Transferable Utility Games with Uncertainty

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Abstract

We introduce the concept of a transferable utility game with uncertainty (TUU-game). In a TUU-game there is uncertainty regarding the payoffs of coalitions. One out of a finite number of states of nature may materialize and conditional on the state, the players are involved in a particular transferable utility game. We consider the case without ex ante commitment possibilities and propose the Weak Sequential Core as a solution concept. We characterize the Weak Sequential Core and show that it is non-empty if all ex post TU-games are convex. We study bankruptcy games with uncertainty and apply the Weak Sequential Core. We find that most of the best-known allocation rules are unstable in this setting, except for the Constrained Equal Awards rule.

Keywords: transferable utility games, uncertainty, weak sequential core, bankruptcy games

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1 Introduction

The vast majority of cooperative game theory has focused on games with deterministic payoffs. Nevertheless, uncertainty plays an inevitable role in in most decision making problems. In this paper we introduce transferable utility games with uncertainty, called TUU-games. A TUU-game consists of two time periods, 0 and 1. In period 1 one out of a finite number of states of nature may materialize and conditional on the state, the players are involved in a particular transferable utility game. An allocation therefore specifies a payoff to each player conditional on each possible state of nature. A utility function is then used to assign a utility level to each profile of state-contingent payoffs.

This new set-up provides a more general treatment of uncertainty than the approach that has appeared in the literature so far. Granot (1977) introduced a cooperative game where the values of the coalitions are random variables with given distribution functions, and players are risk-neutral. This treatment is less complete since it specifies only the marginal distribution of the worths of coalitions, whereas in our approach the complete distribution is specified, implying that for instance correlation between the worths of several coalitions can be incorporated. Suijs and Borm (1999) and Suijs, Borm, De Waegenaere, and Tijs (1999) no longer assume risk neutrality, but keep the specification where only marginal distributions of worths are given. Bossert, Derks, and Peters (2005) consider a pair of TU-games, one of which will be the true game. They do not use utility functions but perform a worst-case analysis. Closest to our set-up is Predtetchinski (2007), where the non-transferable utility case is studied in an infinite horizon setting. His approach is similar to ours in the sense that in both cases the game to be played is determined by the particular realization of the state of nature.

The introduction of uncertainty into cooperative games raises many new and interesting issues. When players can make state-contingent agreements before the resolution of uncertainty, i.e. at period 0, the situation boils down to a non-transferable utility game, and we can apply for instance the classical concept of the Core to determine allocations of payoffs that are stable.

We, on the contrary, are interested in the case where no binding agreements are possible before the state of nature is known. A typical case would be where the state of nature is not verifiable by an outside court. A consequence of the absence of binding agreements is that many ex ante desirable transfers of payoffs across states are not feasible. Indeed, in the absence of binding agreements in period 0, only allocations in the Core of the transferable utility game that results after the state of nature is known, are enforceable.

We are interested in the appropriate definition of the Core in a TUU-game. In this setting coalitions are allowed to form in both periods. Stability requires that a suggested allocation cannot be blocked by any coalition at any period, i.e. both before and after the resolution of uncertainty. We concentrate on agreements which are self-enforcing in the sense that a coalition can only deviate from a given allocation if no sub-coalition ever has a credible counter-deviation. Ray (1989) shows that in a static environment the set of deviations coincides with the set of credible deviations. This is no longer true in our setting, and leads to the solution concept of the Weak Sequential Core.

The Weak Sequential Core was introduced in Kranich, Perea, and Peters (2005) for finite deterministic sequences of TU-games, and it was defined for two-period exchange economies with incomplete markets in Predtetchinski, Herings, and Perea (2006). In Kranich, Perea, and Peters (2005) the Weak Sequential Core was defined as the set of feasible payoff allocations for the grand coalition, from which no coalition ever has a credible deviation. In Habis and Herings (2010) it is demonstrated that the original definition of credibility has to be adapted in order to demonstrate that the Weak Sequential Core has a nice characterization in terms of the cores of appropriately defined subgames. In Predtetchinski, Herings, and Perea (2006) this characterization was used as the definition of the Weak Sequential Core in a two-period exchange economy.

We extend the notion of credible deviation of Habis and Herings (2010) to TUU-games and show that an allocation belongs to the Weak Sequential Core only if conditional on the state of nature it belongs to the Core of the TU-game related to that state. This result follows from the absence of credible deviations in period 1. The absence of credible deviations in period 0 is then used to show that an allocation belongs to Weak Sequential Core if moreover there is no coalition in period 0 that can propose state-contingent Core elements of the game restricted to that coalition, which gives each of its members higher expected utility. In this way we obtain a characterization of the Weak Sequential Core.

A problem of the Weak Sequential Core concept is that the existing literature has failed to provide a general non-emptiness result, whereas moreover both Kranich, Perea, and Peters (2005) and Predtetchinski, Herings, and Perea (2006) give examples where the Weak Sequential Core is empty. We provide a general result on the non-emptiness of the Weak Sequential Core of TUU-games. We show that if the TU-game that is played conditional on the state of nature is convex, then the Weak Sequential Core is non-empty. This result does not impose any assumptions on the utility functions of the players beyond continuity and state-separability.

An important application of convex games is the bankruptcy problem. We study bankruptcy problems with uncertainty, i.e. both the estates and the claims are allowed to be state-dependent. Solutions to bankruptcy problems with uncertainty can be obtained by allocating the payoff in each state according to one of the rules proposed in the literature, like the Proportional rule, the Adjusted Proportional rule, the Constrained Equal Awards rule, the Constrained Equal Losses rule, or the Talmud rule. We refer to Thomson (2003) for an excellent overview of the literature on the bankruptcy problem. The question we ask is which one of these solutions belongs to the Weak Sequential Core of the game, implying that such a solution is stable both in an ex ante and an ex post sense. We demonstrate that the Constrained Equal Awards rule is the only one leading to allocations in the Weak Sequential Core.

The outline of the paper is as follows. We specify the model in Section 2 and give the formal definition of the Weak Sequential Core in Section 3, followed by its characterization in Section 4. We show the non-emptiness result in Section 5. The bankruptcy problem is analyzed in Section 6 and Section 7 concludes.

2 Preliminaries

Consider a game with two time periods, $t \in T = \{0, 1\}$. In period 1 one state s out of a finite set of states of nature $\{1, \ldots, S\}$ occurs. Since no confusion can arise, we also denote this set by S. We define the state of nature for period 0 as state 0, so the set of all states is $S' = \{0\} \cup S$. In period 1 the players are involved in a *cooperative game with transferable utility*, or briefly *TU-game*, where the game itself is allowed to be state-dependent. Period 0 serves as a point in time prior to the resolution of uncertainty.

The TU-game Γ_s played in state $s \in S$ is a pair (N, v_s), where $N = \{1, 2, \ldots, n\}$ is the set of players and $v_s : 2^N \to \mathbb{R}$ is a characteristic function which assigns to each coalition $C \subseteq N$ its *worth* $v_s(C)$, with the convention that $v_s(\emptyset) = 0$. Player $i \in N$ evaluates his payoffs by a utility function $u^i : \mathbb{R}^S \to \mathbb{R}$, which assigns to every profile of payoffs $x^i = (x^i_1, \ldots, x^i_S) \in \mathbb{R}^S$ a utility level $u^i(x^i)$ and is assumed to be continuous and state-separable, i.e. $u^i(x^i) = \sum_{s \in S} u^i_s(x^i_s)$, where $u^i_s(x^i_s)$ is monotonically increasing.

Von Neumann-Morgenstern utility functions are a prominent example of utility functions satisfying these assumptions.

A TU-game with uncertainty is defined as follows.

Definition 2.1. A *TU-game with uncertainty* (*TUU-game*) Γ is a tuple (N, S, v, u) where $v = (v_1, \ldots, v_S)$ and $u = (u^1, \ldots, u^n)$.

Note that there are no payoffs in state 0. State 0 is merely introduced as a point in time when the players face the uncertainty in the future and may decide to cooperate. Payoffs in state 0 could be incorporated into our model but our main interest is to get insight into the effect of future uncertainty on the stability of payoff allocations.

Another observation is that when the cardinality of S is one, the concept of a TUU-game collapses with the one of a TU-game. In the absence of uncertainty, all monotonic transformations of utility functions are equivalent, and it is without loss of generality to take $u^i(x^i) = x^i$. Our interest is obviously in the cases with non-degenerate uncertainty.

The central question in a TUU-game is how the worth $v_s(N)$ of the grand coalition is distributed among its members in every state $s \in S$. A distribution of worth, represented by a matrix $x = (x^1, \ldots, x^n) \in \mathbb{R}^{S \times N}$, is called an *allocation*. The state-s component $x_s = (x_s^1, \ldots, x_s^n) \in \mathbb{R}^N$ of an allocation is referred to as the allocation in state $s \in S$. The total worth obtained by coalition C in state s is $x_s(C) = \sum_{i \in C} x_s^i$. An allocation for a coalition C is a matrix $x^C = (x^i)_{i \in C} \in \mathbb{R}^{S \times C}$, with a state-$s$ component $x_s^C \in \mathbb{R}^C$. The restriction of a TUU-game Γ to coalition C is a TUU-game itself and is denoted by (Γ, C).

3 The Weak Sequential Core

We study which allocations in the game Γ are stable. In general, $\bar{x}$ is stable if there is no state $s' \in S'$ and no coalition $C \subseteq N$ which has a profitable deviation from $\bar{x}$ at state s'. There are various ways in which the notion of profitable deviation might be formulated. Here we concentrate on the Weak Sequential Core, introduced in Kranich, Perea, and Peters (2005) for finite deterministic sequences of TU-games and in Predtetchinski, Herings, and Perea (2006) for two-period exchange economies with incomplete markets. We define the Weak Sequential Core for TUU-games.

When the classical definition of the Core (Gillies, 1959) is adapted to situations with time and uncertainty, it is typically assumed that agents can fully commit to any state-contingent allocation. In this case one would define the set of utilities for a coalition $C \subset N$ as

$$V(C) = \{(u^i(x^i))_{i\in C} \in \mathbb{R}^C \mid \exists x^C \in \mathbb{R}^{S\times C} \text{ such that } \forall s \in S,\ x_s^C(C) \leq v_s(C)\},$$

thereby obtaining an NTU-game. Full commitment may be a strong and unrealistic assumption in the presence of time and uncertainty. Once the state of nature is known, there are typically players which have no incentives to stick to the previously arranged allocation of payoffs. One problem with full commitment is that the state of nature may not be verifiable by an outside court, implying that previously made arrangements cannot be enforced. Here we analyze the case with the absence of commitments and look for agreements which are self-enforcing.

First we define what allocations and thereby deviations are feasible for coalitions at different states, then we formalize the notion of credible deviations and finally we define the Weak Sequential Core of a TUU-game. We start with feasibility at future states.

Definition 3.1. Let some allocation $\bar{x}$ be given. The allocation x^C is *feasible* for a coalition C at state $s \in S$ if

$$\begin{aligned} x_{-s}^C &= \bar{x}_{-s}^C, \\ x_s(C) &\leq v_s(C). \end{aligned}$$

The first condition requires that the members of a coalition take allocations outside state s as given. Since utility functions are assumed to be state-separable, this assumption is harmless. According to the second condition, in state s the members of a coalition can redistribute at most their worth.

We turn next to feasibility as state 0.

Definition 3.2. The allocation x^C is *feasible* for a coalition C at state 0 if

$$x(C) \leq v(C).$$

Note that feasibility at state 0 requires that the allocation must be feasible for coalition C in every state; it requires $\sum_{i\in C} x_s^i \leq v_s(C)$ to hold for all states in period 1.

We continue by defining deviations as feasible allocations that improve the utility of every coalition member.

Definition 3.3. Let some allocation $\bar{x}$ be given. A coalition C can *deviate* from $\bar{x}$ at state $s' \in S'$ if there exists a feasible allocation x^C for C at s' such that

$$u^i(x^i) > u^i(\bar{x}^i), \text{ for all } i \in C.$$

The allocation x^C in Definition 3.3 is referred to as a *deviation.* Definition 3.3 can be extended in an obvious way to define deviations from an allocation x^C by a sub-coalition D of C.

We show in the following example that deviations are not necessarily self-enforcing.

Example 3.4. Consider a TUU-game with two players and with two states in period 1 with equal probability of occurrence. The players are assumed to be strictly risk-averse expected utility maximizers. Let the state-dependent characteristic function be the following: $v_1(\{1,2\}) = v_2(\{1,2\}) = 1$, $v_1(\{1\}) = v_2(\{2\}) = 1$, $v_1(\{2\}) = v_2(\{1\}) = 0$.

Let the allocation

$$\bar{x} = (\bar{x}^1, \bar{x}^2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

be given. Now consider the allocation

$$x = (x^1, x^2) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

which is feasible for the grand coalition in state 0. Since both players are risk-averse, x is a deviation from $\bar{x}$ at state 0 by coalition $\{1,2\}$.

The allocation x is not self-enforcing though, since after the resolution of uncertainty it will always be blocked by a singleton coalition; at state 1 player 1 can block $x_1^1 = \frac{1}{2}$ by $\hat{x}_1^1 = v_1(\{1\}) = 1$ and at state 2 player 2 can block $x_2^2 = \frac{1}{2}$ by $\hat{x}_2^2 = v_2(\{2\}) = 1$.

Since deviations should be self-enforcing, we introduce the notion of credible deviations. In defining credibility, we follow the approach developed in Ray (1989) for the static case. Ray (1989) shows that in a static environment the set of deviations coincides with the set of credible deviations. This is no longer true in our setting.

Credible deviations are defined recursively and by backwards induction. At any future state, any deviation by a singleton coalition is credible. A 2-player coalition has a credible deviation at a future state if there is no singleton sub-coalition with a credible counter-deviation at that state. A credible deviation at a future state for an arbitrary coalition is then defined by recursion. More formally, a recursive definition of a credible deviation at state $s \in S$ by a coalition C is as follows.

Definition 3.5. Let some allocation $\bar{x}$ be given. Any deviation x^C from $\bar{x}$ at state $s \in S$ by a singleton coalition is *credible*. A deviation x^C from $\bar{x}$ at state s by coalition C is *credible* if there is no sub-coalition $D \subsetneq C$ such that D has a credible deviation from x^C at state s.

At state 0, again, any deviation by a singleton coalition is credible. A 2-player coalition has a credible deviation at state 0 if there is no singleton sub-coalition with a credible counter-deviation at any state, current or future. A credible deviation at state 0 by an arbitrary coalition is then defined by recursion. More formally, we have the following definition.

Definition 3.6. Let some allocation $\bar{x}$ be given. Any deviation x^C from $\bar{x}$ at state 0 by a singleton coalition is *credible*. A deviation x^C from $\bar{x}$ at state 0 by coalition C is *credible* if there is no sub-coalition $D \subsetneq C$ and state $s' \in S'$ such that D has a credible deviation from x^C at s'.

Definition 3.7. The *Weak Sequential Core* WSC(Γ) of the game Γ is the set of feasible allocations $\bar{x}$ for the grand coalition from which no coalition ever has a credible deviation.

Our definition of the Weak Sequential Core is different from the one in Kranich, Perea, and Peters (2005) and the one in Predtetchinski, Herings, and Perea (2006). Kranich, Perea, and Peters (2005) do not require the counter-deviation by a sub-coalition to be credible, which leads to problems as demonstrated in Habis and Herings (2010). We adapt the definition in Habis and Herings (2010) to TUU-games. The definition of the Weak Sequential Core in Predtetchinski, Herings, and Perea (2006) for an incomplete markets exchange economy is based directly on the characterization we present in Theorem 4.4.

Example 3.4 (continued). We show that $\bar{x}$ is the only allocation which belongs to the Weak Sequential Core of the game. For an allocation x to belong to the Weak Sequential Core, it must hold that $x_1^1 \geq 1$, since otherwise player 1 could credibly block x in state 1 by $\hat{x}_1^1 = v_1(\{1\}) = 1$. An analogous reasoning implies that $x_1^2 \geq 0$. Similarly, $x_2^2 \geq 1$ must hold, since otherwise player 2 could credibly block x in state 2 by $\hat{x}_2^2 = v_2(\{2\}) = 1$, and by analogous reasons we have $x_2^1 \geq 0$. Now it follows from feasibility for the grand coalition that $\bar{x}$ is the only candidate element of WSC(Γ).

Clearly, singleton coalitions cannot deviate from $\bar{x}$ at any state. The same is obviously true for the grand coalition at any future state. The arguments already used to derive that $\bar{x}$ is the only candidate as a Weak Sequential Core element, imply that the grand coalition does not have a credible deviation at state 0.

4 Characterization

In this section we provide a useful characterization for the Weak Sequential Core. Consider a particular credible deviation at state 0 by some coalition. We show that the set consisting of all credible deviations which improve the utility of all coalition members by the same amount or more is a compact set.

Lemma 4.1. *Let $\bar{x}$ be a feasible allocation and let $\hat{x}^C$ be a credible deviation from $\bar{x}$ at state 0 by coalition C. Let X be the set of credible deviations x^C from $\bar{x}$ at state 0 by coalition C such that $u^i(x^i) \geq u^i(\hat{x}^i)$ for all $i \in C$. Then the set X is compact.*

Proof. First we show that X is closed. Consider a sequence $(x_n^C)_{n\in\mathbb{N}}$ with $x_n^C \in X$ converging to $\tilde{x}^C$. We need to show that $\tilde{x}^C \in X$, so

(i) $\tilde{x}^C$ is a credible deviation from $\bar{x}$ at state 0 by C,

(ii) $u^i(\tilde{x}^i) \geq u^i(\hat{x}^i)$ for all $i \in C$.

The continuity of u^i implies $u^i(\tilde{x}^i) \geq u^i(\hat{x}^i)$ for all $i \in C$, thus (ii) holds.

Clearly, $\tilde{x}^C$ is a deviation from $\bar{x}$ at state 0 by C, so if $\tilde{x}^C$ is not a credible deviation then there is a credible deviation y^D from $\tilde{x}^C$ at $s' \in S'$ by a sub-coalition $D \subsetneq C$. Since $u^i(\tilde{x}^i) < u^i(y^i)$ for all $i \in D$ there must be an $\hat{n}$ such that if $n > \hat{n}$ then for all $i \in D$, $u^i(x_n^i) < u^i(y^i)$. This makes y^D a credible deviation from x_n^C at state s' by coalition D, a contradiction, so (i) holds. Hence, X is closed.

Now we show that X is bounded. For all $x^C \in X$ it holds that

$$x^i \geq v(\{i\}), \quad i \in C,$$

since $x^i_s = v_s(\{i\})$ for all $s \in S$ if $C = \{i\}$, and no player in C should have a credible deviation from x^C at any $s \in S$ if C is not a singleton. Therefore X is bounded from below. Since $x^C(C) = v(C)$, it follows that X is also bounded from above.

$\square$

Note that Lemma 4.1 is not true for the set of deviations rather then the set of credible deviations, since in the case of deviations it might be possible to compensate arbitrarily negative payoffs in one state by sufficiently high positive payoffs in other states.

Our characterization makes use of the classical notion of the Core of a TU-game.

Definition 4.2. A coalition C can *improve* upon an allocation $\bar{x}$ in a TU-game (N, v) if $\bar{x}(C) < v(C)$.

Definition 4.3. The *Core* $\mathrm{C}(N, v)$ of a TU-game (N, v) is the collection of allocations $\bar{x}$ such that $\bar{x}(N) = v(N)$ and there is no coalition C that can improve upon $\bar{x}$.

The Weak Sequential Core can be characterized by means of the Core of suitably chosen subgames.

Theorem 4.4. *The following two statements are equivalent:*

(a) $\bar{x} \in \mathrm{WSC}(\Gamma)$,

(b) $\bar{x}$ *is such that* $\bar{x}_s \in \mathrm{C}(\Gamma_s)$ *for all* $s \in S$, *and there is no* $C \subset N$ *and allocation* x^C *such that* $x^C_s \in \mathrm{C}(\Gamma_s, C)$ *for all* $s \in S$, *and* $u^i(x^i) > u^i(\bar{x}^i)$ *for all* $i \in C$.

Proof.
$(a) \Rightarrow (b)$. Consider some state $s \in S$ and suppose there is a coalition $C \subset N$ that can improve upon $\bar{x}_s$ by x^C_s. We define $x^C_{-s} = \bar{x}_{-s}$. Either x^C is a credible deviation from $\bar{x}$ at state s by coalition C or there is sub-coalition $D \subsetneq C$ such that D has a credible deviation y^D from x^C at s. In the latter case y^D is also a credible deviation from $\bar{x}$ at state s by coalition D. We have a contradiction with $\bar{x} \in \mathrm{WSC}(\Gamma)$. It follows that $\bar{x}_s \in \mathrm{C}(\Gamma_s)$.

Suppose there is $C \subset N$ and x^C such that $x^C_s \in \mathrm{C}(\Gamma_s, C)$ for all $s \in S$, and $u^i(x^i) > u^i(\bar{x}^i)$ for all $i \in C$. We show that if such a deviation exists then there also exists a credible deviation, thereby contradicting (a). If x^C is a credible deviation from $\bar{x}$ at 0 by C, then we are done, so suppose this is not the case. Since $x^C_s \in \mathrm{C}(\Gamma_s, C)$ holds for all $s \in S$, there cannot be a credible deviation from x^C at $s \in S$ by some coalition $D \subsetneq C$, so there must be a credible deviation y^D from x^C at state 0 by some coalition $D \subsetneq C$. But then y^D is also a a credible deviation from $\bar{x}$ at state 0 by D since $u^i(y^i) > u^i(x^i) > u^i(\bar{x}^i)$ for all $i \in D$.

$(b) \Rightarrow (a)$. Suppose (a) does not hold. Since $\bar{x}_s \in \mathrm{C}(\Gamma_s)$ for all $s \in S$, no coalition has a credible deviation from $\bar{x}$ at $s \in S$ and so there must be a credible deviation $\hat{x}^C$ from $\bar{x}$ at state 0 by a coalition C. We will show that then there also exists a credible deviation $\tilde{x}^C$ from $\bar{x}$ at state 0 by coalition C such that $\tilde{x}^C_s \in \mathrm{C}(\Gamma_s, C)$ for all $s \in S$, thereby violating (b).

Let X be the set of credible deviations x^C from $\bar{x}$ at state 0 by C with the property that $u^i(x^i) \geq u^i(\hat{x}^i)$ for all $i \in C$. Let $\tilde{x}^C$ be a solution of the problem

$$\max_{x^C \in X} \sum_{i \in C} u^i(x^i). \tag{1}$$

Since the allocation $\hat{x}^C$ belongs to X, X is non-empty. We know from Lemma 4.1 that X is compact. Therefore the set of maximizers in (1) is non-empty.

We show that $\tilde{x}^C_s$ belongs to $\mathrm{C}(\Gamma_s, C)$ for all $s \in S$. Suppose there exists a state $s \in S$ for which $\tilde{x}^C_s \notin \mathrm{C}(\Gamma_s, C)$. Then there is a coalition $D \subset C$ that can improve upon $\tilde{x}^C_s$ by means of y^D_s. Since $\tilde{x}^C$ is a credible deviation from $\bar{x}$, it is not possible that $D \subsetneq C$, so $D = C$.

We define the allocation $\tilde{y}^C$ by $\tilde{y}^C_s = y^C_s$ and $\tilde{y}^C_{-s} = \tilde{x}^C_{-s}$, and show that $\tilde{y}^C$ belongs to X. By the separability of the utility function it holds that $u^i(\tilde{y}^i) > u^i(\tilde{x}^i) \geq u^i(\hat{x}^i)$ for all $i \in C$.

It also holds that $\tilde{y}^C$ is a credible deviation from $\bar{x}$ at state 0 by C. Suppose not. Since $u^i(\tilde{y}^i) > u^i(\tilde{x}^i) \geq u^i(\hat{x}^i)$ for all $i \in C$, for $\tilde{y}^C$ not to be a credible deviation from $\bar{x}$, there should be a sub-coalition $D \subsetneq C$ with a credible deviation z^D from $\tilde{y}^C$ at $s' \in S'$. This leads to a contradiction when $s' = s$ since y^C_s is credible. When $s' \neq s$ we get a contradiction since $\tilde{x}^C$ is credible. We have shown that $\tilde{y}^C \in X$.

It follows that $\sum_{i \in C} u^i(\tilde{y}^i) > \sum_{i \in C} u^i(\tilde{x}^i)$, which contradicts that $\tilde{x}^C$ is a maximizer. We have shown that $\tilde{x}^C_s \in \mathrm{C}(\Gamma_s, C)$ for all $s \in S$. $\square$

For an allocation to belong to the Weak Sequential Core of the TUU-game Γ, the allocation should belong to the Core of the TU-game Γ_s in every state $s \in S$. Moreover, no coalition should be able to pick an element of the Core of the game restricted to C in every state, and in doing so improve utility in an ex ante sense.

It follows immediately from Theorem 4.4 that the Weak Sequential Core of a TUU-game with one state coincides with the Core of that game.

In a TUU-game one can distinguish ex ante and ex post efficiency.

Definition 4.5. An allocation $\bar{x}$ is *ex ante efficient* in the game Γ if:

(i) $\bar{x}(N) \leq v(N)$.

(ii) There does not exist an allocation x with $x(N) \leq v(N)$ such that $u^i(x^i) > u^i(\bar{x}^i)$ for all $i \in N$.

Definition 4.6. An allocation $\bar{x}$ is *ex post efficient* in the game Γ if $\bar{x}(N) = v(N)$.

Note, that the concept of ex post efficiency says more than the usual feasibility conditions in TU-games, since it requires $\sum_{i \in N} \bar{x}^i_s = v_s(N)$ to hold at all states $s \in S$, but contrary to ex ante efficiency it does not imply Pareto-efficiency, since it does not consider reallocation possibilities across states.

Corollary 4.7. *If* $\bar{x} \in \mathrm{WSC}(\Gamma)$, *then* $\bar{x}$ *is ex post efficient.*

Observe that Example 3.4 demonstrates that an allocation in the Weak Sequential Core might not be ex ante efficient.

5 Non-emptiness

Kranich, Perea, and Peters (2005) show that the Weak Sequential Core of a finite deterministic sequence of TU-games is non-empty if all utility functions are linear. Predtetchinski, Herings, and Perea (2006) give sufficient conditions for non-emptiness for the case of an exchange economy with two agents. These are the only results in the literature so far regarding non-emptiness of the Weak Sequential Core. Both papers present examples where the Weak Sequential Core is empty.

The Weak Sequential Core can also be empty in a TUU-game, as shown in the following example.

Example 5.1. Consider a TUU-game Γ with three players and two future states. The characteristic function v is presented in Table 2.

Table 1: Characteristic function

v	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
v_1	0	5	50	5	140	20	140	150
v_2	0	50	5	5	140	140	20	150

Players have the utility function

$$u^i(x^i) = 1/2(1 - e^{-0.1x_1^i}) + 1/2(1 - e^{-0.1x_2^i}), \quad i = 1, 2, 3.$$

By Theorem 4.4 only allocations in the Core of Γ_1 and Γ_2 can be stable. The Core of each of these TU-games consists of exactly one vector:

$$\begin{aligned} \mathrm{C}(\Gamma_1) &= \{(10, 130, 10)\}, \\ \mathrm{C}(\Gamma_2) &= \{(130, 10, 10)\}. \end{aligned}$$

The resulting allocation

$$\bar{x} = (\bar{x}^1, \bar{x}^2, \bar{x}^3) = \begin{pmatrix} 10 & 130 & 10 \\ 130 & 10 & 10 \end{pmatrix}$$

leads to high uncertainty for players 1 and 2, which could be completely eliminated if they cooperated. Coalition $\{1, 2\}$ can credibly deviate from $\bar{x}$ by perfect pooling at state 0, using

$$x^{\{1,2\}} = (x^1, x^2) = \begin{pmatrix} 70 & 70 \\ 70 & 70 \end{pmatrix},$$

and so achieving a higher utility:

$$u^1(10, 130) = u^2(130, 10) \approx 0.8161 \ll u^1(70, 70) = u^2(70, 70) \approx 0.9991.$$

We have shown that $\mathrm{WSC}(\Gamma) = \emptyset$.

We will show next that if Γ_s is convex for all $s \in S$, then the Weak Sequential Core is non-empty.

Definition 5.2. A TU-game (N, v) is *convex* if for any $C \subset N$ and for all $S \subsetneq T \subset N \backslash C$ it holds that $v(S \cup C) - v(S) \leq v(T \cup C) - v(T)$.

Theorem 5.3. *Let Γ_s be convex for all $s \in S$. Then* $\mathrm{WSC}(\Gamma) \neq \emptyset$.

Proof. Let $\pi : N \to N$ be a permutation, assigning rank number $\pi(i)$ to any player $i \in N$. For a player $i \in N$, we define $\pi^i = \{j \in N \mid \pi(j) \leq \pi(i)\}$ as the set of predecessors of player i. For every $s \in S$, the marginal vector $m^\pi(\Gamma_s) \in \mathbb{R}^N$ is given by

$$m^{\pi,i}(\Gamma_s) = v_s(\pi^i) - v_s(\pi^i \backslash \{i\}), \quad i \in N,$$

and thus assigns to player i his marginal contribution to the worth of the coalition consisting of all his predecessors in π. We show that $\bar{x}$ defined by $\bar{x}_s = m^\pi(\Gamma_s)$, $s \in S$, belongs to $\mathrm{WSC}(\Gamma)$.

Since Γ_s is convex, it holds that $\bar{x}_s \in \mathrm{C}(\Gamma_s)$ for all $s \in S$ (Shapley, 1971). Using Theorem 4.4, it remains to be shown that there is no $C \subset N$ and allocation x^C such that $x_s^C \in \mathrm{C}(\Gamma_s, C)$ for all $s \in S$, and $u^i(x^i) > u^i(\bar{x}^i)$ for all $i \in C$.

Consider $C \subset N$ and x^C with $x_s^C \in \mathrm{C}(\Gamma_s, C)$ for all $s \in S$. Let i be the player in C with the highest $\pi(i)$. It holds that

$$x_s^i \leq v_s(C) - v_s(C \setminus \{i\}) \leq v_s(\pi^i) - v_s(\pi^i \setminus \{i\}) = \bar{x}_s^i,$$

where the first inequality follows since $x_s^C \in \mathrm{C}(\Gamma_s, C)$ and the second inequality since by the choice of i as the highest ranked player in C according to π it holds that $C \setminus \{i\} \subset \pi^i \setminus \{i\}$ and Γ_s is convex. By monotonicity of u^i we have that $u^i(x^i) \leq u^i(\bar{x}^i)$, which completes the proof. □

An interesting feature of Theorem 5.3 is that we do not need to make additional assumptions on the utility functions of the players. Within the framework of expected utility, we allow for both risk-averse and risk-loving players. Also many theories of non-expected utility maximization are covered by our result. This is in contrast to the classical definition of the Core, which might be empty-valued under the same assumptions. Considering the lack of results on non-emptiness of the Weak Sequential Core in the literature so far, this comes as a surprise.

6 Bankruptcy games

The class of convex TU-games admits a wide range of interesting applications. Examples are airport games (Littlechild and Owen, 1973), bankruptcy games (Aumann and Maschler, 1985), sequencing games (Curiel, Pederzoli, and Tijs, 1989) and standard tree games (Granot, Maschler, Owen, and Zhu, 1996). In this section we analyze the application of the Weak Sequential Core to bankruptcy games.

Bankruptcy games originate in a fundamental paper by O'Neill (1982). The problem is based on a Talmudic example, where a man dies, leaving behind an estate, E, which is worth less than the sum of his debts. The question is how the estate should be divided among the creditors.

A bankruptcy problem is defined as a pair (E, d), where $d = (d^1, \ldots, d^n)$ is the vector of individual debts, and $\sum_{i \in N} d^i \geq E \geq 0$. Following Aumann and Maschler (1985), the problem can be transformed into a cooperative game. The characteristic function $v^{E,d}$ is defined to be

$$v^{E,d}(C) = \max\{E - \sum_{i \in N \setminus C} d^i, 0\}, \quad C \subset N \tag{2}$$

so the worth of a coalition C in the game $v^{E,d}$ is that amount of the estate which is not claimed by the complement of C. It has been shown by Curiel, Maschler, and Tijs (1987) that $v^{E,d}$ is convex.

A *rule* is a function that associates with each (E, d) an allocation $x \in \mathbb{R}^N$ such that $\sum_{i \in N} x_i = E$ and $0 \leq x \leq d$. A thorough inventory of the rules can be found in Thomson (2003). The best-known rule is the *Proportional rule* (P) which allocates the estate proportional to the claims. The *Adjusted Proportional rule* (AP) selects the allocation at which each claimant i receives his minimal right $\max\{E - \sum_{j \neq i} d^j, 0\}$, then each claim is revised down accordingly, and finally, the remainder of the estate is divided proportionally to the revised claims. The *Constrained Equal Awards rule* (CEA) is in the spirit of equality; it assigns equal amounts to all claimants subject to no one receiving more than his claim. More formally, we have the following.

Definition 6.1 (Constrained Equal Awards rule)**.** For each bankruptcy problem (E, d), $\text{CEA}^i(E, d) = \min\{d^i, \alpha\}$, $i \in N$, where $\alpha \leq \max_{i \in N} d^i$ is chosen so that $\sum_{i \in N} \min\{d^i, \alpha\} = E$.

The *Constrained Equal Losses rule* (CEL), as opposed to the CEA rule, is focusing on losses claimants incur, and makes these losses equal, with no one receiving a negative amount. The recommendation of the Talmud, later formalized in Aumann and Maschler (1985) as the *Talmud rule* (TR) is a combination of the CEA rule and the CEL rule, depending on the relation of the half-claims and the value of the estate. The *Piniles' rule* (Piniles, 1861) is an application of the CEA rule to the half-claims in two different ways, again depending on the relation of the half-claims and the value of the estate. In our set-up it coincides with the Talmud rule. The *Constrained Egalitarian rule* (Chun, Schummer, and Thomson, 2001) also gives a central role to the half-claims, and guarantees that the awards are ordered as the claims are. In our case it also coincides with the Talmud rule. The *Random Arrival rule* (RA) takes all the possible orders of claimants arriving one at a time, compensates them fully until money runs out, and takes the arithmetic average over all orders of arrival.

The claims-truncated version of the rules are also considered in the literature. Truncating the claims at the value of the estate does not change the result of the CEA, TR and RA rules. The truncated-CEL and truncated-P can be blocked in the stochastic game.

Many rules are related to the solutions of bankruptcy games. The AP rule corresponds to the τ-value (Curiel, Maschler, and Tijs, 1987), the CEA rule to the Dutta-Ray solution (Dutta and Ray, 1989), the TR rule to the prenucleolus (Aumann and Maschler, 1985), and the RA rule to the Shapley value (O'Neill, 1982).

Any rule belongs to the Core of the bankruptcy game. Let $\bar{x}$ be the allocation that the rule associates to the bankruptcy problem (E, d). It holds that $\bar{x}(N) = v(N) = E$. Moreover, we have

$$v(C) = \max\{0, E - \sum_{i \in N \setminus C} d^i\} \leq \max\{0, E - \sum_{i \in N \setminus C} \bar{x}^i\} = \max\{0, \sum_{i \in C} \bar{x}^i\} = \bar{x}(C),$$

so no coalition can improve upon $\bar{x}$.

In the original *estate division problem* of the Talmud a man has 3 wives whose marriage contracts specify that upon his death they should receive 100, 200 and 300 respectively. When the man dies, his estate is found to be worth 100, 200 or 300 in three different scenarios.

The characteristic function of the resulting TU-games, with $d =$100, 200 and 300, is shown in Table 2.

Table 3 summarizes the outcomes of a number of rules applied to the estate division problem.

We are interested in the question to what extent the rules lead to allocations that are self-enforcing in the presence of uncertainty regarding the value of the estate and the size of the debts. A stochastic bankruptcy problem is defined as a tuple (S, E, d, u), where S is

Table 2: Characteristic function of estate division

$v^{E,d}$	$\emptyset$	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$v^{100,d}$	0	0	0	0	0	0	0	100
$v^{200,d}$	0	0	0	0	0	0	100	200
$v^{300,d}$	0	0	0	0	0	100	200	300

Table 3: Estate allocation

Player	Estate	TR	P	AP	CEA	CEL	RA
	100	33 1/3	16 2/3	33 1/3	33 1/3	0	33 1/3
$d^1 = 100$	200	50	33 1/3	40	66 2/3	0	33 1/3
	300	50	50	50	100	0	50
	100	33 1/3	33 1/3	33 1/3	33 1/3	0	33 1/3
$d^2 = 200$	200	75	66 2/3	80	66 2/3	50	83 1/3
	300	100	100	100	100	100	100
	100	33 1/3	50	33 1/3	33 1/3	100	33 1/3
$d^3 = 300$	200	75	100	80	66 2/3	150	83 1/3
	300	150	150	150	100	200	150

a finite set of states of nature, $E = (E_s)_{s \in S}$ is the value of the estate in state s, $d = (d_s)_{s \in S}$ is the state-dependent vector of debts, and $u = (u^i)_{i \in N}$ are the utility functions of the claimants, where $u^i : \mathbb{R}^S \to \mathbb{R}$. Extending the approach of Aumann and Maschler (1985) to the stochastic case, we can transform a stochastic bankruptcy problem into a stochastic bankruptcy game $\Gamma = (N, S, v, u)$, the TUU-game where we set

$$v_s(C) = \max\{E_s - \sum_{i \in N \setminus C} d_s^i, 0\}, \qquad s \in S,\ C \subset N.$$

We have already argued that a rule leads to an allocation in the Core of the bankruptcy game. This implies that for a stochastic bankruptcy game, blocking is not possible after the resolution of uncertainty. However, it might be possible to block ex ante.

As an example, consider the estate division problem of the Talmud, where the claims of the three wives are fixed to 100, 200, and 300, respectively, but the exact value of the estate is uncertain, and the possible values 100, 200, and 300 are equally likely. After the uncertainty regarding the estate's value is resolved in period 1, one of the three TU-games is played, arising from the original three scenarios of the problem. Suppose the wives evaluate the payoffs with the utility function

$$u^i(x^i) = \sum_{s \in S} \frac{1}{3}(1000x_s^i - (x_s^i)^2),$$

whenever $0 \leq x_s^i \leq 300$ for $s \in S$.[1]

We demonstrate that the grand coalition has a credible deviation from the allocation specified by all the rules mentioned before at state 0, with the exception of the Constrained Equal Awards rule. Table 4 lists the credible deviations, denoted by x, and Table 5 the implied utilities, as well as the utilities of the allocations $\bar{x}$ implied by the various rules.

Table 4: Credible deviations

Player	E	TR	P	AP	CEL	RA
$d^1 = 100$	100	25	0	25	0	29
	200	40	33	35	0	29
	300	70	70	65	0.01	60
$d^2 = 200$	100	25	30	25	0	34
	200	75	63	80	49	83
	300	110	108	110	101.13	100
$d^3 = 300$	100	50	70	50	100	37
	200	85	104	85	151	88
	300	120	122	125	198.86	140

Table 5: Utilities

Player	u	TR	P	AP	CEL	RA
$d^1 = 100$	$u^1(\bar{x}^1)$	42407.41	32037.04	39374.07	0.00	37314.81
	$u^1(x^1)$	42625.00	32337.00	39641.67	3.33	37572.67
$d^2 = 200$	$u^2(\bar{x}^2)$	63865.74	61481.48	65274.07	45833.33	66203.70
	$u^2(x^2)$	63883.33	61489.00	65291.67	45833.91	66318.33
$d^3 = 300$	$u^3(\bar{x}^3)$	76365.74	88333.33	77774.07	125833.33	78703.70
	$u^3(x^3)$	76958.33	88466.67	78216.67	125837.90	78762.33

We show next that the Constrained Equal Awards rule belongs to the Weak Sequential Core of the stochastic bankruptcy game under rather general circumstances.

[1] Outside this domain the utility function can be anything, as long as it is continuous, state-separable and monotonically increasing.

Theorem 6.2. *For $i \in N$, let the utility function be given by $u^i(x^i) = \sum_{s\in S} \rho_s w(x^i_s)$, where ρ_s is the objective probability of state s and w is a strictly concave function. Then the allocation implied by the Constrained Equal Awards rule belongs to the Weak Sequential Core of the stochastic bankruptcy game.*

Proof. Let $\bar{x}$ be the allocation implied by the Constrained Equal Awards rule. We have already argued that $\bar{x}_s \in \mathrm{C}(\Gamma_s)$ holds for all $s \in S$. By Theorem 4.4 it remains to be shown that there is no x^C such that $x^C_s \in \mathrm{C}(\Gamma_s, C)$ for all $s \in S$, and $u^i(x^i) > u^i(\bar{x}^i)$ for all $i \in C$

Consider a stochastic bankruptcy problem with set of players $C \subset N$, estate in state s equal to $\max\{E_s - \sum_{i\in N\setminus C} d^i_s, 0\}$ and claims equal to d^i_s for $i \in C$. The corresponding stochastic bankruptcy game is denoted by $(C, S, v^C, (u^i)_{i\in C})$. Let $\bar{y}^C$ be the allocation resulting from the CEA rule.

Note that for $D \subset C$, the worth of coalition D in game v^C_s coincides with its worth in the original game, since

$$\begin{aligned} v^C_s(D) &= \max\{v^C_s(C) - \sum_{i\in C\setminus D} d^i_s, 0\} \\ &= \max\{\max\{E_s - \sum_{i\in N\setminus C} d^i_s, 0\} - \sum_{i\in C\setminus D} d^i_s, 0\}, \end{aligned}$$

where either (a) $E_s - \sum_{i\in N\setminus C} d^i_s > 0$, and so $v^C_s(D) = \max\{E_s - \sum_{i\in N\setminus D} d^i_s, 0\} = v_s(D)$, or (b) $E_s - \sum_{i\in N\setminus C} d^i_s \leq 0$, and so $v^C_s(D) = 0 = v_s(D)$.

We have that $\bar{y}^C_s \in \mathrm{C}(\Gamma_s, C)$ for all $s \in S$. We show next that $\bar{y}^C$ maximizes the sum of the players utilities over allocations x^C with $x^C_s \in \mathrm{C}(\Gamma_s, C)$ for all $s \in S$.

Consider the following constrained maximization problem,

$$\max_{x^C} \sum_{i\in C} u^i(x^i)$$

$$\text{s.t.} \quad \sum_{i\in C} x^i_s = v_s(C), \quad s \in S, \tag{3}$$

$$\sum_{i\in D} x^i_s \geq v_s(D), \quad s \in S,\ \emptyset \neq D \subsetneq C, \tag{4}$$

where condition (3) is required for ex post efficiency and inequality (4) is a no-blocking condition. A solution to the maximization problem maximizes the sum of the players' utilities among those allocations that belong to $\mathrm{C}(\Gamma_s, C)$ for all $s \in S$.

We form the Lagrangian,

$$\mathcal{L}(x,\lambda,\mu)=\sum_{i\in C}\sum_{s\in S}\rho_s w(x_s^i)-\sum_{s\in S}\mu_s(\sum_{i\in C}x_s^i-v_s(C))-\sum_{s\in S}\sum_{D\subsetneq C}\lambda_s^D(\sum_{i\in D}x_s^i-v_s(D)).$$

The first-order conditions, which are necessary and sufficient for a maximum, are given by

$$\begin{aligned}
\rho_s w'(x_s^i)-\mu_s-\sum_{D\subsetneq C|D\ni i}\lambda_s^D &= 0, \quad s\in S,\ i\in C, && (5)\\
\sum_{i\in C}x_s^i-v_s(C) &= 0, \quad s\in S, && (6)\\
\lambda_s^D(\sum_{i\in D}x_s^i-v_s(D)) &= 0, \quad s\in S, \emptyset\neq D\subsetneq C, && (7)\\
\sum_{i\in D}x_s^i-v_s(D) &\geq 0, \quad s\in S, \emptyset\neq D\subsetneq C, && (8)\\
\lambda_s^D &\leq 0, \quad s\in S, \emptyset\neq D\subsetneq C. && (9)
\end{aligned}$$

We will show that together with an appropriate choice of λ and μ, $\bar{y}^C$ satisfies these first-order conditions. Conditions (6) and (8) hold since $\bar{y}_s^C\in \mathrm{C}(\Gamma_s,C)$ for all $s\in S$. To show that the remaining conditions hold as well, we introduce two subsets of players for each state, and distinguish two cases. For $s\in S$, let $I_s=\{i\in C|\bar{y}_s^i=d_s^i\}$ be the set of those agents whose claim is fully paid in state s.

1. $I_s=\emptyset$
   For all $\emptyset\neq D\subsetneq C$ we set $\lambda_s^D=0$, thereby satisfying conditions (7) and (9). Since $I_s=\emptyset$, it holds for all $i\in C$ that $\bar{y}_s^i>d_s^i$. By the definition of the CEA rule, $\bar{y}_s^i$ is independent of i. It follows that $\rho_s w'(\bar{y}_s^i)$ is also independent of i, thus we can define $\mu_s=\rho_s w'(\bar{y}_s^i)$ for all $i\in C$ to satisfy condition (5).

2. $I_s\neq\emptyset$
   Let $C=\{i_s^1,i_s^2,\ldots,i_s^c\}$, where $d_s^{i^1}\leq d_s^{i^2}\leq\ldots\leq d_s^{i^c}$ and c denotes the cardinality of C. Then, using the definition of the CEA rule, for some $k\geq 1$, $I_s=\{i_s^1,\ldots,i_s^k\}$. For $1<j\leq k+1$ we define $D_s^j=\{i_s^j,\ldots,i_s^c\}$, so $C\setminus D_s^j\subset I_s$ and $C\setminus D_s^{k+1}=I_s$. We define $\mu_s=\rho_s w'(\bar{y}_s^{i^1})$, i.e. the marginal utility of the player with the lowest claim in state s. For $1<j\leq k+1$ we define $\lambda_s^{D_s^j}=\rho_s w'(\bar{y}_s^{i^j})-\rho_s w'(\bar{y}_s^{i^{j-1}})$. By the definition of the CEA rule it holds that $\bar{y}_s^{i^j}\geq\bar{y}_s^{i^{j-1}}$, so $\lambda_s^{D_s^j}\leq 0$. For other coalitions D we set $\lambda_s^D=0$. It follows that condition (9) is satisfied. The definition of the CEA rule and

equation (2) imply that

$$\sum_{i\in D_s^j} \bar{y}_s^i = v_s(C) - \sum_{i\in C\setminus D_s^j} \bar{y}_s^i = \max\{E_s - \sum_{i\in N\setminus C} d_s^i, 0\} - \sum_{i\in C\setminus D_s^j} d_s^i. \tag{10}$$

Since $\sum_{i\in D_s^j} \bar{y}_s^i \geq 0$,

$$\max\{E_s - \sum_{i\in N\setminus C} d_s^i, 0\} - \sum_{i\in C\setminus D_s^j} d_s^i = \max\{E_s - \sum_{i\in N\setminus D^j} d_s^i, 0\} = v_s(D_s^j). \tag{11}$$

It follows from equation (10) and (11) that $\sum_{i\in D_s^j} \bar{y}_s^i - v_s(D_s^j) = 0$, so condition (7) is satisfied.

It only remains to show that condition (5) is satisfied as well. All coalitions D that contain player i^1 have $\lambda_s^D = 0$, so for player i^1 this is immediate. Consider player $i^{j'}$ for $1 < j' \leq k$. The only coalitions D such that $i^{j'} \in D$ and $\lambda_s^D \neq 0$ are of the form $\{i^j, \ldots, i^c\}$, for $1 < j \leq j'$. Equation (5) reduces to

$$\rho_s w'(\bar{y}_s^{i^{j'}}) - \rho_s w'(\bar{y}_s^{i^1}) - \sum_{j=2}^{j'} (\rho_s w'(\bar{y}_s^{i^j}) - \rho_s w'(\bar{y}_s^{i^{j-1}})) = 0.$$

Finally, consider $i \in C\setminus I_s$. Note that all such players receive the same payoff in state s, equal to $\bar{y}^{i^{k+1}}$. Since player i is part of all the coalitions D_s^j, we have that equation (5) reduces to

$$\rho_s w'(\bar{y}_s^{i^{k+1}}) - \rho_s w'(\bar{y}_s^{i^1}) - \sum_{j=2}^{k+1} (\rho_s w'(\bar{y}_s^{i^j}) - \rho_s w'(\bar{y}_s^{i^{j-1}})) = 0.$$

Thus $\bar{y}^C$ satisfies all the first-order conditions. It follows that there is no x^C such that $x_s^C \in \mathrm{C}(\Gamma_s, C)$ for all $s \in S$, and $u^i(x^i) > u^i(\bar{y}^i)$ for all $i \in C$.

We show next that $u^i(\bar{x}^i) \geq u^i(\bar{y}^i)$ for all $i \in C$. For all $s \in S$, it follows from the definition of a rule that $\sum_{i\in C} \bar{y}_s^i = v_s(C)$ and $v_s(C) \leq \sum_{i\in C} \bar{x}_s^i$. Using the definition of the CEA rule it follows that for all $s \in S$ and $i \in C$, $\bar{y}_s^i \leq \bar{x}_s^i$. Since the utility function is monotonically increasing, we have that $u^i(\bar{x}^i) \geq u^i(\bar{y}^i)$ for all $i \in C$. Therefore, there is no x^C such that $x_s^C \in \mathrm{C}(\Gamma_s, C)$ for all $s \in S$, and $u^i(x^i) > u^i(\bar{x}^i)$ for all $i \in C$, thereby showing part (b) of Theorem 4.4.

□

The introduction of uncertainty into the bankruptcy problem leads to additional insights into the nature of the division rules. While in the original static game, all the proposed solutions are stable against deviations, it is only the Constrained Equal Awards rule for which this crucial property carries over to the stochastic setting.

7 Conclusion

In this paper we have introduced uncertainty into transferable utility games. Since in reality most decisions are made under uncertainty, this is a natural and important extension. It is not straightforward though, how to define an appropriate Core concept for this stochastic setting. In this paper we consider allocations that are stable in the absence of commitment possibilities. These requirements lead to the notion of credibility. A credible deviation is self-enforcing in the sense that a coalition can credibly deviate from a given allocation if no sub-coalition ever has a credible counter-deviation. These considerations lead to the definition of the Weak Sequential Core.

This notion of stability leads to a characterization of the Weak Sequential Core; all allocations in the Weak Sequential Core belong to the Core of the transferable utility game played after the resolution of uncertainty. Moreover, no coalition can block an allocation in the Weak Sequential Core ex ante by means of an allocation that belongs to the Core of all the ex post games reduced to the coalition. This property facilitates the application of the concept and the proof of its non-emptiness. We show that convexity of the ex post games is sufficient for the non-emptiness of the Weak Sequential Core.

A famous application which leads to convex games is the bankruptcy problem. We introduce the stochastic bankruptcy problem and transform it into a TUU-game. We show that most of the best-known allocation rules are unstable for stochastic bankruptcy games. The Constrained Equal Awards rule is the only exception, and its application leads to allocations in the Weak Sequential Core.

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