbitrageurs' positions. When positions are larger, arbitrageur wealth is more sensitive to changes in risk premia, and therefore the indirect effect is stronger. Instead, at high values of wealth, arbitrageurs' positions are less sensitive to wealth. The main effect of an increase in wealth is to render risk premia less sensitive to wealth (Corollary 2), implying a weaker indirect effect. The hump-shaped indirect effect drives the U-shaped pattern of liquidity.

4.3.2 Volatility

We next examine how arbitrageur wealth affects the volatility of assets. The volatility of asset i is given by Eq. (42). All assets are more volatile than absent financial constraints. And again, it is intuitive that volatility should depend on arbitrageur wealth. Indeed asset i's volatility depends on factors affecting the asset's dividend, σ_i and σ_i^f, but also on factors affecting the supply of and demand for the asset, $\sigma_{i,j,t}^{f\phi}$ and $\sigma_{i,j,t}^{u\phi}$. Unlike the former, the latter do depend on arbitrageur wealth, so that asset volatilities do too. We show however that they do so in a non-trivial and generally non-monotonic way.

Proposition 7 *There exists $\epsilon > 0$ going to zero when $\{\sigma_j^f, \sigma_j^u, \kappa_j^u\}_{j\in\mathcal{A}}$ go to zero such that:*

- *If $W_t > W_{c,t} + \epsilon$, asset i's volatility $\sigma_{i,t}^R$ decreases with arbitrageur wealth W_t.*
- *If $\underline{W} \leq W_t < W_{c,t} - \epsilon$, the component of asset i's volatility $\sigma_{i,t}^R$ due to supply shocks $dB_{j,t}^u$ increases in arbitrageur wealth W_t, and that due to the fundamental shock $dB_{j,t}^f$, $j \in \mathcal{A}$, increases if*

$$\frac{a_j \sigma_j^2 u_{j,t}}{m_j} \leq \frac{\sum_{k\in\mathcal{A}} m_k \mu_k u_{k,t}}{\sum_{k\in\mathcal{A}} \frac{m_k^2 \mu_k}{a_k \sigma_k^2}}. \tag{54}$$

The intuition is as follows. A supply shock to asset $j \neq i$ has no direct effect on asset i's risk premium. However, as for a supply shock to asset i itself, its indirect effect is a hump-shaped function of wealth. Therefore, absent fundamental shocks, volatility would be hump-shaped.

The fundamental shock $dB_{j,t}^f$ also generates hump-shaped volatility if j satisfies condition (54). This condition is satisfied by a non-empty subset of $\mathcal{A}$, and by all assets in $\mathcal{A}$ if they are homogenous (and in particular if there is only one opportunity). It is not satisfied when $a_j\sigma_j^2 u_{j,t}/m_j$ is large relative to a weighted average of this variable across assets, and in that case the volatility due to $dB_{j,t}^f$ decreases with arbitrageur wealth. The intuition is that an increase in wealth leads to an increase in arbitrageur positions (implying larger volatility), but to a reduction in the wealth-sensitivity of risk premia (implying smaller volatility). When $u_{j,t}$ is large, arbitrageurs are invested heavily in opportunity $(j, -j)$, and the second effect dominates because the shock $dB_{j,t}$ has a large impact on wealth.[15]

4.3.3 Correlations

We now turn to asset correlations which again differ from the unconstrained case. First, some assets have uncorrelated fundamentals, i.e., dividends and supply. In our model, these are assets not in the same pair. Absent constraints or segmentation, these assets' returns would be uncorrelated. With constraints however they are correlated because arbitrageur wealth is a common factor affecting all asset returns. Second, assets in the same pair have correlated fundamentals but their returns' correlation is below that absent constraints. We show that correlations depend on arbitrageur wealth in a non-trivial way.

Proposition 8 *Consider $(i, i') \in \mathcal{A}^2$, $i \neq i'$. There exists $\epsilon > 0$ going to zero when $\{\sigma^f_j, \sigma^u_j, \kappa^u_j\}_{j\in\mathcal{A}}$ go to zero such that*

- *If $W_t > W_{c,t} + \epsilon$, the correlation between assets i and i' decreases with arbitrageur wealth W_t.*
- *If $W \leq W_t < W_{c,t} - \epsilon$, the component of the correlation between assets i and i' due to supply shocks $dB^f_{j,t}$ increases with arbitrageur wealth W_t, and that due to the fundamental shock $dB^f_{j,t}$, $j \in \mathcal{A}$, increases if (54) is satisfied.*
- *The opposite holds for the correlation between assets i and $-i$, and for that between assets i and $-i'$.*

The intuition is as follows. Assume that (as for riskless arbitrage) arbitrageurs long all opportunities ($x_{i,t} > 0$ for all $i \in \mathcal{A}$), and risk premia decrease with arbitrageur wealth.

Consider first two assets $i \neq i'$ that arbitrageurs long. For such assets, correlation is positive despite their fundamentals' independence. For high levels of wealth, as wealth increases, their correlation converges to that absent constraint, i.e., zero. Things are different for low levels of wealth. Indeed, a given increase in arbitrageur wealth translates into a larger increase in arbitrageurs' positions, and hence in their exposure to supply shocks. Since arbitrageur wealth is a factor common to all assets, this increases the correlation between i and i'. Hence, the correlation between i and i' tends to be hump-shaped or decreasing in wealth.

[15] Condition (54) is not needed for supply shocks because the direct effect of $dB^f_{j,t}$ is through arbitrageurs' position in opportunity $(j, -j)$, while that of $dB^u_{j,t}$ concerns all opportunities.

Consider now assets i and $-i'$. For such assets, correlation is negative despite their fundamentals' independence. Because fundamental and supply shocks have opposite effects on assets i' and $-i'$, the correlation between assets i and $-i'$ tends to be inverse U-shaped or increasing.

Finally, consider assets i and $-i$. These assets tend to be less correlated than absent constraints. Because fundamental and supply shocks have opposite effects on assets i and $-i$, the correlation between assets i and $-i$ tends to be inverse hump-shaped or increasing.

One interesting aspect of these results is that the effect of a change in arbitrageur wealth on correlations is not uniform across asset pairs or across wealth levels. For instance, a reduction in arbitrage capital (e.g., as during a financial crisis) does not necessary lead to an increase in correlations across all assets, a phenomenon often viewed as contagion, and this for two distinct reasons. First, arbitrageurs' activity tends to bring the prices of assets with correlated fundamentals (e.g., i and $-i$) in line with each other. When they are poorer, they may be able to perform that role, and the correlation between such assets decreases. Second, for low levels of arbitrage wealth, arbitrageurs hold small positions and this weakens the transmission of shocks through arbitrageur wealth, reducing the correlation between assets with uncorrelated fundamentals.

4.3.4 Arbitrage Positions

We next examine how arbitrageur positions depend on their wealth and on the risk of investment opportunities. When arbitrageurs long all opportunities (i.e., $x_{i,t} > 0$ for $i \in \mathcal{A}$), Eq. (46) and Proposition 5 imply that for all $i \in \mathcal{A}$, the expected excess return from opportunity $(i, -i)$ is

$$\Phi_{i,t} = m_i \Pi_t + \frac{Cov_t(dR_{i,t} - dR_{-i,t}, dW_t)}{2W_t dt}. \tag{55}$$

The first term is a compensation for tying up capital as collateral. The risk-adjusted return on collateral Π_t is positive when the financial constraint binds and zero when it is slack. The second term is a compensation for risk. It is positive because both fundamental and supply shocks induce positive correlation between the return on opportunity $(i, -i)$ and arbitrageur wealth. Indeed, a positive fundamental shock $dB^f_{j,t}$ to $j \in \mathcal{A}$ raises arbitrageur wealth, leading to lower risk premia and higher returns from all opportunities. A positive supply shock $dB^u_{j,t}$ to $j \in \mathcal{A}$ raises premia, leading to lower arbitrageur wealth and lower returns from all opportunities.

Lemma 4 *The financial constraint becomes slack at a lower level of wealth than under riskless arbitrage. More precisely, if $\{\sigma^f_j, \sigma^u_j, \kappa^u_j\}_{j \in \mathcal{A}}$ are small, financial constraint (9) holds as an equality if and only if $W_t \leq W_{c,t} - \epsilon$, where $\epsilon > 0$.*

The intuition is as follows. Contrary to the riskfree arbitrage case, aggregate risk is not zero. Hence optimal risk sharing does not involve full insurance for outside investors. Said differently, because arbitrageurs require positive compensation for risk from each opportunity, they do not drive expected excess returns down to zero even when they have enough wealth to do so.

Proposition 9 *For $\{\sigma_j^f, \sigma_j^u, \kappa_j^u\}_{j\in\mathcal{A}}$ small, consider $(i, i') \in \mathcal{A}^2$ such that $(\sigma_i, a_i, \mu_i, u_{i,t}) = (\sigma_{i'}, a_{i'}, \mu_{i'}, u_{i',t})$.*

- *If $m_i > m_{i'}$ and $\sigma_i^f = \sigma_{i'}^f$, changes in arbitrageur wealth impact more strongly their position in opportunity $(i, -i)$ than $(i', -i')$.*
- *If $m_i = m_{i'}$ and $\sigma_i^f > \sigma_{i'}^f$, changes in arbitrageur wealth impact more strongly their position in opportunity $(i, -i)$ than $(i', -i')$ when $W_t > W_{c,t} + \epsilon$ for $\epsilon > 0$ that converges to zero when $\{\sigma_j^f, \sigma_j^u, \kappa_j^u\}_{j\in\mathcal{A}}$ go to zero.*

When arbitrageurs are unconstrained, their positions are limited only by risk aversion. If arbitrageur wealth decreases within that region, risk-aversion increases (the coefficient of absolute risk aversion is $1/W_t$) and returns become more volatile (Proposition 7). These mutually-reinforcing effects induce arbitrageurs to scale down their positions, especially in opportunities that involve more risk. These are the opportunities with high collateral requirements (high m_i) and high fundamental risk (high σ_i^f). Note that opportunities with high collateral requirements are more affected not because the opportunity cost of collateral increases, but because their returns are more volatile.

Consider next the region where the financial constraint binds. Under riskless arbitrage, arbitrageurs scale down more their positions in opportunities with high collateral requirements (Corollary 7). Under risky arbitrage, arbitrageurs are also concerned about the risk of each opportunity, but the variation of this effect with wealth is ambiguous. On the one hand, when wealth decreases, arbitrageurs become more risk-averse. On the other hand, return volatility can decrease (Proposition 7). As a consequence, arbitrageurs can scale down their positions less in riskier opportunities. For small arbitrage risk, the effect of m_i is unambiguous (same as under riskless arbitrage), but the effect of σ_i^f is ambiguous.

5 Conclusion (to be written)

This paper develops a framework to examine the relation between intermediary capital, financial market liquidity and asset prices. Its main features are as follows. First, arbitrageurs are sophisticated investors with better investment opportunities than other investors, but they face financial constraints. Second, ours is a dynamic general equilibrium model capturing the dynamic interaction between arbitrage and arbitrage capital. Third, arbitrageurs face multiple arbitrage opportunities with different characteristics, across which they must allocate their scarce capital.

Our main results are...

Our analysis has left aside a number of important questions which we intend to address in future research.

Modelling: Endogenous constraints/segmentation.

Economics: Imperfect competition; Entry; Integration/Segmentation.

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Appendix

A Riskless Arbitrage

Proof of Proposition 1 and Corollary 1: We solve a given arbitrageur A's problem $\mathcal{P}_t$ using dynamic programming. We distinguish between arbitrageur A's wealth $\hat{W}_t$, and the arbitrageurs' total wealth W_t. In equilibrium $\hat{W}_t = W_t$, but distinguishing $\hat{W}_t$ from W_t is important as W_t influences prices while arbitrageur A can affect only $\hat{W}_t$. We denote $\{\hat{x}_{i,t}\}_{i\in\mathcal{A}}$ arbitrageur A's positions and $\hat{c}_t$ his consumption to distinguish them from the arbitrageurs' total positions $\{x_{i,t}\}_{i\in\mathcal{A}}$ and consumption c_t. We conjecture the value function

$$V(\hat{W}_t, W_t) = \frac{\log(\hat{W}_t)}{\beta} + v(W_t). \tag{A.1}$$

For riskless arbitrage, $(\hat{W}_t, W_t)$ are deterministic, and the Bellman equation is

$$\max_{\hat{x}_{i,t},\hat{c}_t} \left[\log(\hat{c}_t) + V_{\hat{W}}\left(r\hat{W}_t + 2\sum_{i\in\mathcal{A}} \hat{x}_{i,t}\Phi_{i,t} - \hat{c}_t\right) + V_W\mu_t^W - \beta V\right] = 0, \tag{A.2}$$

where μ_t^W denotes the drift of W_t. The first-order condition with respect to $\hat{c}_t$ yields $\hat{c}_t = \beta\hat{W}_t$. Optimizing over $\{\hat{x}_{i,t}\}_{i\in\mathcal{A}}$ amounts to maximizing $\sum_{i\in\mathcal{A}} \hat{x}_{i,t}\Phi_{i,t}$ subject to financial constraint (20). Since $\Phi_{i,t} \geq 0$ for $i \in \mathcal{A}$, the first-order condition yields the policy in the proposition. The maximum value of $2\sum_{i\in\mathcal{A}} \hat{x}_{i,t}\Phi_{i,t}$ is $\hat{W}_t \max_{j\in\mathcal{A}}\left(\frac{\Phi_{j,t}}{m_j}\right)$. Substituting into (A.2), the terms in $\hat{W}_t$ cancel out. Setting the remaining terms to zero determines the function $v(W_t)$. ■

Proof of Lemma 1: Eqs. (10) and (18) imply that

$$\Phi_{i,t} = a_i\sigma_i^2\left(u_i - \frac{x_{i,t}}{\mu_i}\right). \tag{A.3}$$

For $\Phi_{i,t}/m_i = \Pi_t$ and $x_{i,t} > 0$, (A.3) implies $a_i\sigma_i^2 u_i/m_i > \Pi_t$. Solving (A.3) for $x_{i,t}$ yields (22). ■

Proof of Lemma 2: From Corollary 1, $\forall i \in \mathcal{I}$, $x_{i,t} = 0$ or $\Phi_{i,t} = m_i\Pi_t$, which implies $x_{i,t}\Phi_{i,t} = x_{i,t}m_i\Pi_t$. Substituting together with $c_t = \beta W_t$ into (19) and yields

$$dW_t = \left[(r-\beta)W_t + 2\Pi_t\sum_{i\in\mathcal{A}} m_i x_{i,t}\right]dt. \tag{A.4}$$

Eq. (23) follows from (20) and (A.4) by noting that when (20) is slack, $\Pi_t = 0$.

For (20) to be slack, arbitrageurs must be able to hold $x_{i,t} = \mu_i u_i$ for all $i \in \mathcal{I}$, which requires $W_t \geq 2\sum_{i\in\mathcal{A}} m_i\mu_i u_i \equiv W_c$. This also implies $y_{i,t} = -u_i$ for all $i \in \mathcal{I}$, and therefore $\Phi_{i,t} = 0$ from (19). $x_{i,t} > 0$ for all $i \in \mathcal{A}$ and $\Phi_{i,t} = 0$ implies $\Pi_t = 0$ (Corollary 1).

If $W_t < W_c$, $\exists i \in \mathcal{A}$ such that $x_{i,t} < \mu_i u_i$, which implies $y_{i,t} > u_i$ and $\Phi_{i,t} > 0$ (from Eq. (19)). This implies $\Pi_t > 0$ (Corollary 1). Moreover if arbitrageurs invest in all opportunities, their position $x_{i,t}$ in each of them is given by (22). Substituting into (20) (which holds as an equality) yields

$$W_t = 2\sum_{i\in\mathcal{A}} m_i\mu_i\left(u_i - \frac{m_i\Pi_t}{a_i\sigma_i^2}\right) = W_c - \frac{\Pi_t}{B}, \tag{A.5}$$

which implies (24). ■

Proof of Proposition 2: We first determine W_s for $s \geq t$ such that $W_t < W_c$. Using (24), we can write (23) as

$$dW_t = (A - BW_t)W_t dt. \tag{A.6}$$

To integrate (A.6), we note that

$$\frac{d}{dt}\left(\frac{W_t e^{-At}}{A - BW_t}\right) = \frac{Ae^{-At}}{(A - BW_t)^2}\left[\frac{dW_t}{dt} - (A - BW_t)W_t\right] = 0,$$

where the second step follows from (A.6). Therefore, for $s > t$,

$$\frac{W_s e^{-As}}{A - BW_s} = \frac{W_t e^{-At}}{A - BW_t}.$$

Solving for W_s, we find (27).

We next determine W_s for $s \geq t$ such that $W_t \geq W_c$. If for $s > t$, W_s is in the unconstrained region, then it is given by (28), obtained by integrating (23). The boundary $W_s = W_c$ is reached for $s = t_c$. For $s > t_c$, W_s is given by (27) by setting $t = t_c$ and $W_t = W_c$. This is (28). ■

Proof of Proposition 3: Eq. (33) stems from (27) and (32), and Eq. (34) from (28)-(30) and (32). ■

Proof of Proposition 4: From Lemma 1 and Lemma 2. ■

Proof of Corollary 2: Eq. (33) implies that in the constrained region is

$$\frac{\partial\phi_{i,t}}{\partial W_t} = -m_i B\int_0^\infty \frac{e^{As}}{\left[\frac{B}{A}W_t\left(e^{As} - 1\right) + 1\right]^2} e^{-rs} ds. \tag{A.7}$$

Eq. (34) implies that in the unconstrained region is

$$\frac{\partial \phi_{i,t}}{\partial W_t} = -\frac{rm_iB}{(\beta-r)W_t}\left(\frac{W_c}{W_t}\right)^{\frac{r}{\beta-r}} \int_0^\infty \left[W_c - \frac{W_c e^{As}}{\frac{B}{A}W_c\left(e^{As}-1\right)+1}\right] e^{-rs}ds. \tag{A.8}$$

In both cases, $\partial\phi_{i,t}/\partial W_t$ is negative and increases as W_t increases. To show strict convexity, we also need to show that $\partial\phi_{i,t}/\partial W_t$ is continuous at $W_t = W_c$. Integrating (A.7) by parts, we find

$$\begin{aligned}\frac{\partial \phi_{i,t}}{\partial W_t} &= \left[\frac{m_i}{W_t}\frac{e^{-rs}}{\frac{B}{A}W_t\left(e^{As}-1\right)+1}\right]_0^\infty + \frac{m_i}{W_t}\int_0^\infty \frac{re^{-rs}}{\frac{B}{A}W_t\left(e^{As}-1\right)+1}ds \\ &= -\frac{m_i}{W_t}\left(1-\int_0^\infty \frac{re^{-rs}}{\frac{B}{A}W_t\left(e^{As}-1\right)+1}ds\right), \end{aligned} \tag{A.9}$$

and therefore,

$$\left.\frac{\partial \phi_{i,t}}{\partial W_t}\right|_{W_t=W_c^-} = -\frac{m_i}{W_c} + \frac{m_i}{W_c}\int_0^\infty \frac{re^{-rs}}{\frac{B}{A}W_c\left(e^{As}-1\right)+1}ds. \tag{A.10}$$

Moreover, (A.8) implies that

$$\begin{aligned}\left.\frac{\partial \phi_{i,t}}{\partial W_t}\right|_{W_t=W_c^+} &= -\frac{rm_iB}{\beta-r}\int_0^\infty \left[1-\frac{e^{As}}{\frac{B}{A}W_c\left(e^{As}-1\right)+1}\right]e^{-rs}ds \\ &= -\frac{rm_iB}{\beta-r}\int_0^\infty \left[1-\frac{A}{BW_c}-\frac{1-\frac{A}{BW_c}}{\frac{B}{A}W_c\left(e^{As}-1\right)+1}\right]e^{-rs}ds. \end{aligned} \tag{A.11}$$

Using the definition of A (Eq. (25)), we find that (A.11) coincides with (A.10). ■

Proof of Corollary 3: The variable u_j affects $\phi_{i,t}$ through W_c and $A \equiv r-\beta+BW_c$. Since $\partial W_c/\partial u_j$ is a positive constant, it suffices to show the corollary for W_c rather than u_j.

To determine the sign of the cross-effect, we examine how the effect of W_t on $\phi_{i,t}$ depends on W_c. Consider first the constrained region. Since A is increasing in W_c, (A.9) implies that $\partial\phi_{i,t}/\partial W_t$ is decreasing in W_c, i.e., $\partial^2\phi_{i,t}/\partial W_c\partial W_t < 0$. Consider next the unconstrained region. Eq. (A.8) implies that $\partial^2\phi_{i,t}/\partial W_c\partial W_t < 0$ if

$$\frac{\partial}{\partial W_c}\left[W_c - \frac{W_c e^{As}}{\frac{B}{A}W_c\left(e^{As}-1\right)+1}\right] > 0. \tag{A.12}$$

For a general value of W_t,

$$W_c - \frac{W_t e^{As}}{\frac{B}{A}W_t\left(e^{As}-1\right)+1} = W_c - \frac{A}{B} - \frac{W_t-\frac{A}{B}}{\frac{B}{A}W_t\left(e^{As}-1\right)+1} = \frac{\beta-r}{B} - \frac{W_t-\frac{A}{B}}{\frac{B}{A}W_t\left(e^{As}-1\right)+1}. \tag{A.13}$$

Therefore, for $W_t = W_c$,

$$W_c - \frac{W_c e^{As}}{\frac{B}{A}W_c\left(e^{As}-1\right)+1} = \frac{\beta - r}{B} - \frac{\beta - r}{B}\frac{1}{\frac{B}{A}W_c\left(e^{As}-1\right)+1}.$$

This expression is increasing in W_c since A is increasing in W_c. Thus, in both the constrained and unconstrained regions, $\partial^2\phi_{i,t}/\partial W_c \partial W_t = \partial^2\phi_{i,t}/\partial W_t \partial W_c < 0$. To conclude that the effect of W_c on $\phi_{i,t}$ is more negative the larger W_t is, we also need to show that $\partial\phi_{i,t}/\partial W_c$ is continuous at $W_t = W_c$. Eq. (33) implies that in the constrained region

$$\frac{\partial\phi_{i,t}}{\partial W_c} = m_i B \int_0^\infty \frac{\partial}{\partial W_c}\left[W_c - \frac{W_t e^{As}}{\frac{B}{A}W_t\left(e^{As}-1\right)+1}\right] e^{-rs} ds. \tag{A.14}$$

Eq. (34) implies that in the unconstrained region

$$\begin{aligned}\frac{\partial\phi_{i,t}}{\partial W_c} =& \frac{r m_i B}{(\beta - r)W_c}\left(\frac{W_c}{W_t}\right)^{\frac{r}{\beta - r}} \int_0^\infty \left[W_c - \frac{W_c e^{As}}{\frac{B}{A}W_c\left(e^{As}-1\right)+1}\right] e^{-rs} ds \\ &+ m_i B \left(\frac{W_c}{W_t}\right)^{\frac{r}{\beta - r}} \int_0^\infty \frac{\partial}{\partial W_c}\left[W_c - \frac{W_c e^{As}}{\frac{B}{A}W_c\left(e^{As}-1\right)+1}\right] e^{-rs} ds. \end{aligned} \tag{A.15}$$

Eqs. (A.14) and (A.15) imply that

$$\begin{aligned} &\left.\frac{\partial\phi_{i,t}}{\partial W_c}\right|_{W_t = W_c^-} = \left.\frac{\partial\phi_{i,t}}{\partial W_c}\right|_{W_t = W_c^+} \\ \Leftrightarrow\ & 0 = \frac{r m_i B}{\beta - r}\int_0^\infty \left[1 - \frac{e^{As}}{\frac{B}{A}W_c\left(e^{As}-1\right)+1}\right] e^{-rs} ds - m_i B \int_0^\infty \frac{e^{As}}{\left[\frac{B}{A}W_c\left(e^{As}-1\right)+1\right]^2} e^{-rs} ds \\ \Leftrightarrow\ & \left.\frac{\partial\phi_{i,t}}{\partial W_t}\right|_{W_t = W_c^-} = \left.\frac{\partial\phi_{i,t}}{\partial W_t}\right|_{W_t = W_c^+}, \end{aligned}$$

which holds.

We next show that $\partial\phi_{i,t}/\partial W_c > 0$. Eq. (A.13) implies that $\partial\phi_{i,t}/\partial W_c > 0$ in the constrained region if the function

$$G : A \longrightarrow \frac{W_t - \frac{A}{B}}{\frac{B}{A}W_t\left(e^{As}-1\right)+1}$$

is decreasing in A. Since the denominator is increasing in A, $G(A)$ is decreasing if $W_t > A/B$. Since $\partial^2\phi_{i,t}/\partial W_t \partial W_c < 0$, inequality $\partial\phi_{i,t}/\partial W_c > 0$ holds also if $W_t < A/B$. Finally, (A.12) and (A.15) imply $\partial\phi_{i,t}/\partial W_c > 0$ in the unconstrained region. ■

Proof of Corollaries 4, 5 and 6: The first result follows from (33) and (34) by observing that the only asset-specific term in each equation is m_i. Using the same observation, we can derive the second result from (A.7) and (A.8), and the third result from (A.14) and (A.15). ■

Proof of Corollary 7: Follows from Proposition 4. ■

B Risky Arbitrage

Proof of Proposition 5: We proceed as in the proof of Proposition 1, conjecturing the value function (A.1). The Bellman equation is

$$\max_{\hat{x}_{i,t},\hat{c}_t} \left\{ \log(\hat{c}_t) + V_{\hat{W}} \left(r\hat{W}_t + 2\sum_{i\in\mathcal{A}} \hat{x}_{i,t}\Phi_{i,t} - \hat{c}_t \right) \right.$$

$$+\frac{1}{2}V_{\hat{W}\hat{W}} \left[\sum_{j\in\mathcal{A}} \left(\hat{x}_{j,t}\sigma_j^f - \sum_{i\in\mathcal{A}} \hat{x}_{i,t}\sigma_{i,j,t}^{f\phi} \right)^2 + \sum_{j\in\mathcal{A}} \left(\sum_{i\in\mathcal{A}} \hat{x}_{i,t}\sigma_{i,j,t}^{u\phi} \right)^2 \right]$$

$$\left. +V_W\mu_t^W + \frac{1}{2}V_{WW} \left[\sum_{j\in\mathcal{A}} \left(\sigma_{j,t}^{fW} \right)^2 + \sum_{j\in\mathcal{A}} \left(\sigma_{j,t}^{uW} \right)^2 \right] - \beta V \right\} = 0, \tag{B.1}$$

where μ_t^W denotes the drift of W_t. The first-order condition with respect to $\hat{c}_t$ yields $\hat{c}_t = \beta\hat{W}_t$. Optimization over $\{\hat{x}_{i,t}\}_{i\in\mathcal{A}}$ amounts to maximizing

$$\sum_{i\in\mathcal{A}} x_{i,t}\Phi_{i,t} - \frac{1}{2\hat{W}_t} \left[\sum_{j\in\mathcal{A}} \left(\hat{x}_{j,t}\sigma_j^f - \sum_{i\in\mathcal{A}} \hat{x}_{i,t}\sigma_{i,j,t}^{f\phi} \right)^2 + \sum_{j\in\mathcal{A}} \left(\sum_{i\in\mathcal{A}} \hat{x}_{i,t}\sigma_{i,j,t}^{u\phi} \right)^2 \right] \tag{B.2}$$

subject to the financial constraint (9). The first-order condition yields the policy in the proposition. The policy $\{\hat{x}_{i,t}\}_{i\in\mathcal{A}}$ and the maximum value of (B.2) are linear in $\hat{W}_t$. Substituting into (B.1), the terms in $\hat{W}_t$ cancel. Setting the remaining terms to zero, determines the function $v(W_t)$. ■

Proof of Lemma 3: Substituting $\sigma_{i,j,t}^{f\phi}$ from (48) into (45) and solving for $\sigma_{j,t}^{fW}$, we find (49). Substituting $\sigma_{i,j,t}^{u\phi}$ from (48) into (45) and solving for $\sigma_{j,t}^{uW}$, we find (52). ■

Proof of Proposition 6: When $\{\sigma_j^f, \sigma_j^u, \kappa_j^u\}_{j\in\mathcal{A}}$ are small, the highest-order term in (52) is

$$\sigma_{i,j,t}^{u\phi 0} \equiv -\frac{\partial\phi_{i,t}^0}{\partial W_t} \frac{2\sum_{k\in\mathcal{A}} x_{k,t}^0 \frac{\partial\phi_{k,t}^0}{\partial u_{j,t}}\sigma_j^u}{1 + 2\sum_{k\in\mathcal{A}} x_{k,t}^0 \frac{\partial\phi_{k,t}^0}{\partial W_t}} + \frac{\partial\phi_{i,t}^0}{\partial u_{j,t}}\sigma_j^u, \tag{B.3}$$

where $(\phi_{k,t}^0, x_{k,t}^0)$ denote the functions $(\phi_{k,t}, x_{k,t})$ evaluated under riskless arbitrage at the point

$(W_t, \{u_{k,t}\}_{k\in\mathcal{A}})$. The proposition will follow if we show that $\sigma^{u\phi 0}_{i,j,t}$ is positive, decreasing in W_t for $W_t > W_{c,t}$, and increasing in W_t for $A/B \leq W_t < W_c$.

When $W_t > W_{c,t}$, $x^0_{k,t} = \mu_k u_{k,t}$, and the denominator in (B.3) is equal to

$$1 + 2\sum_{k\in\mathcal{A}} \mu_k u_{k,t} \frac{\partial \phi^0_{k,t}}{\partial W_t} = 1 + \frac{2\sum_{k\in\mathcal{A}} m_k \mu_k u_{k,t}}{W_t}\left(-1 + \int_0^\infty \frac{re^{-rs}}{\frac{B}{A}W_t\left(e^{As}-1\right)+1}ds\right)$$
$$= 1 - \frac{W_{c,t}}{W_t} + \frac{W_{c,t}}{W_t}\int_0^\infty \frac{re^{-rs}}{\frac{B}{A}W_t\left(e^{As}-1\right)+1}ds > 0, \tag{B.4}$$

where the first step follows from (A.9), and the second from $A \equiv BW_{c,t} - (\beta - r) > 0$. The variable $\sigma^{u\phi 0}_{i,j,t}$ is positive and decreasing in W_t because (B.4) is positive, $\frac{\partial \phi^0_{k,t}}{\partial u_{j,t}}$ is positive and decreasing in W_t (Corollary 3), and $\frac{\partial \phi^0_{k,t}}{\partial W_t}$ is negative and increasing in W_t (Corollary 2).

When $W_t < W_{c,t}$, (A.9) implies that

$$\frac{\partial \phi^0_{i,t}}{\partial W_t} = -\frac{m_i}{W_t} + \frac{m_i}{W_t}\int_0^\infty \frac{re^{-rs}}{\frac{B}{A}W_t\left(e^{As}-1\right)+1}ds, \tag{B.5}$$

and (A.13) and (A.14) imply that

$$\frac{\partial \phi^0_{i,t}}{\partial u_{j,t}} = -m_i m_j \mu_j B \int_0^\infty \frac{\partial}{\partial W_{c,t}}\left[\frac{W_t - \frac{A}{B}}{\frac{B}{A}W_t\left(e^{As}-1\right)+1}\right] e^{-rs}ds. \tag{B.6}$$

Moreover, since $\{x^0_{k,t}\}_{k\in\mathcal{A}}$ satisfy the financial constraint (20), (A.9) implies that

$$1 + 2\sum_{k\in\mathcal{A}} x^0_{k,t} \frac{\partial \phi^0_{k,t}}{\partial W_t} = 1 + \frac{2\sum_{k\in\mathcal{A}} m_k x^0_{k,t}}{W_t}\left(-1 + \int_0^\infty \frac{re^{-rs}}{\frac{B}{A}W_t\left(e^{As}-1\right)+1}ds\right)$$
$$= \int_0^\infty \frac{re^{-rs}}{\frac{B}{A}W_t\left(e^{As}-1\right)+1}ds, \tag{B.7}$$

and (A.13) and (A.14) imply that

$$2\sum_{k\in\mathcal{A}} x^0_{k,t} \frac{\partial \phi^0_{k,t}}{\partial u_{j,t}} = \left(2\sum_{k\in\mathcal{A}} m_k x^0_{k,t}\right) m_j \mu_j B \int_0^\infty \frac{\partial}{\partial W_{c,t}}\left[W_{c,t} - \frac{W_t e^{As}}{\frac{B}{A}W_t\left(e^{As}-1\right)+1}\right] e^{-rs}ds.$$
$$= -W_t m_j \mu_j B \int_0^\infty \frac{\partial}{\partial W_{c,t}}\left[\frac{W_t - \frac{A}{B}}{\frac{B}{A}W_t\left(e^{As}-1\right)+1}\right] e^{-rs}ds. \tag{B.8}$$

Substituting (B.5)-(B.8) into (B.3), we find

$$\sigma^{u\phi 0}_{i,j,t} = -m_i m_j \mu_j B \sigma^u_j \frac{\int_0^\infty \frac{\partial}{\partial W_{c,t}} \left[\frac{W_t - \frac{A}{B}}{\frac{B}{A} W_t (e^{As}-1)+1} \right] e^{-rs} ds}{\int_0^\infty \frac{re^{-rs}}{\frac{B}{A}W_t(e^{As}-1)+1} ds}$$

$$= m_i m_j \mu_j B \sigma^u_j \left\{ \frac{1}{r} + \frac{\int_0^\infty \frac{\left(W_t - \frac{A}{B}\right)\frac{B}{A^2} W_t \left(Ase^{As} - e^{As} + 1\right)}{\left[\frac{B}{A}W_t(e^{As}-1)+1\right]^2} e^{-rs} ds}{\int_0^\infty \frac{re^{-rs}}{\frac{B}{A}W_t(e^{As}-1)+1} ds} \right\}, \tag{B.9}$$

where the second step follows from $A \equiv r - \beta + BW_{c,t}$. Since the function

$$W_t \longrightarrow \frac{\left(W_t - \frac{A}{B}\right) \frac{B}{A^2} W_t \left(Ase^{As} - e^{As} + 1\right)}{\left[\frac{B}{A} W_t \left(e^{As} - 1\right) + 1\right]^2}$$

is positive for $W_t > A/B$ and increasing in W_t for $W_t > A/(2B)$, and the function

$$W_t \longrightarrow \int_0^\infty \frac{re^{-rs}}{\frac{B}{A}W_t\left(e^{As}-1\right)+1} ds$$

is positive and decreasing in W_t, $\sigma^{u\phi 0}_{i,j,t}$ is positive and increasing in W_t for $A/B \leq W_t < W_c$. ■

Proof of Proposition 7: The loading $\sigma^{f\phi}_{i,j,t}$ for $(i,j) \in \mathcal{A}$ is given by (50). When $\{\sigma^f_j, \sigma^u_j, \kappa^u_j\}_{j\in\mathcal{A}}$ are small, the highest-order term in (50) is

$$\sigma^{f\phi 0}_{i,j,t} \equiv \frac{\partial \phi^0_{i,t}}{\partial W_t} \frac{2x^0_{j,t}\sigma^f_j}{1 + 2\sum_{k\in\mathcal{A}} x^0_{k,t} \frac{\partial \phi^0_{k,t}}{\partial W_t}}. \tag{B.10}$$

The proposition will follow from (42) and the properties of $\sigma^{u\phi 0}_{i,j,t}$ shown in the proof of Proposition 6, if we show that $\sigma^{f\phi 0}_{i,j,t}$ is negative, increasing in W_t for $W_t > W_{c,t}$, and decreasing in W_t for $W_t < W_{c,t}$ and j satisfying (54).

When $W_t > W_{c,t}$, $x^0_{k,t} = \mu_k u_{k,t}$ and the denominator in (B.10) is equal to (B.4). Since σ^f_j, $x^0_{k,t}$, and (B.4) are positive, and $\frac{\partial \phi^0_{k,t}}{\partial W_t}$ is negative and increasing in W_t, $\sigma^{f\phi 0}_{i,j,t}$ is negative and increasing in W_t. When $W_t < W_{c,t}$,

$$x^0_{j,t} = \mu_j \left(u_{j,t} - \frac{m_j \Pi^0_t}{a_j \sigma^2_j} \right) = \mu_j \left[u_{j,t} - \frac{m_j B(W_{c,t} - W_t)}{a_j \sigma^2_j} \right], \tag{B.11}$$

where the first step follows from (22) and the second from (24). Substituting (B.5), (B.7) and

(B.11) into (B.10), we find

$$\sigma^{f\phi 0}_{i,j,t} = -2m_i\mu_j\sigma^f_j\left[\frac{u_{j,t} - \frac{m_j B W_{c,t}}{a_j\sigma_j^2}}{W_t} + \frac{m_j B}{a_j\sigma_j^2}\right]\left[\frac{1}{\int_0^\infty \frac{re^{-rs}}{\frac{B}{A}W_t(e^{As}-1)+1}ds} - 1\right]. \tag{B.12}$$

The first square bracket is positive since $x^0_{j,t} > 0$, and increases with W_t for j satisfying (54). Since the second square bracket is positive and increasing in W_t, $\sigma^{f\phi 0}_{i,j,t}$ is negative and decreasing in W_t. ■

Proof of Proposition 8: The correlation between assets $(i, i') \in \mathcal{A}^2$ is

$$\rho_{i,i',t} \equiv \frac{\left(\sigma^{f\phi}_{i,i,t} - \sigma^f_i\right)\sigma^{f\phi}_{i',i,t} + \sigma^{f\phi}_{i,i',t}\left(\sigma^{f\phi}_{i',i',t} - \sigma^f_{i'}\right) + \sum_{j\in\mathcal{A}/\{i,i'\}}\sigma^{f\phi}_{i,j,t}\sigma^{f\phi}_{i',j,t} + \sum_{j\in\mathcal{A}}\sigma^{u\phi}_{i,j,t}\sigma^{u\phi}_{i',j,t}}{\sigma^R_{i,t}\sigma^R_{i',t}}. \tag{B.13}$$

When $\{\sigma^f_j, \sigma^u_j, \kappa^u_j\}_{j\in\mathcal{A}}$ are small, the highest-order term in (B.13) is

$$\frac{\left(\sigma^{f\phi 0}_{i,i,t} - \sigma^f_i\right)\sigma^{f\phi 0}_{i',i,t} + \sigma^{f\phi 0}_{i,i',t}\left(\sigma^{f\phi 0}_{i',i',t} - \sigma^f_{i'}\right) + \sum_{j\in\mathcal{A}/\{i,i'\}}\sigma^{f\phi 0}_{i,j,t}\sigma^{f\phi 0}_{i',j,t} + \sum_{j\in\mathcal{A}}\sigma^{u\phi 0}_{i,j,t}\sigma^{u\phi 0}_{i',j,t}}{\sigma_i\sigma_{i'}}. \tag{B.14}$$

The properties of $\rho_{i,i',t}$ follow from (B.14) and the properties of $(\sigma^{u\phi 0}_{i,j,t}, \sigma^{f\phi 0}_{i,j,t})$ shown in the proofs of Propositions 6 and 7. The properties of $\rho_{i,-i',t}$ follow from $\rho_{i,-i',t} = -\rho_{i,i',t}$, which is implied from symmetry. To show the properties of $\rho_{i,-i,t}$, we note that symmetry implies that

$$\rho_{i,-i,t} = \frac{\sigma_i^2 - \left(\sigma^f_i - \sigma^{f\phi}_{i,i,t}\right)^2 - \sum_{j\in\mathcal{A}/\{i\}}\left(\sigma^{f\phi}_{i,j,t}\right)^2 - \sum_{j\in\mathcal{A}}\left(\sigma^{u\phi}_{i,j,t}\right)^2}{\left(\sigma^R_{i,t}\right)^2}.$$

When $\{\sigma^f_j, \sigma^u_j, \kappa^u_j\}_{j\in\mathcal{A}}$ are small, $\rho_{i,-i,t}$ is close to one. Using (42), we find that the highest-order term in $1 - \rho_{i,-i,t}$ is

$$\frac{2\left[\left(\sigma^f_i - \sigma^{f\phi 0}_{i,i,t}\right)^2 + \sum_{j\in\mathcal{A}/\{i\}}\left(\sigma^{f\phi 0}_{i,j,t}\right)^2 + \sum_{j\in\mathcal{A}}\left(\sigma^{u\phi 0}_{i,j,t}\right)^2\right]}{\sigma_i^2}. \tag{B.15}$$

The comparative statics of (B.15) are the same as for (B.14). Therefore, the properties of $\rho_{i,-i,t}$ are opposite to those of $\rho_{i,i',t}$. ■

Proof of Lemma 4: When $\{\sigma^f_j, \sigma^u_j, \kappa^u_j\}_{j\in\mathcal{A}}$ are small, arbitrageurs long all opportunities and their first-order condition is (55). Combining with (43) and using (46), we find

$$x_{i,t} = \mu_i\left[u_{i,t} - \frac{\Psi_{i,t} + m_i\Pi_t}{a_i\left(\sigma^R_{i,t}\right)^2}\right], \tag{B.16}$$

where

$$\Psi_{i,t} \equiv \frac{1}{W_t}\left[(\sigma_i^f - \sigma_{i,i,t}^{f\phi})\sigma_{i,t}^{fW} - \sum_{j\in\mathcal{A}/\{i\}} \sigma_{i,j,t}^{f\phi}\sigma_{j,t}^{fW} - \sum_{j\in\mathcal{A}}\sigma_{i,j,t}^{u\phi}\sigma_{j,t}^{uW}\right].$$

Multiplying (B.16) by m_i and summing over $i \in \mathcal{A}$, we find that the financial constraint (9) holds as an equality if and only if

$$W_t = W_{c,t} - 2\sum_{i\in\mathcal{A}} \frac{m_i\mu_i\left(\Psi_{i,t} + m_i\Pi_t\right)}{a_i\left(\sigma_{i,t}^R\right)^2}. \tag{B.17}$$

For small $\{\sigma_j^f, \sigma_j^u, \kappa_j^u\}_{j\in\mathcal{A}}$, the highest-order term in $\Psi_{i,t}$ is

$$\Psi_{i,t}^0 \equiv \frac{1}{W_t}\left[(\sigma_i^f - \sigma_{i,i,t}^{f\phi 0})\sigma_{i,t}^{fW0} - \sum_{j\in\mathcal{A}/\{i\}} \sigma_{i,j,t}^{f\phi 0}\sigma_{j,t}^{fW0} - \sum_{j\in\mathcal{A}}\sigma_{i,j,t}^{u\phi 0}\sigma_{j,t}^{uW0}\right], \tag{B.18}$$

where

$$\sigma_{j,t}^{fW0} \equiv \frac{2x_{j,t}^0\sigma_j^f}{1 + 2\sum_{i\in\mathcal{A}} x_{k,t}^0 \frac{\partial\phi_{k,t}^0}{\partial W_t}} \quad \text{and} \quad \sigma_{j,t}^{uW0} \equiv -\frac{2\sum_{k\in\mathcal{A}} x_{k,t}^0 \frac{\partial\phi_{k,t}^0}{\partial u_{j,t}}\sigma_j^u}{1 + 2\sum_{k\in\mathcal{A}} x_{k,t}^0 \frac{\partial\phi_{k,t}^0}{\partial W_t}} \tag{B.19}$$

are the highest-order terms in $(\sigma_{j,t}^{fW}, \sigma_{j,t}^{uW})$, respectively. Since $\sigma_{i,j,t}^{f\phi 0} < 0$, $\sigma_{i,j,t}^{u\phi 0} > 0$, $\sigma_{j,t}^{fW0} > 0$ and $\sigma_{j,t}^{uW0} < 0$, $\Psi_{i,t}^0 > 0$. Lemma 4 follows from (B.17), $\Pi_t \geq 0$ and $\Psi_{i,t}^0 > 0$. ■

Proof of Proposition 9: In the region where arbitrageurs are unconstrained, $\Pi_t = 0$. Eq. (B.16) implies that when $\{\sigma_j^f, \sigma_j^u, \kappa_j^u\}_{j\in\mathcal{A}}$ are small, the two highest-order terms in $x_{i,t}$ are

$$x_{i,t}^1 \equiv \mu_i\left(u_{i,t} - \frac{\Psi_{i,t}^0}{a_i\sigma_i^2}\right). \tag{B.20}$$

Since $\sigma_{i,j,t}^{f\phi 0}$ and $\sigma_{j,t}^{uW0}$ are negative and increasing in W_t, and $\sigma_{i,j,t}^{u\phi 0}$ and $\sigma_{j,t}^{fW0}$ are positive and decreasing in W_t, (B.18) implies that $\Psi_{i,t}^0$ is decreasing in W_t, and therefore, $x_{i,t}^1$ is increasing. Eq. (B.20) implies that

$$x_{i,t}^1 - x_{i',t}^1 = \frac{\mu}{a\sigma^2}\left(\Psi_{i',t}^0 - \Psi_{i,t}^0\right), \tag{B.21}$$

where $(\sigma, a, \mu, u_t) \equiv (\sigma_i, a_i, \mu_i, u_{i,t}) = (\sigma_{i'}, a_{i'}, \mu_{i'}, u_{i',t})$. Noting that $\phi_{i,t}^0/m_i = \phi_{i',t}^0/m_{i'}$ and $x_{i,t}^0 = \mu_i u_{i,t}$, and using (B.3), (B.10), (B.18), (B.19) and (B.21), we find

$$x_{i,t}^1 - x_{i',t}^1 = \frac{\mu}{a\sigma^2 W_t}\left\{\frac{2\mu u_t\left[\left(\sigma_{i'}^f\right)^2 - \left(\sigma_i^f\right)^2\right]}{1 + 2\sum_{i\in\mathcal{A}}\mu u_{k,t}\frac{\partial\phi_{k,t}^0}{\partial W_t}} + \left(1 - \frac{m_{i'}}{m_i}\right)\left(\sum_{j\in\mathcal{A}}\sigma_{i,j,t}^{f\phi 0}\sigma_{j,t}^{fW0} + \sum_{j\in\mathcal{A}}\sigma_{i,j,t}^{u\phi 0}\sigma_{j,t}^{uW0}\right)\right\}.$$

(B.22)

If $m_i > m_{i'}$ and $\sigma_i^f = \sigma_{i'}^f$, the first term in the curly bracket is zero. Since the second term is negative and increasing in W_t, $x_{i,t}^1 - x_{i',t}^1$ is increasing in W_t. If $m_i = m_{i'}$ and $\sigma_i^f > \sigma_{i'}^f$, the second term in the curly bracket is zero. Since the first term is negative and increasing in W_t, $x_{i,t}^1 - x_{i',t}^1$ is increasing in W_t. In both cases, $\partial x_{i,t}^1/\partial W_t > \partial x_{i',t}^1/\partial W_t > 0$, i.e., changes in arbitrageur wealth impact more strongly their position in opportunity $(i, -i)$ than $(i', -i')$.

In the region where arbitrageurs are constrained, $\Pi_t > 0$. Eq. (B.16) implies that when $\{\sigma_j^f, \sigma_j^u, \kappa_j^u\}_{j\in\mathcal{A}}$ are small, the two highest-order terms in $x_{i,t}$ are

$$x_{i,t}^1 = \mu_i \left\{ u_{i,t} - \frac{\Psi_{i,t} + m_i \Pi_t}{a_i \sigma_i^2} \left[1 + \frac{\left(\sigma_{i,t}^R\right)^2 - \sigma_i^2}{\sigma_i^2} \right] \right\}. \tag{B.23}$$

The comparative statics with respect to m_i follow by considering the highest-order term

$$x_{i,t}^0 = \mu_i \left(u_{i,t} - \frac{m_i \Pi_t}{a_i \sigma_i^2} \right),$$

i.e., as in the case of riskless arbitrage. (The ambiguous comparative statics with respect to σ_i^f follow by considering the term in the next order.) ■